PHYSICAL REVIEW VOLUME 177, NUMBER 1 5 JANUARY 1969

Theory of Photons in a Fully Ionized Gas. I. Photon Momentum Distribution*†

In Kil Hwang and W. T. Grandy, Jr.
Physics Department, University of Wyoming, Laramie, Wyoming
(Received 5 August 1968)

The role of the photon momentum distribution and its relation to the thermal radiation spectrum in a fully ionized gas is studied. An analytic continuation of the theory to low-momentum quantum numbers is developed which is valid for the nonrelativistic system at high temperature and low density. A mass renormalization is carried out in order to treat the electromagnetic self-energies systematically, after which several new results are calculated for the number density, energy density, and momentum distribution for photons. Leading-order effects due to Coulomb interactions are also found for the thermodynamic functions of the radiation field. Of particular importance is a careful analysis of the dependence of the radiation spectrum on the photon momentum distribution, and this relation is found to be far from straightforward.

I. INTRODUCTION

Although considerable work has been devoted to the study of the dielectric constant for an ionized medium,[1] or for its electron-gas approximation, little effort has been expended on the explicit study of the thermal radiation spectrum of a fully ionized gas at high temperatures and low densities. In particular, the relation of Planck's law for the energy density of blackbody radiation to the radiation spectrum from an actual ionized gas is not at all clear. As is well known, the energy density in momentum space is a functional of the photon momentum distribution, and the purpose of this paper is to take a first step toward a microscopic understanding of the radiation spectrum by presenting a detailed study of the momentum distribution at nonrelativistic temperatures and densities. In the following paper, the second step is taken and the explicit relation of the momentum distribution to the radiation from the medium is found. It is quite possible that the results of this investigation may have important astrophysical implications, in that the interpretation of radiation observations might be enhanced.

Several authors[2–6] have indeed studied the photon momentum distribution in the fully ionized gas, but the expressions obtained generally suffered from the fact that they are only valid for small momentum values. Chappell[4] indicated how an analytic continuation to low-momentum quantum numbers might be made; however, he did not develop the theory in detail, nor apply it explicitly. In a sense, this paper takes up that problem. Both Nakai[3] and Grandy[6] investigated the effect of charged-particle interactions on the photon momentum distribution, an effect which is found to be almost negligible. We shall again investigate this question here.

In Sec. II the formal equations of the quantum statistics of multicomponent systems are reviewed, and those pertinent to a study of the photon momentum distribution exhibited. The approximations necessary for a systematic calculation are then discussed in the context of a diagrammatic analysis. Of particular interest is the straightforward iterative technique, which is outlined briefly and which uncovers the inadequacy of such an approach for small values of photon momentum quantum numbers. The section concludes with a careful physical description of the system to be studied, along with a detailed parameter analysis of the nonrelativistic, high-temperature, low-density gas.

The analytic continuation of the theory so as to encompass low-momentum quantum numbers is developed in Sec. III. This continuation is based on a Bogoliubov-type transformation first discovered by Chappell *et al.*,[7] and which effectively removes unphysical zero momentum transfers from the interaction Hamiltonian. These formal developments allow us to calculate quite easily the momentum distribution and radiation spectrum, as well as relevant thermodynamic quantities, in Sec. IV.

As is well known, the nonrelativistic Hamiltonian describing the interaction between photons and charged particles contains two types of interaction: a term in $\vec{p}\cdot\vec{A}$, and a term in $\vec{A}^2$, where $\vec{A}$ is the vector potential in the Coulomb gauge ($\vec{\nabla}\cdot\vec{A}=0$), and $\vec{p}$ is the momentum of a charged particle. When one studies the higher-order $\vec{A}^2$ diagrams occurring in the expansion of the photon momentum distribution, one encounters a large number of divergent integrals in the momentum representation. The divergences are intrinsic to any theory of quantum electrodynamics; indeed, they occur in such a systematic manner that they can be eliminated by a straightforward mass renormalization. Such a procedure for removing these self-energy terms in statistical mechanics was developed in detail by Mohling and Grandy[8] (MG), and in Sec. V it is shown that this method provides for a very simple treatment of these higher-order terms, and is carried out explicitly through second order in the fine-structure constant. Further higher-order contributions arise due to Coulomb interactions among charged particles, and radiative corrections to these interactions, and the nature of these effects, is discussed in Sec. VI.

In Sec. VII we attempt to analyze the physical meaning of the photon momentum distribution which has been calculated, along with its relation to the thermal radiation spectrum. In particular, we discuss the relative merits of introducing the notion of "quasiphotons" versus that of a macroscopic index of refraction; what can precisely be said about the function $u(\beta, k)$, Eq. (40); and finally, the interpretation to be attached to the energy-momentum relation for photons.

Our aim here is to analyze very carefully the momentum distribution for photons in a nonrelativistic, fully ionized gas, and to investigate the relation of this function to the radiation spectrum of the system. It will be seen that it is quite difficult to reach a satisfactory conclusion in the latter respect, which is unfortunate, because the momentum distribution is so relatively easy to calculate. However, the theory presented here forms a sound foundation for and contains the germ of the solution to the physical problem, and in the following paper a solution is suggested.

II. QUANTUM STATISTICS OF MULTICOMPONENT SYSTEMS

Prior to developing a theory of the photon momentum distribution, it is first necessary to review briefly some of the equations of the quantum statistical mechanics of multicomponent systems. Such a theory, which specifically incorporated charged particles and photons into the system, was described in some detail in MG, and we shall refer the reader to that paper for a more detailed discussion and derivation of the theory.[8] The electromagnetic interactions are most conveniently discussed in Fock space and the momentum representation, in which case the Hamiltonian takes the form

$$H=H_0+V_C+V_\gamma, \tag{1}$$

where

$$H_0=\sum_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}\omega_{\vec{k}}, \tag{2}$$

$$V_C=\frac{1}{2}\sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} a_1^\dagger a_2^\dagger\langle\vec{k}_1\vec{k}_2|V_C|\vec{k}_3\vec{k}_4\rangle a_4a_3, \tag{3}$$

$$V_\gamma=V_{1\gamma}+V_{2\gamma}+V_{1\gamma}^\dagger+V_{2\gamma}^\dagger, \tag{4}$$

corresponding to the free-particle Hamiltonian, the Coulomb interaction, and the interaction of charged particles with photons, respectively. We have adopted the notation that the sums over states $\vec{k}$ range over all free-particle states (including spin) of all the different kinds of particles in the system, so that these sums implicitly include sums over particle types α.[9] Thus the creation and annihilation operators satisfy either fermion or boson commutation relations, as the case may be. The quantity $\omega_{\vec{k}}=\omega_\alpha(k)=\hbar^2k^2/2M_\alpha$ for α-type particles, and $\omega_{\vec{k}}=\omega_\gamma(k)=\hbar ck$ for photons, where M_α is the mass of an α-type particle. Explicit expressions for the interaction terms of Eq. (4) are given as follows:

$$V_{1\gamma}=\sum_{\vec{k}_1\vec{k}_2\vec{k}_3} a_1^\dagger\langle\vec{k}_1|V_{1\gamma}|\vec{k}_2\vec{k}_3\rangle a_2a_3^{(\gamma)}, \tag{5a}$$

$$V_{2\gamma}=\sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} a_1^\dagger\langle\vec{k}_1|V_{2\gamma}|\vec{k}_2\vec{k}_3\vec{k}_4\rangle a_2a_3^{(\gamma)}a_4^{(\gamma)} + \sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} a_1^\dagger a_2^{(\gamma)\dagger}\langle\vec{k}_1\vec{k}_2|V_{2\gamma}|\vec{k}_3\vec{k}_4\rangle a_3a_4^{(\gamma)}, \tag{5b}$$

where the (real) matrix elements of $V_{1\gamma}$ and $V_{2\gamma}$ are given in the Coulomb gauge by

$$\langle\vec{k}_1|V_{1\gamma}|\vec{k}_2\vec{k}_3\rangle = -Z_1(\hbar^2/M)(2\pi\alpha/\Omega k_3)^{1/2}(\vec{k}_1\cdot\hat{\epsilon}_3)\times\delta(\vec{k}_1,\vec{k}_2+\vec{k}_3)\delta(m_1,m_2), \tag{6a}$$

$$\langle\vec{k}_1|V_{2\gamma}|\vec{k}_2\vec{k}_3\vec{k}_4\rangle = Z_1^2(\hbar^2\pi\alpha/M\Omega)(k_3k_4)^{-1/2}(\hat{\epsilon}_3\cdot\hat{\epsilon}_4)\times\delta(\vec{k}_1,\vec{k}_2+\vec{k}_3+\vec{k}_4)\delta(m_1,m_2), \tag{6b}$$

$$\langle\vec{k}_1\vec{k}_2|V_{2\gamma}|\vec{k}_3\vec{k}_4\rangle = Z_1^2(\hbar^2\pi\alpha/M\Omega)(k_2k_4)^{-1/2}(\hat{\epsilon}_2\cdot\hat{\epsilon}_4)\times\delta(\vec{k}_1+\vec{k}_2,\vec{k}_3+\vec{k}_4)\delta(m_1,m_3). \tag{6c}$$

