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Theory of Photons in a Fully Ionized Gas. II. Planck's Law and the Dielectric Constant*

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A theory of the radiation spectrum in a nonrelativistic fully ionized gas in thermal equilibrium is suggested, based on Planck's law of radiation. From an observation regarding the photon momentum distribution, an exact and simple dispersion relation for transverse electromagnetic waves in the medium is deduced, and a new dielectric "constant" is obtained. An important feature of the theory is the relative ease with which the thermodynamic functions describing the radiation field can be calculated.

I. INTRODUCTION AND THEORY

In the preceding paper[1] a systematic theory of the photon momentum distribution in a fully ionized gas was constructed. The microscopic theory was developed in a somewhat standard way and applied to the calculation of several thermodynamic functions pertaining to radiation in the medium. Of particular interest was the roundabout procedure required to find the energy density of radiation in the gas. Although new results were obtained, it is clear that not much was gained in the way of insight into the manner in which the presence of charged particles affects the over-all view of the radiation field. More precisely, the discussion in Sec. I – VII has underscored the difficulties involved in attempting to discuss Planck's law of radiation in the context of a fully ionized gas. It was found very difficult to resolve the apparent conceptual conflicts with classical thermodynamics. In the present paper, a conceptually different approach is developed which, it is to be hoped, will lead to a definitive view of the radiation field in a fully ionized medium.

In his well-known treatise on thermal radiation,[2] Planck proved that the energy density of radiation as a function of temperature and frequency, $u(\beta, \omega)$, is a universal function independent of the medium. That is, for two different media

$$q_1^2 u_1(\beta, \omega) = q_2^2 u_2(\beta, \omega) , \tag{1}$$

where $\beta = (\kappa T)^{-1}$, κ is Boltzman's constant, T is the absolute temperature, and q is the speed of propagation in the medium. For transparent media one can relate q to the index of refraction through $q^2 = c^2/\eta^2$. The very important quantities q_i are often omitted in otherwise quite good textbooks, where one finds the statement that

$$u(\beta, \omega) = (\hbar\omega^3/\pi^2 c^3)\nu_\gamma(\omega) , \tag{2}$$

$$\nu_\gamma(\omega) = e^{-\beta\hbar\omega}/(1 - e^{-\beta\hbar\omega}), \tag{3}$$

is a universal function of β and ω. An exception is the book by Landau and Lifshitz,[3] in which it is observed that for a dispersive medium, $\eta = \eta(\omega)$, the universal function is really

$$u(\beta, \omega) = \hbar\omega\nu_\gamma(\omega)\frac{\omega^2\eta^2}{\pi^2 c^3}\frac{d}{d\omega}(\omega\eta). \tag{4}$$

This expression is equivalent to (1), corresponding to Planck's original statement. The proof of Eq. (1) is the well known one based on the Second Law of Thermodynamics[2] and, in view of the certain validity of the argument, one must conclude that Eq. (4) is the only possible modification of Planck's law for an ionized gas in thermal equilibrium.

The utility of Eq. (4) has not, of course, gone entirely unnoticed, and a fashionable procedure has been to discuss the ionized medium in terms of its dielectric "constant," $\epsilon = \eta^2$. Several calculations of ϵ have been made,[4–9] but these approaches have generally not focused on the relevance to the radiation spectrum *per se*, nor have they admitted calculations of high precision. In general applications, it is usually assumed that the dispersion relation

$$\hbar\omega(k) \simeq \omega_\Gamma(k) \equiv \hbar(c^2k^2 + \omega_p^2)^{1/2} , \tag{5}$$

obtained from microscopic principles in I, leads to a good approximation to the dielectric constant (or, more appropriately, dielectric function). We have introduced here the plasma frequency, defined by

$$\omega_p^2 = \sum_\alpha \omega_p^2(\alpha) = \sum_\alpha (4\pi\rho_\alpha Z_\alpha^2 e^2)/M_\alpha , \tag{6}$$

where the sum is over all types of charged particles in the system, ρ_α is the number density of α-type (charged) particles, M_α is their mass, and Z_α their charge number.

