Theory of Waves in Inhomogeneous Warm Plasmas*

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A theory is developed for the behavior of waves in a warm inhomogeneous plasma in the presence of a magnetic field perpendicular to the density gradients of the unperturbed plasma. The full set of the Maxwell-Vlasov equations is solved by use of perturbation methods. The theory is valid for arbitrary directions of wave propagation and both for strong as well as for weak magnetic fields, including zero. It is shown that this theory possesses as limiting cases previously derived results, and we work out as an example of the theory a quantitative analysis of the O'Brien, Gould, and Parker experiment.

I. INTRODUCTION

Kinetic equations have been proposed in the past few years in order to account for the behavior of an electromagnetic field in a plasma. Buchsbaum and Hasegawa[1] derived an equation for the radial electrostatic modes in the positive column which properly accounted for the observed absorption spectrum.[2] The problem of wave coupling to the external fields was not considered until quite recently; some theories were suggested[3-8] which accounted for the coupling mechanism and gave explanation to the fine structure observed.

The equations developed by Pearson[7] differed from those by Allis and Azevedo[6] as well as those by Bernstein[3] by having terms such as $\nabla[\omega_p^2(\vec{x})\nabla\cdot\vec{E}]$ instead of $\nabla^2[\omega_p^2(\vec{x})\vec{E}]$. In Sec. II, it is shown that if proper account is taken of the anisotropic nature of the equilibrium distribution function in strong magnetic fields we obtain terms in $\nabla[\omega_p^2(\vec{x})\nabla\cdot\vec{E}]$; when these equations are regarded as expansions in r_c/L and r_c/λ (r_c is the electron Larmor radius, λ is an effective wavelength, and L is the scale length of the density gradients), they are correct to order $r_c^2/\lambda L$. The asymmetric nature of the equilibrium distribution function is brought forth by the diffusion in the presence of the magnetic field, and the equations derived are valid in the strong-magnetic-field limit. However, it is easy to obtain equations that are correct to zero order in r_c/L and are valid for arbitrary magnetic fields including zero. These equations have terms such as $\nabla^2[\omega_p^2(\vec{x})\vec{E}]$ and are found by use of the dielectric tensor formalism,[4,6,11] which we show by an elementary argument to be valid in this approximation; in this case, the ambipolar fields in the weak-magnetic-field limit and the anisotropy in the equilibrium distribution function due to diffusion in a strong magnetic field can be neglected. Then the equations should be regarded[4,11] as expansions in L_D/λ and L_D/L and are valid to terms of second order in L_D/λ.

We came to the present theory in the process of trying to understand quantitatively the nature of the axial waves observed by O'Brien, Gould, and Parker[9]; Sec. III is devoted to the derivation of dispersion relations for the dipole modes described by the above mentioned experiment. The purpose of this section is to show that the derived equations give account for the experimental results in the zero-magnetic-field case better than other existing theories. The theory explains also experimental results in the strong-magnetic-field limit, and in order to prove this, we show that the Buchsbaum and Hasegawa equation may be obtained as a particular case when we restrict our considerations to longitudinal waves and perpendicular propagation.

II. THEORY

We consider the nonrelativistic Vlasov equation

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\nabla f-\frac{e}{m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial\vec{v}}=0 \tag{1}$$

and look for solutions of the form

$$f(\vec{x},\vec{v},t;\vec{E})=N_0 g(\vec{x})f_0(\vec{x},\vec{v})+e^{i(\omega t-k_3 z)}f_1(\vec{x},\vec{v};\vec{E}). \tag{2}$$

The first term on the right is the equilibrium distribution function in the presence of the magnetic field and inhomogeneity. $N_0 g(\vec{x})$ is the electron density at $\vec{x}$. The second term is a small perturbation around equilibrium with a separated z dependence in the form of a wave; this is not strictly necessary and is only made to alleviate the algebra. We also separate the magnetic field $\vec{B}$ into two parts:

$$\vec{B}=B_0\hat{z}+\vec{B}_1(\vec{x},t). \tag{3}$$

Here B_0 is the external static magnetic field and $\vec{B}_1(\vec{x},t)$ is a small perturbation, and we neglect both the electrostatic field that sustains the plasma and the ambipolar field.[1,4]