In these equations, $\delta(a,b)$ is a Kronecker δ function of a and b, Z_1 is the charge number of particle (1) in units of $|e|$, m_1 is the spin-projection quantum number of that particle, $\alpha=e^2/\hbar c$ is the fine-structure constant, Ω is the volume of the system, and $\hat{\epsilon}_i$ is the polarization unit vector of the photon in the ith state and satisfies the transversality relation

$$\vec{k}_i\cdot\hat{\epsilon}_i=0. \tag{7}$$

An explicit from for $\langle\vec{k}_1\vec{k}_2|V_C|\vec{k}_3\vec{k}_4\rangle$ is given in (MG-108).

One then proceeds as is done at some length in MG by deriving the quantum Ursell expansion and exhibiting the entire theory as a diagrammatic expansion from which all thermodynamic and distribution functions can be obtained. The final expression of the theory is in terms of "master graphs," which are defined in Sec. VII of MG. Of particular interest are master L graphs, which have one incoming and one outgoing external line carrying momentum (and spin) $\vec{k}$, and which obviously bear some relation to the momentum distribution.[10] We can then define a function (for α-type particles)

$$L_\alpha(t_2,t_1,\vec{k})=\sum\begin{pmatrix}\text{all } L \text{ graphs with}\\ \text{given external lines}\end{pmatrix}_{\vec{k}_\alpha}, \tag{8}$$

where t_i is an inverse-temperature variable ranging from 0 to $\beta=(\kappa T)^{-1}$, with κ = Boltzmann's constant, and T the absolute temperature. The sum of Eq. (8) is a quite fundamental component of the theory, because, as demonstrated in MG, the momentum distribution for α-type particles can be written

$$\langle n_\alpha(\vec{k})\rangle=N_\alpha(\vec{k})[1+\int_0^\beta L_\alpha(\beta,s,\vec{k})ds]. \tag{9}$$

The function L_α is itself a functional of $N_\alpha(\vec{k})$, so that the latter function is defined by the following set of coupled, nonlinear integral equations for a multicomponent system:

$$N_\alpha(\vec{k}) = \nu_\alpha(\vec{k})[1 + N_\alpha(\vec{k}) \int_0^\beta L_\alpha(\beta, s, \vec{k}) ds], \tag{10}$$

where

$$\nu_\alpha(\vec{k}) = e^{\beta g_\alpha - \beta\omega_\alpha(k)} (1 - \epsilon_\alpha e^{\beta g_\alpha - \beta\omega_\alpha(k)})^{-1}. \tag{11}$$

Here, g_α is the chemical potential for α-type particles, and $\epsilon_\alpha = +1$ for bosons and -1 for fermions. In the noninteracting system $\nu_\alpha(\vec{k})$, of course, is equal to the momentum distribution for α-type particles. Finally we can combine Eqs. (9) and (10) to obtain the more useful form

$$\langle n_\alpha(\vec{k})\rangle = \nu_\alpha(\vec{k}) \frac{1 + \int_0^\beta L_\alpha(\beta, s, \vec{k}) ds}{1 - \nu_\alpha(\vec{k}) \int_0^\beta L_\alpha(\beta, s, \vec{k}) ds}. \tag{12}$$

Figure 1 exhibits the only one- and two-vertex L graphs in the theory, when the external lines are photon lines, and both diagrams are of first order in the fine-structure constant α. One can find the explicit rules for writing down the analytic expressions for these diagrams in MG, and we will merely point out here for orientation purposes that the vertex in $L_\gamma^{(1)}$ corresponds to an $\vec{A}^2$ interaction, while both vertices in $L_\gamma^{(2)}$ correspond to $(\vec{p}\cdot\vec{A})$ interactions. In the next order, α^2, there are 12 contributing diagrams, and these are shown in Figs. 2 and 3. The master L graphs of these three figures represent the entire contribution to the sum of Eq. (8) through second order in the fine-structure constant, for photons.

As a final matter for this formal section, we shall clearly define the model in which we are interested by specifying the limits on the parameters describing the system. We envision a fully ionized gas consisting of any number of species of ions and the appropriate number of electrons to provide charge neutrality in the volume Ω. The infinite volume or thermodynamic limit will be consistently taken, so that the total number of particles in the system is to be considered very large in order that the density remain finite. This implies that we shall always convert sums over states to integrals. The system of interest here is one of low density and high, but nonrelativistic, temperature. Consequently it can be shown quite easily[8] that the fugacity

$$z_\alpha \equiv e^{\beta g_\alpha} \simeq \rho_\alpha \lambda_\alpha^3 (2S_\alpha + 1)^{-1} \ll 1, \tag{13}$$

where ρ_α is the number density of α-type particles, S_α is their spin in units of $\hbar$, and λ_α is their thermal wavelength: $\lambda_\alpha^2 = 2\pi\hbar^2\beta/M_\alpha$. Equation (13), clearly, is characteristic of a nondegenerate system.

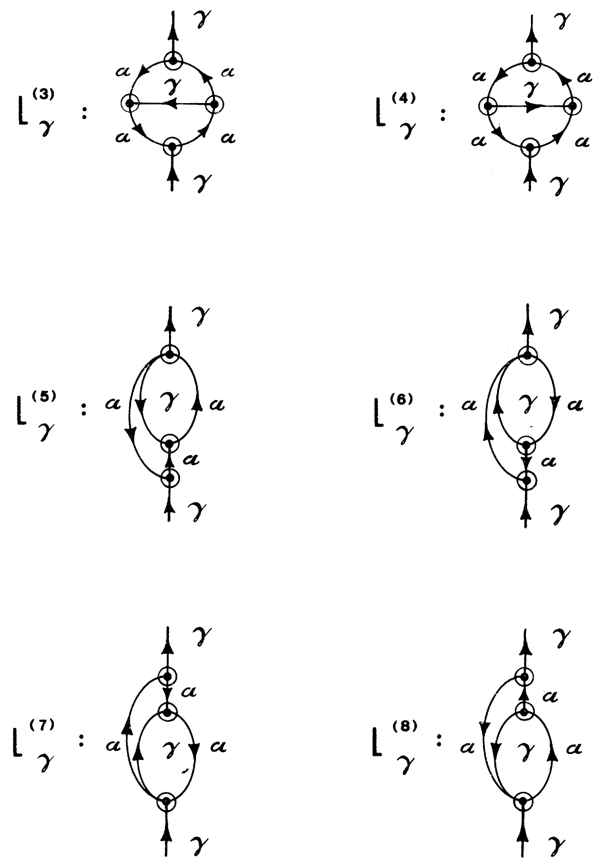

FIG. 2. The six master L graphs of order e^4 which contain no double bonds and, therefore, introduce no divergence difficulties.

The nonrelativistic nature of the system can, in the first place, be characterized by the parameter

$$\eta_\alpha = (\beta M_\alpha c^2)^{-1} \ll 1, \tag{14}$$

which merely states that the thermal energy is less than particle rest energies. Secondly, one should not expect photon energies in the nonrelativistic system to be large enough to induce pair production; that is, $\hbar ck \ll 2M_\alpha c^2$. This restriction can be expressed more conveniently as a relation between single-particle and photon energies:

$$\omega_\alpha(k) \ll \omega_\gamma(k). \tag{15}$$

Or one can say that photon momenta must be much smaller than an inverse Compton wavelength for any of the α-type particles.