Nevertheless, if we wish to calculate the energy density in momentum space, which is generally much simpler than using the energy representation, then it is necessary to write $\omega = \omega(k)$ and replace $\nu_\gamma(\vec{k})$ by the true momentum distribution of photons in the system. In I we developed a systematic theory of the momentum distribution, $\langle n_\gamma(\vec{k})\rangle$, and it was seen that there appear large deviations from the form (2) in higher order. Moreover it is not clear how to specifically obtain the form (4) for the frequencies, since the two distributions in $\vec{k}$ and ω are not necessarily related by a simple Fourier transformation. Therefore, how is one to interpret the photon momentum distribution, such as that obtained in Eq. (I-36), which is calculated in a straightforward manner from microscopic statistical mechanics? Furthermore, how can one possibly obtain the form (4) for Planck's law from such calculations?

In the view of the author, the answers to these questions rest with the lessons to be learned from classical thermodynamics. That is, Eq. (1) seems to be unimpeachable, and should actually be built into the theory. In order to implement this idea, let us recall from I that an exact formal expression for the photon momentum distribution is

$$\langle n_\gamma(\vec{k})\rangle = \nu_\gamma(\vec{k}) \frac{1+\int_0^\beta L_\gamma(\beta,t,\vec{k})\,dt}{1-\nu_\gamma(\vec{k})\int_0^\beta L_\gamma(\beta,t,\vec{k})\,dt}, \tag{7}$$

where $\nu_\gamma(\vec{k})$is given by Eq. (3) with $\omega = ck$, and $L_\gamma(\beta, t, \vec{k})$ is the sum over all master L graphs, characterized by one incoming and one outgoing line carrying photon momenta $\vec{k}$. The sum L_γ can be calculated to an arbitrary degree of precision by evaluating diagrams of consistently higher order,[1] a task which asymptotically approaches the impossible. In momentum space, the energy density of radiation is

$$u(\beta,k) = [\langle n_\gamma(\vec{k})\rangle/\Omega]\,\omega(k)(dN/dk), \tag{8}$$

where dN/dk is the density of states, and Ω is the volume of the system. It should be obvious that as one calculates more and more corrections to the momentum distribution, Eq. (8) will deviate more and more from any functional form remotely resembling Planck's law. One cannot conclude absolutely, though, that a complete evaluation of L_γ would not eventually provide a microscopic derivation of Planck's law. This, however, would probably be impossible, because such a calculation would be tantamount to performing infinite-order perturbation theory. Furthermore, the same situation prevails in any other approach to calculating the momentum distribution.

A conceptually different approach begins with the observation that for *any* kind of particle the distribution in energy should be identical to the distribution in momentum. That is, for any physically reasonable energy-momentum relation, there should exist a one-to-one correspondence between energy and momentum, so that for N particles with momentum $\vec{k}$ there should be N particles with energy $\omega(k)$, in a uniform system. For most particles, however, this observation provides no means for determining what these distributions should be. But, and this is the major point, Eq. (1) provides an additional condition in the case of photons, which is not available for particles with mass. Thus one is impelled to adopt Planck's law from the beginning, and demand that the photon momentum distribution have the form

$$\langle n_\gamma(\vec{k})\rangle = \frac{e^{-\beta\hbar\omega(k)}}{1-e^{-\beta\hbar\omega(k)}} \equiv \nu(\vec{k}). \tag{9}$$

To be sure, Eq. (9) is an assertion, but it seems to have strong support from the preceding discussion. Moreover, similar results appear to have been found in studying quasiparticle descriptions of other systems,[10] and in a sense the photon is already a quasiparticle representing the radiation field. If Eqs. (7) and (9) are equated, it is now an easy matter to show that

$$\hbar\omega(k) = \omega_\gamma(k) - \beta^{-1}\ln[1+\int_0^\beta L_\gamma(\beta,t,\vec{k})\,dt], \tag{10}$$

where $\omega_\gamma(k) = \hbar ck$. This represents an exact and extremely simple expression for the dispersion relation in the medium from which it is then possible to identify the index of refraction to be inserted into Eq. (4). Further, Eq. (10) is valid at all (nonrelativistic) temperatures and frequencies.