Taking account of the prescriptions (2) and (3) together with the assumption that the plasma is quasineutral and neglecting the ion dynamics, we may write the full set of the Maxwell equations as

$$\nabla\cdot\vec{E} = -4\pi e\int d^3v f_1, \quad \nabla\cdot\vec{B}_1 = 0, \quad \nabla\times\vec{E} = -(i\omega/c)\vec{B}_1, \quad \nabla\times\vec{B}_1 = -(4\pi e/c)\int d^3v\,\vec{v}f_1 + (i\omega/c)\vec{E}. \tag{4}$$

We can also write from the above the wave equation

$$(c^2/\omega^2)\nabla\times\nabla\times\vec{E} - \vec{E} = -(4\pi e/i\omega)\int d^3v\,\vec{v}f_1 . \tag{5}$$

It can be seen that choosing cylindrical coordinates in velocity space, $\vec{v} = [v_\perp\cos\phi, v_\perp\sin\phi, v_\parallel]$, we may determine the equilibrium distribution function from the Vlasov equation (1). The equilibrium distribution function satisfies

$$f_0\vec{v}\cdot\nabla g + \omega_c g\frac{\partial f_0}{\partial\phi} = 0, \tag{6}$$

where $\omega_c = eB_0/mc$ is the electron gyrofrequency. In deriving (6) from (1) we assumed that $|g\vec{v}\cdot\nabla f_0| \ll |f_0\vec{v}\cdot\nabla g|$ which is correct when $r_c/L \ll 1$; here r_c is the electron Larmor radius and L is the scale length of the density gradient. At this point, we restrict the density gradients to the (x_1, x_2) plane; the solution of (6) is given by

$$f_0(\vec{x},\vec{v}) = (2\pi v_T^2)^{-3/2}\exp - [\tfrac{1}{2}(\vec{v}/v_T)^2 + \hat{z}\times(\nabla g/g)\cdot(\vec{v}/\omega_c)] \tag{7}$$

v_T being the thermal velocity.

From the above equations, we can see that f_1 must obey the following equation:

$$[\partial/\partial\phi) + H]f_1 = G,$$

$$\text{where}\quad H = (i/\omega_c)(\omega - k_3v_\parallel - i\vec{v}_\perp\cdot\nabla) \quad\text{and}\quad G = -\omega_p^2 f_0\vec{v}\cdot\vec{F}/4\pi e\omega_c v_T^2 . \tag{8}$$

We have also set

$$\vec{F}_\perp = g\vec{E}_\perp - \frac{iv_T^2(\nabla g)}{\omega\omega_c}\hat{z}\cdot\nabla\times\vec{E}, \quad F_3 = gE_3 - \frac{v_T^2}{\omega\omega_c}\hat{z}\times\nabla g\cdot[(k_3 - \frac{\omega}{v_\parallel})\vec{E}_\perp - i\nabla E_3],$$

and $\omega_p^2 = 4\pi N_0e^2/m$ is the plasma frequency. We are only interested in the particular solution of Eq. (8); this is given in terms of an operator U by

$$f_1 = U^{-1}\int_0^\phi d\phi' UG, \quad\text{where}\quad U = \exp\int_0^\phi d\phi' H. \tag{9}$$

If we expand U in (9) in an infinite series of Bessel operators and notice that $e^{A+B} = e^Ae^B$ is valid for commuting operators A and B we obtain

$$f_1 = \frac{\omega_p^2\exp[-\tfrac{1}{2}(\vec{v}/v_T)^2]}{e\omega_c v_T^5(2\pi)^{3/2}}\left\{\exp\left[\frac{-v_\perp}{\omega_c}\left(\sin\phi\frac{\partial}{\partial x_1} - \cos\phi\frac{\partial}{\partial x_2}\right)\right]\right\}\sum_{m,n=-\infty}^{\infty} i^{n+1}I_m(\alpha_1)I_n(\alpha_2)$$

$$\times\left(\frac{e^{i(m-n-1)\phi}}{m-n-1+\eta}v_\perp F_+ + \frac{e^{i(m-n+1)\phi}}{m-n+1+\eta}v_\perp F_- + \frac{e^{i(m-n)\phi}}{m-n+\eta}v_\parallel F_3\right). \tag{10}$$

In the above expression, we have set

$$F_\pm = \tfrac{1}{2}(F_1 \pm iF_2); \quad \alpha_i = (v_\perp/\omega_c)g(\partial/\partial x_i)g^{-1}, \; i = 1, 2; \quad \eta = (\omega/\omega_c) - (k_3/\omega_c)v_\parallel ,$$

and the I_m are modified Bessel operators of the first kind and order m. Equation (10) is an *exact* particular solution of Eq. (8). This same procedure can be used for any other type of equilibrium distribution function. The above expression for f_1 must be substituted on the right-hand side of Eq. (5) in order to obtain the desired expression for the current density.