In the next section we shall see that an important parameter in the system is the plasma frequency for α-type particles,

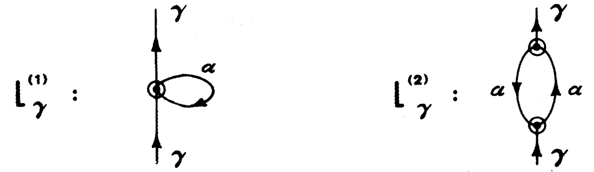

FIG. 1. The one- and two-vertex master L graphs with external photon lines. In the Γ theory the first of these is identically zero.

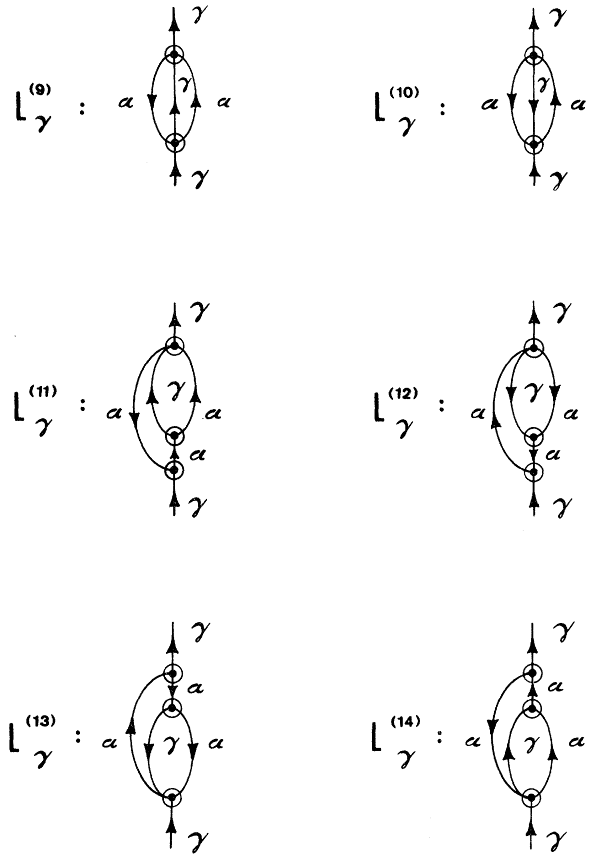

FIG. 3. The six master L graphs of order e^4 which admit the possibility of wiggly-line double bonds, and which contain divergent parts.

$$\omega_p(\alpha) = (4\pi\rho_\alpha Z_\alpha{}^2 e^2/M_\alpha)^{1/2}, \tag{16}$$

in terms of which we define a plasma frequency for the system

$$\omega_p{}^2 = \sum_\alpha \omega_p{}^2(\alpha). \tag{17}$$

Thus as can be shown quite simply from, say, (MG-152), it is required in the nonrelativistic, nondegenerate system that

$$\beta\hbar\omega_p \equiv \zeta = \sum_\alpha \zeta_\alpha = \sum_\alpha \beta\hbar\omega_p(\alpha) \ll 1. \tag{18}$$

The parameters ζ and ζ_α will prove to be quite important in what follows.

III. ANALYTIC CONTINUATION TO $\vec{k} = 0$

It is a simple matter to evaluate the one-vertex diagram of Fig. 1, following the rules given in MG, and one finds that[6]

$$L_\gamma{}^{(1)}(t_2, t_1, \vec{k}) \simeq -\frac{\hbar^2\omega_p{}^2}{2\hbar ck}\,\theta(t_2 - t_1), \tag{19}$$

where $\theta(x) = 1$ for $x > 1$, and zero otherwise. The dependence on the momentum $\vec{k}$ in this result is a general feature of all the photon L graphs of the theory, such as those in Figs. 1–3. Hence, it is clear that for $|\vec{k}| \ll 1$ a straightforward interation of the integral equation (10), and therefore of $\langle n_\gamma(\vec{k})\rangle$, will be in powers of k^{-1}, and thus unsatisfactory. This unpleasant situation, then, requires that an analytic continuation for $\langle n_\gamma(\vec{k})\rangle$ be found to the region of small-momentum quantum numbers. It is probable that such a procedure could be carried out in the diagrammatically expanded theory, but attempts to do so[5] seem to be fraught with conceptual difficulties. Instead, it seems more fruitful to return to the Hamiltonian, as described in Eqs. (1)–(6), and introduce a transformation found by Chappell *et al.*[7]

As motivation for the transformation to be used, let us observe that the single-vertex diagram of Fig. 1 appears to represent a photon-charged particle interaction in which zero momentum is transferred. In general this would seem to be a very unphysical process and, indeed, has been found to be a noncontributing factor in other contexts.[11] In fact such "interactions" appear to be a bothersome phenomena in all many-body theories in one way or the other. Thus it would be advantageous to formally remove such diagrams from the theory.

Zero-momentum interactions arise in the $V_{2\gamma}$ part of the Hamiltonian, Eq. (5b), which we now rewrite as[12]

$$V_{2\gamma} = V_{2\gamma}{}' + V_0, \tag{20}$$

where

$$\begin{aligned} V_{2\gamma}{}' = &\sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} (1 - \delta_{\vec{k}_1\vec{k}_2}\delta_{\vec{k}_3, -\vec{k}_4}) \\ &\times a_1{}^\dagger \langle \vec{k}_1 | V_{2\gamma} | \vec{k}_2\vec{k}_3\vec{k}_4\rangle a_2 a_3{}^{(\gamma)} a_4{}^{(\gamma)} \\ &+ \sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} (1 - \delta_{\vec{k}_1\vec{k}_3}\delta_{\vec{k}_2\vec{k}_4}) \\ &\times a_1{}^\dagger a_2{}^{(\gamma)\dagger} \langle \vec{k}_1\vec{k}_2 | V_{2\gamma} | \vec{k}_3\vec{k}_4\rangle a_3 a_4{}^{(\gamma)}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} V_0 = &\sum_{\vec{k}_1\vec{k}_2} \langle \vec{k}_1 | V_{2\gamma} | \vec{k}_1\vec{k}_2, -\vec{k}_2\rangle a_1{}^\dagger a_1 a_2{}^{(\gamma)} a_{-2}{}^{(\gamma)} \\ &+ \sum_{\vec{k}_1\vec{k}_2} \langle \vec{k}_1\vec{k}_2 | V_{2\gamma} | \vec{k}_1\vec{k}_2\rangle a_1{}^\dagger a_1 a_2{}^{(\gamma)\dagger} a_2{}^{(\gamma)}. \end{aligned} \tag{22}$$

In defining V_0 in terms of zero-momentum transfers, we have paid due attention to the δ functions in Eqs. (6). These latter equations also demonstrate that the matrix elements in V_0 are actually independent of the variable $\vec{k}_1$, so that substitution of Eqs. (6b) and (6c) into Eq. (22) allows the $\vec{k}_1$ sum to be performed. This sum is then proportional to $\hbar^2\omega_p{}^2/4$, due to the appearance of number operators

$$\sum_{\vec{k}_1} a_{\vec{k}_1}{}^\dagger a_{\vec{k}_1}.$$

We now follow Chappel *et al*,[7] and introduce new photon operators by means of a canonical transformation of the type first applied to the many-body problem by Bogoliubov[13]:

$$b_{\vec{k}} = u_k a_{\vec{k}} + v_k a_{-\vec{k}}^{\dagger}, \quad b_{\vec{k}}^{\dagger} = u_k a_{\vec{k}}^{\dagger} + v_k a_{-\vec{k}}, \tag{23}$$

with the inverse transformation

$$a_{\vec{k}} = u_k b_{\vec{k}} - v_{\vec{k}} b_{-\vec{k}}^{\dagger}, \quad a_{\vec{k}}^{\dagger} = u_k b_{\vec{k}}^{\dagger} - v_k b_{-\vec{k}}. \tag{24}$$

The constants in these transformation equations are determined by the requirement that the transformation be canonical; that is, that $b_{\vec{k}}$ and $b_{\vec{k}}^{\dagger}$ satisfy the same commutation relations as do the original photon operators. Thus we find that

$$\left.\begin{matrix} u_k^2 \\ v_k^2 \end{matrix}\right\} = \frac{(\omega_\Gamma \pm \omega_\gamma)^2}{4\omega_\Gamma \omega_\gamma}, \tag{25}$$

where $\omega_\Gamma(k) \equiv (\omega_\gamma^2 + \hbar^2 \omega_p^2)^{1/2}$, (26)

and, as usual, $\omega_\gamma(k) = \hbar ck$.