It is difficult to appreciate the power and simplicity of Eq. (10) until one actually uses it and compares the results with previous techniques and calculations, for its importance rests with the predictions it can make. First of all, the function L_γ can be calculated in a straightforward manner by simply evaluating diagrams as indicated in I and elsewhere.[11,12] The first dozen or so diagrams are relatively easy to evaluate, but the algebra becomes progressively more tedious in higher order, as is to be expected. However, in practical applications one probably does not need the dispersion relation beyond, say, order e^6.

Secondly, one need not be concerned now with actually summing all the L graphs contributing to L_γ, because we have adopted the hypothesis that the radiation spectrum is given exactly by Eq. (4). There is no need to derive microscopically Planck's law, because it is true *ab initio*.

Finally, Eq. (10) allows one to reproduce previous results, such as those of the preceding paper, almost trivially, and obtain new results in an equally effortless manner. This potency will be demonstrated in the following section.

Prior to verifying these statements, though, it is first necessary to rectify a calculational defect intrinsic to the expression (10). In I it was shown that the diagrammatic iteration of the function L_γ was ill-behaved at low-momentum values, being essentially a series expansion in $|\vec{k}|^{-1}$. Thus one must find an analytic continuation to $|\vec{k}| = 0$. But this is precisely what is accomplished by the Bogoliubov transformation of Eq. (I-24), and the redescription of the theory developed in I can be adapted wholesale here. Hence in place of Eq. (10) we write the completely equivalent expression

$$\hbar\omega(k) = \omega_\Gamma(k) - \beta^{-1}\ln[1+\int_0^\beta L_\Gamma(\beta,t,\vec{k})\,dt], \tag{11}$$

where $\omega_\Gamma(k)$ is defined in Eq. (5). The essentials of the Γ theory are described in Sec. III of I, the physically important point being the screening of the photon-charged particle interactions by the medium.

II. CALCULATIONS

We are interested in the nonrelativistic, homogeneous, fully ionized gas at high temperature and low density. In the preceding paper[1] a detailed parameter analysis of this system was given, and the reader is referred to that paper for the limits on the various parameters of the theory in this region of the temperature-density spectrum. We shall also refer to that work for minor calculational details which will be omitted here, at the same time assuming the necessary mass renormalization to be carried out implicitly at all stages of the calculations.

As a first example of the physical content of the dispersion relation (11), let us note that ignoring the logarithm yields the well-known approximate result,

$$\omega(k) \simeq (c^2k^2 + \omega_p^{\,2})^{1/2} \ . \tag{12}$$

It is usually stated[4-9] that this relationship is correct to order (v^2/c^2); that is, correct in the nonrelativistic limit.[13] This statement is true only in the "bubble-diagram" approximation which, in the present formalism, corresponds to summing over all contributions from zero-momentum transfer diagrams. These diagrams were effectively summed by the Bogoliubov transformation introduced in I, and this procedure yields precisely the relation (12). To be sure, Eq. (12) is correct up to relativistic corrections in this approximation. However, it is quite clear that Eq. (11) is accurate to any degree of approximation in the plasma frequency, independent of relativistic corrections.

In order to illustrate this point, let us evaluate the next correction term to Eq. (12), which is nonrelativistic and follows trivially from the evaluation of diagrams in I. The lowest-order diagram contributing to the function L_Γ in Eq. (11) is that of Fig. 1, and this quantity was evaluated in Eq. (I-31). Since this term is of order $\zeta^2 = (\beta\hbar\omega_p)^2 \ll 1$, we can expand the logarithm in Eq. (11) and retain only the first term. Then

$$\omega(k) \simeq (c^2k^2 + \omega_p^{\,2})^{1/2} - (\hbar\beta)^{-1}[\omega_p^{\,2}/(c^2k^2 + \omega_p^{\,2})]\,(\cosh\beta\omega_\Gamma - 1)/\beta\omega_\Gamma \ . \tag{13}$$