The integrals can not be evaluated in closed form, and we must resort to approximations. The Bessel operators can be expanded in a power series, and then we may expand f_1 in a power series of the parameters r_c/λ and r_c/L. We choose to keep only terms in $(r_c/\lambda)^0$, $(r_c/L)^0$, r_c/λ, r_c/L, $(r_c/\lambda)^2$, and $r_c^2/\lambda L$, and when $L_D/\lambda \ll 1$ (L_D is the usual Debye length) and $r_c/L < 1$, higher-order terms may be neglected. The case where we have very small magnetic fields $[(r_c/L) > 1]$ will be considered later.

The $v_\perp$ and ϕ integrals can be easily performed. The $v_\parallel$ integrals are singular on the real axis at the points $k_3v_\parallel = \omega - n\omega_c$, $m = 0, \pm1, \ldots ,$ and a prescription must be made to evaluate them. Use of the Plemelj formula analytically continues the distribution function off the real axis and therefore leads to Landau and cyclotron damping. The Cauchy principal-value integrals can be evaluated by means of an expansion in terms of Hermite polynomials.

Performing these integrals we get the following expressions for the current densities:

$$-\frac{4\pi}{i\omega}\vec{J}_\perp=\epsilon_1 g\vec{E}-\frac{i\omega_c}{\omega}\epsilon_2\hat{z}\times g\vec{E}-\Lambda_1\left[\nabla\times g\nabla\times\vec{E}+\frac{i\omega}{\omega_c}(\hat{z}\times\nabla g)\nabla\cdot\vec{E}\right]$$

$$-(\nabla\cdot g\nabla+\hat{z}\times\nabla g\cdot\nabla\hat{z}\times)\left(\Lambda_2\vec{E}-2\frac{i\omega_c}{\omega}\Lambda_3\hat{z}\times\vec{E}\right)+ik_3\left(\Lambda_4-\frac{i\omega_c}{\omega}\Lambda_5\hat{z}\times\right)g\nabla E_3-\frac{k_3\omega}{\omega_c}\Lambda_7(\hat{z}\times\nabla g)E_3 \quad (11)$$

$$-\frac{4\pi}{i\omega}J_3=ik_3\nabla\cdot\left(\Lambda_4-\frac{i\omega_c}{\omega}\Lambda_5\hat{z}\times\right)g\vec{E}+\left(\epsilon_3 g-\Lambda_6\nabla\cdot g\nabla-\frac{i\omega}{\omega_c}\Lambda_8\hat{z}\times\nabla g\cdot\nabla\right)E_3. \quad (12)$$

In the above expressions, we have $\vec{J}=-e\int d^3v\,\vec{v}f_1$ as usual, and the operator ∇ is two dimensional, as well as $\vec{E}$. The ϵ_i and Λ_i will be expressed in terms of the following parameters:

$$\alpha=\omega/\omega_p,\quad \beta=\omega_c/\omega_p,\quad \gamma=\sqrt{3}\,k_3L_D \quad (12a)$$

and $[(\alpha-\beta)^2-\gamma^2][(\alpha+\beta)^2-\gamma^2]=\Gamma_1$, $[(\alpha-2\beta)^2-\gamma^2][(\alpha+2\beta)^2-\gamma^2]=\Gamma_2$,

and are given by

$$\epsilon_1=\frac{2}{3}\left(\frac{1}{\alpha^2-\beta^2}+\frac{\alpha^2-\beta^2-\gamma^2}{2\Gamma_1}\right)+i\frac{(3\pi/2)^{1/2}}{\alpha\gamma}e^{-3(\alpha^2+\beta^2)/2\gamma^2}\cosh\frac{3\alpha\beta}{\gamma^2},$$

$$\epsilon_2=\frac{2}{3}\left(\frac{1}{\alpha^2-\beta^2}+\frac{\alpha^2-\beta^2+\gamma^2}{2\Gamma_1}\right)+i\frac{(3\pi/2)^{1/2}}{\beta\gamma}e^{-3(\alpha^2+\beta^2)/2\gamma^2}\sinh\frac{3\alpha\beta}{\gamma^2},$$