One must now recall that the free-particle Hamiltonian, Eq. (2), contains the Hamiltonian of the free radiation field, $H_{\rm rad}$. Equations (24) are substituted into Eqs. (5a), (21), and (22), and after some lengthy algebra it is found that

$$(H_{\rm rad} + V_0 + V_0^{\dagger}) = \sum_{\vec{k}} b_{\vec{k}}^{\dagger} b_{\vec{k}} \omega_\Gamma(k). \tag{27}$$

Therefore the canonical transformation has diagonalized that portion of the interaction due to V_0 and provided us with the new radiation Hamiltonian of Eq. (27). Because the new single-photon energies reflect some sort of screening by the medium, Chappell *et al.*[7] have interpreted the transformation (23) as a redescription in terms of "quasi-photons." We shall not adopt this nomenclature because there seem to be several objections to this concept, which we shall discuss later.

Nevertheless the canonical transformation has served the dual purpose of removing zero-momentum transfers from the theory, as well as providing an analytic continuation to small values of photon momenta. That is, the single-photon energies

$$\omega_\Gamma(k) = \hbar(c^2k^2 + \omega_p^2)^{1/2} \tag{28}$$

are well behaved as $|\vec{k}| \to 0$. There remains, of course, the question of the effects of the transformation on the remainder of the theory. After some tedious, but elementary algebra, one finds that all of the previous equations are affected in a simple way, and the transformed equations are obtained as follows:

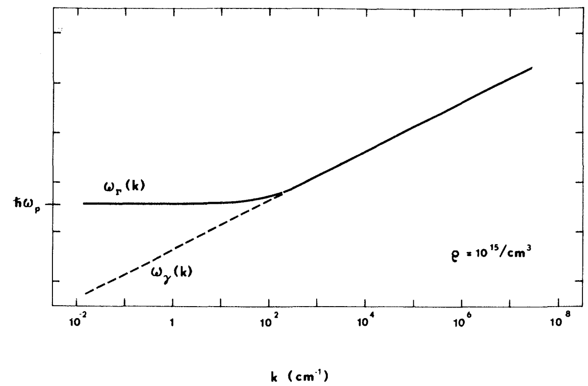

FIG. 4. Modification of the energy-momentum relation for photons after the Bogoliubov transformation of Eq. (24) has been applied to the system Hamiltonian.

(i) The symbolic replacement $\gamma \to \Gamma$ is made everywhere.

(ii) Equations (1), (2), and (4) are respectively replaced by

$$H = H_0 + V_C + V_\Gamma, \tag{1'}$$

$$H_0 = \sum_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} \omega_{\vec{k}} + \sum_{\vec{k}} b_{\vec{k}}^{\dagger} b_{\vec{k}} \omega_\Gamma(k), \tag{2'}$$

$$V_\Gamma = V_{1\Gamma} + V_{2\Gamma} + V_{1\Gamma}^{\dagger} + V_{2\Gamma}^{\dagger}. \tag{4'}$$

(iii) In Eqs. (6) the quantities (α/k_i) are replaced by $e^2/\omega_\Gamma(k_i)$, and in Eqs. (6b) and (6c) we make the replacement $\delta[\;] \to \bar{\delta}[\;]$ in the Kronecker δ functions of momentum variables, where $\bar{\delta}$ is a Kronecker δ function which carries with it the stipulation that closed "loops," such as that of the single-vertex diagram in Fig. 1, are prohibited. Hence rule (iii) implies that in the transformed theory (see Fig. 1)

$$L_\Gamma^{(1)}(t_2, t_1, \vec{k}) \equiv 0. \tag{29}$$

It is to be emphasized that the canonical transformation of the theory we have made in this section has really done nothing more than formally eliminate zero-momentum transfer diagrams from the formalism. However, we have seen that the form of the single-photon energies, Eq. (28), has provided an analytic continuation to small values of $|\vec{k}|$. One verifies that, for $ck > \omega_p$, the leading-order term in the expansion of the radical in (28), along with the re-insertion of closed "loops," leads directly back to the γ theory of the preceding section. Figure 4 exhibits the manner in which the original photon spectrum has been altered by the Bogoliubov transformation, and we shall discuss the meaning of this new energy-momentum relation in Sec. VII.

IV. MOMENTUM DISTRIBUTION AND PHOTON DENSITY

It is now possible to calculate the photon momentum distribution, $\langle n_\Gamma(\vec{k})\rangle$, in the transformed theory by a straightforward approximation procedure. One first sets $\alpha = \Gamma$ in Eqs. (11) and (12), and then makes the replacement $\gamma \to \Gamma$ in Figs. 1–3. To first order in e^2, we then have the approximate result

$$\int_0^\beta L_\Gamma(\beta, s, \vec{k}) ds \simeq \int_0^\beta L_\Gamma^{(2)}(\beta, s, \vec{k}) ds . \tag{30}$$

The right-hand side of this equation represents the contribution from the two-vertex diagram of Fig. 1, and from the prescription given in MG for calculating with master graphs it is a simple matter to find that

$$L_\Gamma^{(2)}(\beta, s, \vec{k}) \simeq \sum_\alpha [\hbar^2 \omega_p^2(\alpha)/2\omega_\Gamma(k)] I_\alpha(s, \vec{k}), \tag{31}$$

where the sum is over all particle types (except photons) in the system, and we have applied the inequality (13). We have also defined a function

$$I_\alpha(s, \vec{k}) = \int_0^{s_1/\beta} \exp\{\beta(s_0 - s)\omega_\Gamma + (s_0 - s)[(s_0 - s) + 1]\lambda_\alpha^2 k^2/4\pi\} ds_0$$
$$+ \int_{s_1/\beta}^1 \exp\{\beta(s_0 - s)\omega_\Gamma + (s_0 - s)[(s_0 - s) - 1]\lambda_\alpha^2 k^2/4\pi\} ds_0, \tag{32}$$

in which the temperature variables have been normalized by β, and we have summed over polarization directions of the photon. It should be emphasized that Eq. (31) is exact to leading order in the particle density, the only approximation being that of dropping terms in ρ_α^2, and higher. The integrals in Eq. (32) can be evaluated exactly in terms of error functions of imaginary argument:

$$I_\alpha(s, \vec{k}) = -i\frac{\pi^{1/2}}{2b}\{e^{-a_+^2}[\text{erf}(ia_+) - \text{erf}(ia_+ - isb)] + e^{-a_-^2}[\text{erf}(ia_- + i(1-s)b) - \text{erf}(ia_-)]\}, \tag{33}$$

where

$$a_\pm \equiv [\beta\omega_\Gamma(k) \pm \beta\omega_\alpha(k)]/2[\beta\omega_\alpha(k)]^{1/2}, \quad b \equiv [\beta\omega_\alpha(k)]^{1/2}. \tag{34}$$

Finally we are interested in the nonrelativistic region as expressed by Eq. (15), which also applies to the case $\gamma \to \Gamma$. In this approximation the parameters in Eq. (34) are very large, and the asymptotic forms of the error functions can be used. Hence,

$$I_\alpha(s, \vec{k}) \simeq \frac{\exp\{-s[1 + (1-s)(\omega_\alpha/\omega_\Gamma)]\beta\omega_\Gamma\}(e^{\beta\omega_\Gamma} - 1)}{\beta\omega_\Gamma[1 + (\omega_\alpha/\omega_\Gamma) - 2s(\omega_\alpha/\omega_\Gamma)]} - \frac{2}{\beta\omega_\Gamma}\frac{\omega_\alpha/\omega_\Gamma}{1 - (\omega_\alpha/\omega_\Gamma)^2}, \quad \omega_\alpha(k) \ll \omega_\Gamma(k); \tag{35a}$$

$$\to (e^{-s\beta\omega_\Gamma}/\beta\omega_\Gamma)(e^{\beta\omega_\Gamma} - 1) \quad \text{as } (\omega_\alpha/\omega_\Gamma) \to 0. \tag{35b}$$