To the author's knowledge, this dispersion relation for transverse electromagnetic waves has not appeared in the literature before; yet, it follows immediately from what is essentially the simplest approximation to the expression (11). The relativistic corrections obtained by previous authors constitute corrections to Eq. (13) also. That is, in evaluating the diagram $L_\Gamma^{(2)}$ in I we obtain Eq. (I-35a), which contains terms restricting charged particle energies to values much smaller than photon energies[14]: $(\hbar^2k^2/2M_\alpha)/\omega_\Gamma(k)$. If only the leading-order relativistic correction is kept and the second term in Eq. (13) is neglected, one obtains[15]

$$\omega(k) \simeq (c^2k^2 + \omega_p^{\,2})^{1/2} + \tfrac{1}{2}[k/(\beta M)^{1/2}]\,\omega_p/(c^2k^2 + \omega_p^{\,2})^{3/2} \tag{14}$$

in keeping with previous results.[4] Nevertheless, Eq. (13) is much more important than (14), and we shall return to the former equation when the dielectric constant is discussed below. For now, it should be noted that the frequency cutoff at ω_p occurring in Eq. (12) is lowered by a factor of $\frac{1}{2}$ in Eq. (13).

The thermodynamic functions describing radiation in the medium which were obtained in I now follow easily from Eqs. (9) and (13). Thus the number density of photons in the medium is

$$\rho_\gamma = \frac{1}{\Omega}\sum_{\vec{k}} \nu(\vec{k}) \simeq \frac{2}{\pi^2}\int_0^\infty \nu_\Gamma(\vec{k})k^2dk - \frac{2}{\pi^2}(\hbar\omega_p)^2\int_0^\infty \frac{\cosh\beta\omega_\Gamma - 1}{\beta\omega_\Gamma}\,\nu_\Gamma(\vec{k})\,[1+\nu_\Gamma(\vec{k})]\frac{k^2dk}{\omega_\Gamma^{\,2}}$$

$$\simeq [2/\pi^2(\beta\hbar c)^3]\,(1.202 - \tfrac{1}{4}\zeta^2\ln\zeta - \tfrac{1}{4}\textstyle\sum_\alpha \zeta_\alpha^{\,2}\ln\eta_\alpha) + O(\zeta^2), \tag{15}$$

as in Eqs. (I-38) and (I-39). The parameters η_α, ζ, and ζ_α are defined in Eqs. (I-14) and (I-18), and the sum is over all types of charged particles in the system. This result was discussed in I.

An approximate expression for the dielectric constant can be extracted from the approximation (13) by an obvious iteration procedure. One finds

$$\epsilon(\omega) \simeq (1 - \omega_p^{\,2}/\omega^2) + 2(\hbar\beta)^{-1}(\omega_p^{\,2}/\omega^3)(\cosh\beta\hbar\omega - 1)/\beta\hbar\omega \to 1, \text{ as } \hbar \text{ or } \omega \to 0. \tag{16}$$

This expression also seems to be new. Moreover the index of refraction can now be obtained in this order of approximation and the spectral energy-density calculated by means of Eqs. (4) and (11). This function has been calculated and compared with that for the vacuum, as well as with that using Eq. (12), in Fig. 2.

It is particularly interesting to calculate the total energy-density of radiation, which is now given

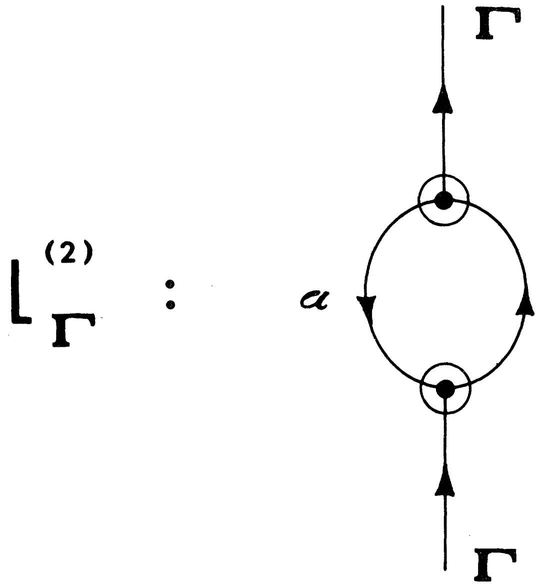

FIG. 1. The lowest-order master L graph contribution to the function $L_{\Gamma}(\beta, t, \vec{k})$ in Eq. (11).