$$\epsilon_3=\frac{1}{\alpha^2-\gamma^2}+i\frac{3\alpha}{\gamma^3}\left(\frac{3\pi}{2}\right)^{1/2}e^{-3\alpha^2/2\gamma^2},$$

$$\Lambda_1=\frac{4L_D^2}{3\alpha^2}\left(\frac{1}{\alpha^2-\beta^2}+\frac{\alpha^2(\alpha^2-\beta^2+3\gamma^2)}{2\Gamma_1(\alpha^2-\gamma^2)}\right)+i\frac{2L_D^2(3\pi/2)^{1/2}}{\alpha\beta^2\gamma}e^{-3\alpha^2/2\gamma^2}\left(e^{-3\beta^2/2\gamma^2}\cosh\frac{3\alpha\beta}{\gamma^2}-1\right),$$

$$\Lambda_2=2L_D^2\left(\frac{1}{(\alpha^2-\beta^2)(\alpha^2-4\beta^2)}+\frac{(\alpha^2-\beta^2+\gamma^2)(\alpha^2-4\beta^2+\gamma^2)}{2\Gamma_1\Gamma_2}\right)$$

$$+i\frac{L_D^2(3\pi/2)^{1/2}}{\alpha\beta^2\gamma}e^{-3\alpha^2/2\gamma^2}\left(\frac{1}{2}e^{-6\beta^2/\gamma^2}\sinh\frac{6\alpha\beta}{\gamma^2}-e^{-3\beta^2/2\gamma^2}\sinh\frac{3\alpha\beta}{\gamma^2}\right),$$

$$\Lambda_3=2L_D^2\left(\frac{1}{(\alpha^2-\beta^2)(\alpha^2-4\beta^2)}+\frac{\alpha^2(\alpha^2-\beta^2+3\gamma^2)(\alpha^2-4\beta^2+3\gamma^2)}{2(\alpha^2-\gamma^2)\Gamma_1\Gamma_2}\right)$$

$$+i\frac{L_D^2(3\pi/2)^{1/2}}{\beta^3\gamma}e^{-3(\alpha^2+\beta^2)/2\gamma^2}\left(\frac{1}{4}e^{-9\beta^2/2\gamma^2}\cosh\frac{6\alpha\beta}{\gamma^2}-\cosh\frac{3\alpha\beta}{\gamma^2}+\frac{3}{4}e^{3\beta^2/2\gamma^2}\right),$$

$$\Lambda_4=\frac{2L_D^2}{\Gamma_1}+i\frac{3L_D^2}{\alpha\beta\gamma^3}\left(\frac{3\pi}{2}\right)^{1/2}e^{-3(\alpha^2+\beta^2)/2\gamma^2}\left(\alpha\sinh\frac{3\alpha\beta}{\gamma^2}-\beta\cosh\frac{3\alpha\beta}{\gamma^2}\right), \quad (12b)$$

$$\Lambda_5=\frac{L_D^2}{\alpha^2(\alpha^2-\beta^2)}+i\frac{3L_D^2(3\pi/2)^{1/2}}{\beta^2\gamma^3}e^{-3(\alpha^2+\beta^2)/2\gamma^2}\left(\alpha\cosh\frac{3\alpha\beta}{\gamma^2}-\beta\sinh\frac{3\alpha\beta}{\gamma^2}-\alpha e^{-3\beta^2/2\gamma^2}\right),$$

$$\Lambda_6=\frac{L_D^2}{\alpha^2(\alpha^2-\beta^2)}+i\frac{3L_D^2(3\pi/2)^{1/2}}{\alpha\beta^2\gamma^3}e^{-3(\alpha^2+\beta^2)/2\gamma^2}\left((\alpha^2+\beta^2)\cosh\frac{3\alpha\beta}{\gamma^2}-2\alpha\beta\sinh\frac{3\alpha\beta}{\gamma^2}-\alpha^2e^{3\beta^2/2\gamma^2}\right),$$

$$\Lambda_7 = \frac{L_D^{\,2}}{\alpha^2(\alpha^2-\gamma^2)} + i\frac{3L_D^{\,2}}{\alpha\gamma^3}\left(\frac{3\pi}{2}\right)^{1/2} e^{-3\alpha^2/2\gamma^2},$$

$$\Lambda_8 = \frac{L_D^{\,2}}{\alpha^2(\alpha^2-\beta^2)} + i\frac{3L_D^{\,2}(3\pi/2)^{1/2}}{\alpha^2\beta\gamma^3} e^{-3(\alpha^2+\beta^2)/2\gamma^2}\left((\alpha^2+\beta^2)\sinh\frac{3\alpha\beta}{\gamma^2} - 2\alpha\beta\cosh\frac{3\alpha\beta}{\gamma^2}\right).$$