Because we are interested solely in the nonrelativistic limit, we shall adopt the form (35b) in calculating the momentum distribution. Thus with Eqs. (30), (31), and (35b), Eq. (12) for the photon momentum distribution becomes in the nonrelativistic case

$$\langle n_\Gamma(\vec{k})\rangle \simeq \nu_\Gamma(\vec{k}) \frac{1 + [\hbar^2\omega_p^2/\omega_\Gamma^2(k)][\cosh\beta\omega_\Gamma(k) - 1]/\beta\omega_\Gamma(k)}{1 - [\hbar^2\omega_p^2/\omega_\Gamma^2(k)]\,\nu_\Gamma(\vec{k})[\cosh\beta\omega_\Gamma(k) - 1]/\beta\omega_\Gamma(k)}, \tag{36}$$

where $\nu_\Gamma(\vec{k})$ is defined by Eq. (11). In Fig. 5, $\langle n_\Gamma(\vec{k})\rangle$ and $\nu_\Gamma(\vec{k})$ are compared with the corresponding quantity for the vacuum.

The number density of photons is defined to be

$$\rho_\Gamma = (1/\Omega)\sum_{\vec{k}} \langle n_\Gamma(\vec{k})\rangle, \tag{37}$$

although we shall later take up the question of whether this should actually be called ρ_γ. If the expression (36) is now substituted into (37), one finds a divergent integral for large values of $|\vec{k}|$. This ultraviolet unpleasantness, however, is due only to the nonrelativistic approximation and, if we use the form (35a) rather than (35b) in Eq. (37), then the integral converges nicely. On the other hand, an equivalent and simpler procedure is to merely introduce a cutoff at the Compton wavelength: $k < \lambda_c^{-1}$. One verifies that this inequality is the same as (15), but it must also be remembered that, had we used the form (35a), there would have remained a sum over α, which will affect the final form of ρ_Γ.

We now assume that the denominator in Eq. (12) can be expanded in powers of ζ defined in Eq. (18), and through order ζ^2 we find[14]

$$\rho_\Gamma \simeq \frac{2}{(2\pi)^3}\int \nu_\Gamma(\vec{k}) d^3k + 2\frac{\hbar^2\omega_p^2}{(2\pi)^3}\int \nu_\Gamma(\vec{k})[1 + \nu_\Gamma(\vec{k})]\frac{\cosh\beta\omega_\Gamma(k) - 1}{\beta\omega_\Gamma(k)}\frac{d^3k}{\omega_\Gamma^2(k)}. \tag{38}$$

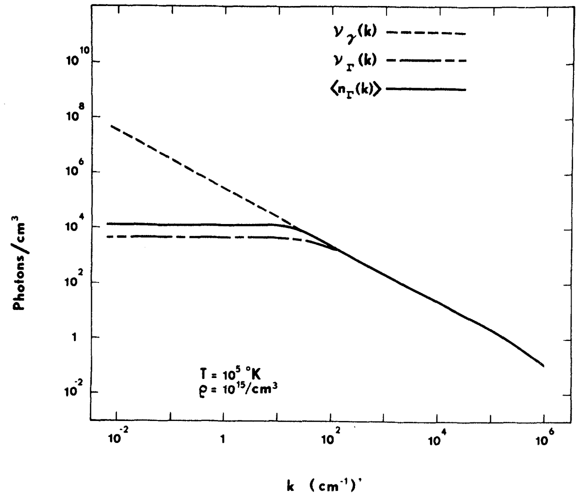

FIG. 5. Demonstration of how the (infinite) momentum distribution for photons is cut off at small $|\vec{k}|$ in the Γ theory, for $T=10^5$ °K and $\rho=10^{15}/\text{cm}^3$. Note that the momentum distribution of Eq. (36) differs very little from the approximation $\nu_\Gamma(\vec{k})$.

The integrals are readily evaluated in a straightforward manner, and the leading-order form of the photon density is

$$\rho_\Gamma \simeq [2/\pi^2(\beta\hbar c)^3]\,(1.202 - \tfrac{1}{4}\zeta^2\ln\zeta - \tfrac{1}{4}\textstyle\sum_\alpha \zeta_\alpha{}^2\ln\eta_\alpha) + O(\zeta^2)\,, \tag{39}$$

where η_α is defined by Eq. (14). The first term in Eq. (39) is plainly the free-photon result, and the remaining positive terms show that the photon density in a fully ionized medium is greater than that in a vacuum at the same temperature. This merely reflects the fact that interacting charged particles create photons. Equation (39) has also been found by Smith by a different approach.[5]

It is the energy density of radiation or energy per unit volume as a function of temperature and wave number (or frequency) which is a measureable quantity, so that we wish to examine

$$u(\beta,k) \equiv [\langle n_\Gamma(\vec{k})\rangle/\Omega]\;\omega_\Gamma(k)(dN/dk)\,. \tag{40}$$

This is certainly a correct and useful expression for $u(\beta,k)$, but only if $\omega_\Gamma(k)$ is truly the single-photon energy in the medium and if the density of states, (dN/dk), is readily calculable. Let us assume momentarily that the single-photon energy is given by Eq. (26), and that the density of states is properly calculated in the usual manner for a cavity,[15]

$$(dN/dk) = \Omega k^2/\pi^2\,. \tag{41}$$

Then let us calculate the total energy density of radiation in Ω by integrating Eq. (40) over all values of momentum magnitude. In the course of pursuing this calculation, one readily proves the following identity:

$$u(\beta) = \int_0^\infty u(\beta,k)dk = (1/\Omega)\textstyle\sum_{\vec{k}}\omega_\Gamma(k)\langle n_\Gamma(\vec{k})\rangle\;, \tag{42}$$

which is precisely the expression we would have adopted on intuitive grounds had we set out to calculate $u(\beta)$ directly. But the second line of Eq. (42) is only correct insofar as both $\omega_\Gamma(k)$ and $\langle n_\Gamma(\vec{k})\rangle$ represent the true photon energies and momentum distribution, respectively. In this sense, if one substitutes Eqs. (26) and (36) into (42), one is not going to obtain correct answers in higher order, because to the extent that $\omega_\Gamma(k)$ is correct, we can only write

$$\langle n_\Gamma(\vec{k})\rangle \simeq \nu_\Gamma(\vec{k}) \tag{43}$$

for the momentum distribution. That is, one can certainly calculate corrections to the momentum distribution using $\omega_\Gamma(k)$, as we have done, but the resulting $\langle n_\Gamma(\vec{k})\rangle$ cannot be used indescriminately in the second line of Eq. (42) when Eq. (26) is used. This point involves simple, but sometimes subtle, considerations, and so we shall return to it in Sec. VII and approach it from yet another direction. At that time, we shall also examine $u(\beta,k)$ in more detail.