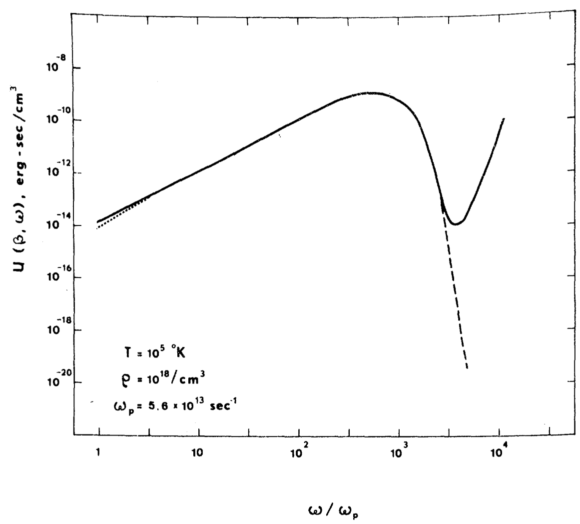

FIG. 2. The spectral energy-density computed from Eqs. (4) and (16) is given by the solid line, and compared with that for the vacuum (dashed line) and that computed from Eq. (12) (dotted line). The turn-up of the solid line is valid as far as indicated, after which relativistic particle effects are important and probably bring the curve down again.

exactly by the simple prescription

$$u(\beta) = (1/\Omega)\sum_{\vec{k}} \omega(k)\nu(\vec{k}) . \tag{17}$$

Contrary to the conclusions reached in I, this relation is precisely correct, because any approximate calculation of $\omega(k)$ automatically makes $\omega(k)$ and $\nu(\vec{k})$ the true photon energy and momentum distribution, respectively, to the *same* order. If Eqs. (13) and (9) are now substituted into (17), one finds

$$\begin{aligned} u(\beta) &\simeq (1/\Omega)\sum_{\vec{k}} \omega_{\Gamma}(k)\nu_{\Gamma}(\vec{k}) - [(\hbar\omega_p)^2/\beta\Omega]\sum_{\vec{k}} [\nu_{\Gamma}(\vec{k})/\omega_{\Gamma}^{2}(k)]\,(\cosh\beta\omega_{\Gamma} - 1)/\beta\omega_{\Gamma} \\ &\simeq (\pi^2\kappa^4/15\hbar^3c^3)T^4[1 + (15/2\pi^4)\textstyle\sum_{\alpha}\zeta_{\alpha}^{2}\ln\eta_{\alpha} + O(\zeta^4)] , \end{aligned} \tag{18}$$

again in agreement with I, Eq. (I-48). Note that the leading-order effect from the interaction of photons with charged particles tends to reduce the total energy density as expected.[8]

A word of caution should be inserted at this point. The parameter $\eta_{\alpha} = \kappa T/M_{\alpha}c^2$, appearing in both Eqs. (15) and (18), is a relativistic parameter, and its smallness measures the nonrelativistic behavior of α-type charged particles. One should note carefully, however, that its appearance here is due to the intrinsic relativistic nature of the radiation field, and is *not* a correction due to particle motion.