In order to conclude this section, which has as its objective the calculation of the thermodynamic functions relating to photons, let us complete the calculation of $u(\beta)$. A correct and unambiguous way to do this is to adopt the statistical mechanical equation of (MG-9), namely

$$u(\beta) = -(\partial f/\partial\beta)|_{z,\Omega}\,, \tag{44}$$

where $f(\beta,N,\Omega)$ is the grand potential for the entire ionized gas, and z and Ω are to be held constant in the differentiation. The leading-order contribution to the grand potential due to photon-charged particle interactions has been calculated previously.[16] Through order ζ^2 the result is the same in either the γ or Γ formulation (see Sec. VI below), and so we can write

$$f_\gamma \simeq (2\alpha/\pi)\sum_\alpha (2S_\alpha+1) z_\alpha \lambda_\alpha^{-3} Z_\alpha^{\,2} \eta_\alpha \ln\eta_\alpha \; . \tag{45}$$

Thus $u(\beta)$ can be written as a sum of two contributions: one from the free-photon grand potential (in the Γ formulation), and one from f_γ. Moreover the former contribution is completely equivalent to using the second line of Eq. (42), with the approximation (43). Therefore

$$u(\beta) \simeq u_1(\beta) + u_2(\beta) \, ,$$

where
$$u_1(\beta) = (1/\Omega)\sum_{\vec{k}} \omega_\Gamma(k) \nu_\Gamma(\vec{k}) \simeq \frac{\pi^2 \kappa^4}{15\hbar^3 c^3} T^4[1+O(\zeta^4)] \, , \tag{46}$$

and

$$u_2(\beta) = -(\partial f_\gamma/\partial\beta)|_{z_\alpha, \Omega} \simeq [1/2\pi^2\beta(\beta\hbar c)^3] \sum_\alpha \zeta_\alpha^{\,2} \ln\eta_\alpha \; . \tag{47}$$

Thus to the order of our approximations, the total energy density of radiation in the fully ionized gas is

$$u(\beta) \simeq \sigma T^4[1 + (15/2\pi^4)\sum_\alpha \zeta_\alpha^{\,2} \ln\eta_\alpha + O(\zeta^4)] \, , \tag{48}$$

where
$$\sigma = \pi^2\kappa^4/15\hbar^3 c^3 \tag{49}$$

is the Stefan-Boltzmann constant. The second term in Eq. (48) appears to be a new result.

As a final point, we mention that, to the order of the calculations considered here, the radiation pressure is

$$P_{\text{rad}} \simeq \tfrac{1}{3} u_1(\beta) + O(\zeta^2) \, . \tag{50}$$

It is interesting, but not too surprising, that only $u_1(\beta)$ contributes to this relation because the factor of $\frac{1}{3}$ is strictly a relativistic phenomenon. Since the interaction of photons is with nonrelativistic charged particles, one expects their effects on the pressure to be intimately interrelated. On the contrary, it is a little surprising that the relation (50) even holds to the order indicated.

V. HIGHER-ORDER DIAGRAMS

It is probably never necessary in any practical application of the theory to include contributions of order higher than those considered in the preceding section. Nevertheless, it is important to formally investigate the higher-order diagrams for the following two reasons: (1) One must verify the efficacy of both the diagrammatic expansion of $L(t_2, t_1, \vec{k})$ and the expansion of $\langle n_\Gamma(\vec{k})\rangle$; and, (2) it is desirable to know that the higher-order diagrams are well behaved in the event that their calculation becomes necessary. Thus in Figs. 2 and 3 we have exhibited all of the diagrams occurring in next order e^4.

There is a good reason for grouping the 12 diagrams as done in these figures, because a short calculation shows that the analytic expressions for those of Fig. 3 contain ultraviolet divergences. This is not, surprising, of course, but only a manifestation of the general self-energy malady inherent in the present description of quantum electrodynamics, both relativistic and nonrelativistic. Indeed, a salient feature of MG was the systematic treatment of these self-energy terms which always occur in systems of interacting charged particles. The analysis of this "one-particle problem" is given in Sec. IV of MG.

As a concrete example for investigating the higher-order terms, one finds for the expression corresponding to the first diagram in Fig. 3

$$L_\Gamma^{(9)}(\beta, t_1, \vec{k}) \simeq \frac{4\pi^2 e^4 \hbar^4}{\Omega^2 \omega_\Gamma(k)} \sum_\alpha \frac{Z_\alpha^{\,4}(2S_\alpha+1)}{M_\alpha^{\,2}} \sum_{\vec{l}_1 \vec{k}_3} \frac{(\hat{\epsilon}_3 \cdot \hat{\epsilon}_{\vec{k}})^2}{\omega_\Gamma(k_3)} \times \frac{1}{W}[\nu_\alpha(\vec{l}_2)\nu_\Gamma(\vec{k}_3) + \nu_\alpha(\vec{l}_1)\nu_\Gamma(\vec{k}_3)$$
$$- \nu_\alpha(\vec{l}_1) + \nu_\alpha(\vec{l}_1)\nu_\Gamma(\vec{k}_3) e^{(\beta-t_1)W} + \nu_\alpha(\vec{l}_1) e^{(\beta-t_1)W} - \nu_\alpha(\vec{l}_2)\nu_\Gamma(\vec{k}_3) e^{-t_1 W}] \, , \tag{51}$$

where $\vec{l}_2 = \vec{l}_1 + \vec{k} - \vec{k}_3$, $W = \omega_\alpha(l_1) + \omega_\Gamma(k) - \omega_\alpha(l_2) - \omega_\Gamma(k_3)$,

and $\nu_\alpha(\vec{k})$ is defined in Eq. (11). With reference to Eq. (51), we can now comment on the two points raised in the first paragraph of this section.

First of all, in the infinite-volume limit the sums can be converted to integrals in the usual way, and the physical parameters extracted by rendering the integration variables dimensionless. One then observes that the entire quantity $L_\Gamma^{(9)}$ is of order ζ^4, and so the expansion scheme adopted in the preceding section is quite likely to be a good one.

Secondly, it is easy to verify that, when the infinite-volume limit is taken, the third term in square brackets in Eq. (51) leads to a divergent integral for large values of $|\vec{k}_3|$. Diagrammatically, this divergence is associated with the case in which the two internal lines of the diagram for $L_\Gamma^{(9)}$ which go in the same direction do not carry factors of $\nu(\vec{k})$. In MG this structure was called a "wiggly-line double bond," and it should be clear that the divergence arises from the dynamics of the system and not the statistics.

These divergences arise throughout the theory in higher order, and they are to be interpreted as electromagnetic self-energies arising from the interaction of the charged particle with its own radiation field, and are considered unphysical. As is customary, these self-energies can be removed from the theory through mass renormalization.[17,8] However, the self-energy problem is not completely restricted to the higher-order terms, because a single charged particle can emit and re-absorb virtual photons, in a manner indicated diagrammatically in Fig. 6. Therefore in any renormalization scheme, we must also account for these contributions to the self-energy.

Mass renormalization is carried out by first defining an "observable" (or "dressed") mass for α-type particles:

$$M_\alpha = M_\alpha^{(0)} + \delta M_\alpha , \tag{52}$$

where $M_\alpha^{(0)}$ is called the "bare mass," and δM_α is the "electromagnetic mass" which is due to the electromagnetic self-energy. Next, in all of the equations of the theory, one now makes the replacement $M_\alpha \to M_\alpha^{(0)}$ and notes the expansions

$$\text{(a)}\ \frac{1}{M_\alpha} \simeq \frac{1}{M_\alpha^{(0)}}\left(1 - \frac{\delta M_\alpha}{M_\alpha^{(0)}}\right), \quad \text{(b)}\ \frac{1}{[M_\alpha]^2} \simeq \frac{1}{[M_\alpha^{(0)}]^2}\left(1 - 2\frac{\delta M_\alpha}{M_\alpha^{(0)}}\right). \tag{53}$$

Then, by noting the divergent term in Eq. (51), we can identify

$$\delta M_\alpha = (4\pi e^2\hbar^2/\Omega) Z_\alpha^{\,2} \sum_{\vec{k}_4} (\hat{\epsilon}_{\vec{k}} \cdot \hat{\epsilon}_4)^2 / \omega_\Gamma(k_4)[\omega_\alpha(k_4) + \omega_\Gamma(k_4)] . \tag{54}$$

This divergent quantity, therefore, can be removed from Eq. (51) by replacing $M_\alpha^{(0)}$ with the dressed mass M_α.

Of course, Eq. (54) is not complete because δM_α must really be the sum of all self-energy terms. Thus, we must include those contributions to δM_α arising from diagrams such as that of Fig. 6, as well as those due to the other diagrams in Fig. 3. Fortunately these latter contributions are all equal to that of Eq. (54); otherwise, the scheme would fail.

A systematic method for carrying out the mass renormalization to all orders has been presented in MG, and the reader can refer to that work for the details. Here we shall merely point out that, through order e^4, the present theory is completely renormalized if we add to every single-particle energy, $\omega_\alpha(k)$, for charged particles, a counter-term

$$S_\alpha(k) \simeq (\pi e^2\hbar^4 k^2/\Omega)(Z_\alpha^{\,2}/M_\alpha^{\,2}) \sum_{\vec{k}_4} [2(\hat{\epsilon}_{\vec{k}} \cdot \hat{\epsilon}_4)^2 + \hbar^2(\hat{k} \cdot \hat{\epsilon}_4)^2] / \omega_\Gamma(k_4)[\omega_\alpha(k_4) + \omega_\Gamma(k_4)] . \tag{55}$$

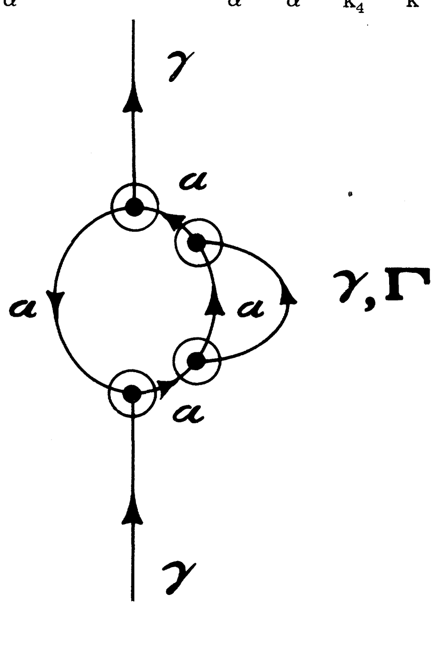

FIG. 6. The leading-order diagrammatic representation of emission and reabsorption of a virtual photon by charged particles.

This quantity has been defined with a factor $\hbar^2k^2/2M_\alpha$ because, as observed by Smith,[5] the renormalization must not alter the form of the free-particle energy.

It is very important to realize that after carrying out the renormalization procedure there will remain finite parts of the originally divergent diagrams. These are known as "radiative corrections" and are of order e^4. There is no reason to calculate these contributions here, however, for we have restricted our calculations of thermodynamic quantities to order e^2. In the next section, though, we shall calculate one radiative correction which is of intrinsic interest.

One important point should be stressed regarding the renormalization program which has been outlined briefly here. In practice, it has proved impossible to carry out the renormalization program in the transformed theory. Rather, it is necessary to first renormalize the untransformed equations (the γ theory), and then perform the Bogoliubov transformation. This is not unexpected, and a major reason is that the single-vertex diagram vanishes in the Γ theory. For instance, one must group $L_\gamma^{(1)}$ with the divergent parts of $L_\gamma^{(9)}$ and $L_\gamma^{(10)}$ to identify the electromagnetic self-energy, and this could not be done in the transformed theory.

VI. RADIATIVE CORRECTIONS AND COULOMB INTERACTIONS

There is one radiative correction which is of importance in the present context, and that is associated with the grand potential. In the discussion preceding Eq. (45), we pointed out that the leading-order contribution to the grand potential due to photon-charged particle interactions was the same in either the transformed or untransformed theory. We should like to prove this statement at this time.

According to MG, the diagrammatic expression for the grand potential is given in terms of master O graphs; that is, diagrams such as those of Figs. 1–3 in which the external lines are closed. The leading-order diagram of this type which is due to photon-charged particle interactions is that of Fig. 7, and its analytical expression will be denoted by f_Γ. The dominant contribution to this diagram is due to the self-energy effect. After carrying out the mass renormalization, one finds from the rules given in MG that

$$f_\Gamma \simeq (\beta^3\hbar^4e^2\Omega/32\pi^5)\sum_\alpha[(2S_\alpha+1)Z_\alpha^2 z_\alpha/M_\alpha^2\lambda_{c,\alpha}]\eta_\alpha I(\eta_\alpha,\mu_\alpha), \tag{56}$$

where we have momentarily introduced the notation $\mu_\alpha=\zeta_\alpha\eta_\alpha$, $\lambda_{c,\alpha}$ is the Compton wavelength for α-type particles, and

$$I(\eta_\alpha,\mu_\alpha)=8\pi^2\int_0^\infty x^4e^{-x^2/2\eta_\alpha}dx\int_{-1}^{+1}(1-w^2)\,dw$$
$$\times\int_0^\infty\frac{y^2dy}{(y^2+\mu_\alpha^2)^{1/2}}\;\frac{\exp\{-[y^2-2xyw+2(y^2+\mu_\alpha^2)^{1/2}]/2\eta_\alpha\}-1}{[y^2-2xyw+2(y^2+\mu_\alpha^2)^{1/2}]^2/4\eta_\alpha^2}. \tag{57}$$

If one now sets $\mu_\alpha=0$, this integral is identical to that encountered in the γ theory.[11,16] The integral $I(\eta_\alpha,0)$ can be evaluated asymptotically for $\eta_\alpha\ll 1$, and one finds it is proportional to $\eta_\alpha\ln\eta_\alpha$ in lowest order. Thus,

$$f_\Gamma\simeq(2e^2/\pi\hbar c)\sum_\alpha(2S_\alpha+1)z_\alpha\lambda_\alpha^{-3}Z_\alpha^2\eta_\alpha\ln\eta_\alpha[1+O(\zeta_\alpha)]\,,\;=f_\gamma[1+O(\zeta)]\,, \tag{58}$$

thereby verifying Eq. (45).

As a final calculation, we shall find the way in which the Coulomb interactions affect the photon momentum distribution in lowest order. This calculation was performed in the context of the γ theory in I, and proceeds in a similar manner here, in the transformed theory. The relevant diagram is that of Fig. 8, in which the function $L_D^{(2)}$ is the sum over all ring diagrams and represents the screened-Coulomb interaction.[8] We see no purpose to be served in writing down the detailed calculation of this diagram here, because the integrals involved are merely modifications of those encountered in deriving the familiar Debye-Hückel theory.[8] Therefore we shall just give the final result, which involes the classical Debye parameter for α-type charged particles.

$$\Lambda_\alpha=\beta e^2Z_\alpha^2/\lambda_D, \tag{59}$$

where λ_D is the Debye length:

$$\lambda_D\simeq(4\pi\beta e^2\sum_\alpha\rho_\alpha Z_\alpha^2)^{-1/2}. \tag{60}$$

In a form suitable for substituting directly into Eq. (12), the master L graph of Fig. 8 yields to lowest order in the electronic charge

$$\int_0^\beta L_{\Gamma,c}(\beta,t,\vec{k})dt\simeq[f(\beta\omega_\Gamma)/\beta\omega_\Gamma]\sum_\alpha\zeta_\alpha^2\Lambda_\alpha, \tag{61}$$

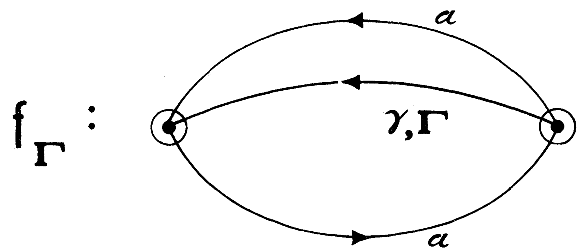

FIG. 7. The lowest-order diagrammatic contribution to the grand potential due to the interaction of photons with charged particles.

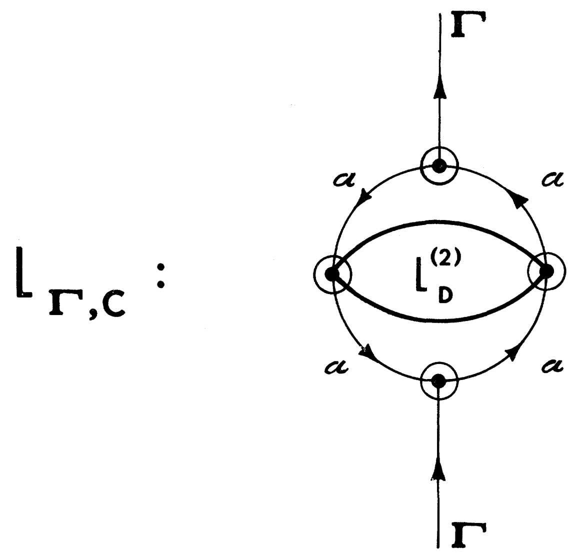

FIG. 8. The leading-order diagrammatic contribution to the photon momentum distribution due to Coulomb interactions among charged particles. The function $L_D^{(2)}$ represents the screened Coulomb interaction obtained by summing over all ring diagrams, and is discussed in detail in MG.

where $f(x)=(2/x^2)[1+6/x^2+(2/x)\sinh x-(6/x^2)\cosh x]$. (62)

The correction (61) to the momentum distribution is to be added to both the numerator and denominator of Eq. (36), but it must be remembered that in the latter case a factor of $\nu_\Gamma(\vec{k})$ must be introduced.