As a final illustrative calculation, let us consider the leading-order effects of Coulomb interactions in the fully ionized gas as they bear on the radiation quantities discussed above. The dominant contribution from charged-particle interactions comes from the diagram of Fig. 3 which was evaluated in Sec. VI of I. The relevant quantity to be substituted into Eq. (11) is that of Eq. (I-61), and we find in place of (13)

$$\omega(k) \simeq (c^2k^2 + \omega_p^{\,2})^{1/2} - \frac{1}{\hbar\beta}\left(\frac{\omega_p^{\,2}}{c^2k^2 + \omega_p^{\,2}}\right)\frac{\cosh\beta\omega_{\Gamma} - 1}{\beta\omega_{\Gamma}} - \frac{1}{\hbar\beta}\,\frac{f(\beta\omega_{\Gamma})}{\beta\omega_{\Gamma}}\sum_{\alpha}\zeta_{\alpha}^{\,2}\Lambda_{\alpha} , \tag{19}$$

where ζ_{α} and Λ_{α} (= Debye-Hückel parameter) are defined in I, and $f(\beta\omega_{\Gamma})$ is given by Eq. (I-62). In the temperature-density region of interest here, this Coulomb-correction is quite negligible, and is probably no more important than the omitted photon-charged particle diagrams of order ζ^4 (see Figs. 2 and 3 of I). The alteration of Eqs. (15), (16), and (18) due to the additional term in the dispersion relation, (19), is straightforward and we shall not write down the explicit equations here.

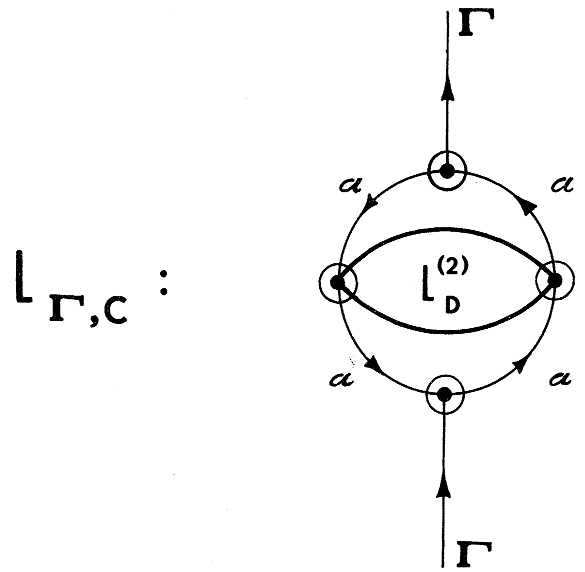

FIG. 3. The leading-order diagram containing Coulomb interactions which contributes to the function $L_\Gamma(\beta, t, \vec{k})$ in Eq. (11).

III. DISCUSSION

The major objective here has been to try to overcome the conceptual difficulties associated with relating the photon momentum distribution to the radiation spectrum in a fully ionized gas. Thus it has been asserted that Eq. (9) must be the true momentum distribution, and this leads directly to the dispersion relation (11). We have presented both physical and calculational support for this assertion, although it is to be emphasized that the next stage is to muster experimental support, say, for Eq. 16. For instance, the second term in this equation should be most prominent at high, but nonrelativistic frequencies.

There are, of course, possible objections which can be raised against the theory, one of which is that unphysical results may appear when higher-order calculations are performed. Therefore it would be of some value to include contributions from the diagrams of Figs. 2 and 3 in I in the quantities calculated here, particularly if experiments sensitive to these terms can be devised.

Finally, the deviation of the spectral energy density from the vacuum curve, indicated in Fig. 2, should be a measurable difference, and such an observation would provide a good test of the theory.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. C. R. Smith for many stimulating discussions concerning the problem studied here. It is also in order to thank Eugene Nosal for helping the author convince himself of the essential role to be played by the Bogoliubov transformation in analytically continuing the theory to small $|\vec{k}|$.

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[*]Supported in part by a grant from the National Science Foundation.

[1]In Kil Hwang and W. T. Grandy, Jr., preceding paper, Phys. Rev. 177, 359 (1969); hereafter referred to as I. The reader is asked to refer to I, if necessary, in order to clarify the notation used in the present paper.

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[13]See, however, Silin (Ref. 5), who has obtained an effective nonrelativistic correction to Eq. (12) in his study of the high-frequency dielectric constant in a plasma.

[14]One cannot, of course, use Eq. (13) for very large, relativistic values of momentum.

[15]The multicomponent subscripts have been omitted in Eq. (14), so as to facilitate comparison with the results of others. Therefore, one should think of (14) as referring to electrons only.