VII. DISCUSSION

It seems appropriate at this point to make some comments regarding the interpretation to be given to the foregoing theory and calculations. Regarding the photon energy-momentum relation, Eq. (26), obtained from the Bogoliubov transformation, we find it difficult to understand the concept of "quasiphotons" introduced earlier.[7] For this concept would not seem to fit physically into the usual picture of quasiparticles, in that collective effects due to photons are difficult to visualize. Rather it appears physically more satisfying to interpret the perturbation of the radiation properties of the gas in terms of the macroscopic index of refraction, or dielectric constant of the medium, a concept which lends itself more readily to experimental interpretation. Of course, the approximate dielectric constant obtained from Eq. (26),

$$\epsilon(\omega)\simeq 1-\omega_p^2/\omega^2, \tag{63}$$

can still be thought of as due to a microscopic screening of the photon charged-particle interaction.

Thus it seems reasonable to assume the existence of "photon integrity"; that is, a photon is always a photon in the ionized gas. One should, therefore, treat the Γ label introduced above as a mere notational device reflecting primarily the procedure used for analytic continuation, although the method most certainly expedites the calculations. With this in mind, the densities ρ_Γ and $u(\beta, k)$ should always be considered as referring to photons.

Equation (63) is really a classical result and well known.[18] This form cannot be completely correct, however, because we have seen that Eq. (26) itself is only approximately the true photon energy in the medium. Thus although the momentum distribution can be calculated quite accurately, it would seem to offer little direct information regarding $\epsilon(\omega)$. That is, a suitable theory of $\langle n_\Gamma(\vec{k})\rangle$ does not in itself provide a *simple* procedure for finding the index of refraction.

In Sec. IV we calculated the total energy density of radiation in the fully ionized gas, culminating in what appears to be a new result in Eq. (48). However, it was found somewhat difficult to derive this result in a straightforward manner, particularly from the function $u(\beta, k)$, Eq. (40). Although one can certainly find the correct momentum distribution to be inserted into this equation, it is not at all clear what should be used for the density of states, dN/dk. Furthermore, no simple criterion seems to exist for aid in making this decision. The difficulty here is compounded, of course, by realizing that $\omega_\Gamma(k)$ is only correct to a first approximation. Hence it is doubtful that a good knowledge of the momentum distribution can provide much insight into the behavior of the radiation spectrum in momentum space. Similar arguments apply, of course, to the frequency spectrum. To the order that

$$u(\beta, k)\simeq (k^2/\pi^2)\omega_\Gamma(k)\nu_\Gamma(\vec{k}) \tag{64}$$

represents a good approximation, one can observe the differences in this function and that for the vacuum in Fig. 9. Equation (64) appears to be the best approximation possible from the theory of the photon momentum distribution outlined here.

To the above pessimistic comments must be added one more observation. It is often stated[19] that the function $u(\beta, k)$, or $u(\beta, \omega)$, is a universal function of the indicated variables. If so, then how can the photon momentum distribution in a fully ionized gas ever differ from that for the vacuum except trivially? If there is a difference, such as exhibited by Eq. (36), then one can construct systems violating the Second Law of Thermodynamics. Therefore a large dilemma arises in applying the momentum distribution to a study of the photon energy density in a fully ionized gas; indeed, one must ask if it even makes sense to calculate $\langle n_\Gamma(\vec{k})\rangle$ in this respect. In the following paper a solution to this problem is suggested, from which one concludes that great care must be exercised in applying any theory of the momentum distribution to a study of the radiation spectrum.

ACKNOWLEDGMENT

In conclusion, we would like to express our appreciation to Dr. C. R. Smith for several illuminating conversations.

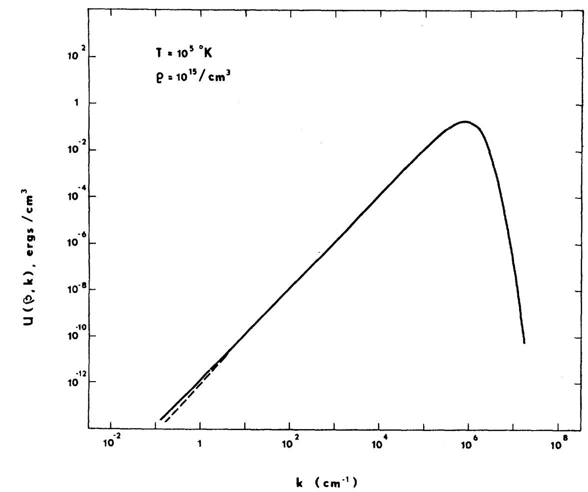

FIG. 9. Comparison of $u(\beta,k)$, Eq. (64), with the analogous function for the vacuum (dashed line). Slight differences occur only for very small values of $|\vec{k}|$.

*Supported in part by a grant from the National Science Foundation.

†Based in part on a thesis (IKH) submitted to the faculty of the graduate school of the University of Wyoming in partial fulfillment of the requirements for the Ph. D. degree, 1968.

[1]For a representative sample, see A. Simon and E. G. Harris, Phys. Fluids 3, 245 (1960); D. F. Dubois *et al.*, Phys. Rev. 129, 2376 (1963); T. H. Dupree, Phys. Fluids 7, 923 (1964); R. E. Aamodt *et al.*, Phys. Fluids 7, 1952 (1964); P. J. Mallozzi and H. Margenau, Ann. Phys. (N. Y.) 38, 177 (1966).

[2]R. K. Osborn and E. H. Klevans, Ann. Phys. (N. Y.) 15, 105 (1961); T. Nakayama, Ph. D. thesis, University of Colorado, 1962 (unpublished).

[3]S. Nakai, Nucl. Phys. 63, 131 (1965).

[4]W. R. Chappell and W. E. Brittin, Phys. Rev. 146, 75 (1966).

[5]C. R. Smith, Ph.D. thesis, University of Colorado, 1967 (unpublished).

[6]W. T. Grandy, Jr., Nuovo Cimento 40, 265 (1965); hereafter referred to as I.

[7]W. R. Chappell *et al.*, Nuovo Cimento 38, 1186 (1965).

[8]F. Mohling and W. T. Grandy, Jr., J. Math. Phys. 6, 348 (1965); hereafter referred to as MG. Equation (n) of this paper will be denoted by (MG-n).

[9]The particle-type label γ will be used exclusively for photons, in both equations and figures.

[10]That is, the probability for finding an α-type particle in the system with momentum (and spin) $\vec{k}$.

[11]W. T. Grandy, Jr., and F. Mohling, Ann. Phys. (N. Y.) 34, 424 (1965); in particular, Appendix B.

[12]It is important to observe that the Hamiltonian has originally been written in normal form; that is, with all annihilation operators to the right of all creation operators. This is standard procedure, to be sure, but must be remembered when the subsequent transformation is introduced.

[13]N. N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947).

[14]The factor of 2 in Eq. (38) is the contribution from the sum over photon-polarization states.

[15]See, e.g., K. Huang, Statistical Mechanics (John Wiley & Sons, Inc., New York, 1963), p. 254.

[16]Reference 11, footnote 4. See also W. T. Grandy, Jr., Anais. Acad. Brasil. Cienc. 39, 65 (1967). The coefficient a_5 in this latter article should be $2/\pi$.

[17]F. Dyson, Phys. Rev. 75, 1736 (1949).

[18]W. K. H. Panofsky and M. Phillips, Classical Electricity and Magnetism (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962), 2nd ed., p. 413.

[19]J. E. and M. G. Mayer, Statistical Mechanics (John Wiley & Sons, Inc., London, 1940), p. 366.