Restricting (11) and (12) to perpendicular propagation, we obtain

$$-\frac{4\pi}{i\omega}\vec{J}_\perp = \epsilon g\left(1 - i\frac{\omega_c}{\omega}\hat{z}\times\right)\vec{E} - \Gamma\left(\nabla\times g\nabla\times\vec{E} + i\frac{\omega}{\omega_c}(z\times\nabla g)\nabla\cdot\vec{E}\right) - \Lambda\left(\nabla\cdot g\nabla + \hat{z}\times\nabla g\cdot\nabla\hat{z}\times\right)\left(1 - 2i\frac{\omega_c}{\omega}z\times\right)\vec{E} \tag{13}$$

$$-\frac{4\pi}{i\omega}J_3 = \alpha^{-2}gE_3 - \frac{\Gamma}{2}\left(\nabla\cdot g\nabla + i\frac{\omega}{\omega_c}z\times\nabla g\cdot\nabla\right)E_3, \tag{14}$$

where $\epsilon = 1/(\alpha^2-\beta^2)$; $\Gamma = 2L_D^{\,2}/\alpha^2(\alpha^2-\beta^2)$; $\Lambda = 3L_D^{\,2}/(\alpha^2-\beta^2)(\alpha^2-4\beta^2)$.

Equation (13) is not exactly the same equation found by Pearson, the difference being due mainly to the term $\omega/\omega_c(z\times\nabla g)\nabla\cdot\vec{E}$, which is not contained in Pearson's equation.

The above equations fail to be valid for low magnetic fields. This is because the diffusion which causes the anisotropy in the equilibrium distribution function in strong magnetic fields becomes ambipolar[10] in weak magnetic fields, and the equilibrium distribution becomes Maxwellian isotropic. In this case the ambipolar field is not negligible[10] and contributes to the current-density expression with terms of first order in r_c/L or L_D/L. Therefore we use the isotropic Maxwellian distribution for the equilibrium configuration and obtain a new G to be used in (8).

$$G = -\omega_p^{\,2} f_0 \vec{v}\cdot g\vec{E}/4\pi e\omega_c v_T^{\,2}. \tag{15}$$

It is easy to verify that in this case $\vec{J}$ can be expressed in the form[4, 11]

$$J_k = \sum_{l=1}^{3} \sigma_{kl}(i\nabla)gE_l, \qquad k = 1, 2, 3, \tag{16}$$

where $\sigma_{kl}(\vec{k})$ is the usual conductivity tensor for a warm plasma.

Notice that (11) and (12) are valid in the homogeneous case as well as (16); the conductivity operator in (16) can therefore be written down by setting $g=1$ in Eqs. (11) and (12). When this is done we obtain from expressions (16)

$$-\frac{4\pi}{i\omega}J_\perp = (\epsilon_1 - i\frac{\omega_c}{\omega}\epsilon_2\hat{z}\times)g\vec{E} - \Lambda_1\nabla\times\nabla\times g\vec{E} - \nabla^2\left(\Lambda_2 - 2i\frac{\omega_c}{\omega}\Lambda_3\hat{z}\times\right)g\vec{E} + ik_3\left(\Lambda_4 - i\frac{\omega_c}{\omega}\Lambda_5\hat{z}\times\right)\nabla(gE_3), \tag{17}$$

$$-\frac{4\pi}{i\omega}J_3 = (\epsilon_3 - \Lambda_6\nabla^2)gE_3 + ik_3\nabla\cdot\left(\Lambda_4 - i\frac{\omega_c}{\omega}\Lambda_5\hat{z}\times\right)g\vec{E}. \tag{18}$$

In the case of strong magnetic fields these equations are valid to zero order in r_c/L and to retain terms of the order of $r_c^{\,2}/\lambda^2$ in the presence of terms of order $r_c^{\,2}/\lambda L$ we must have $\lambda \ll L$ as observed by Pearson.[7] In the low-magnetic-field limit the above equations should be regarded[4, 11] as expansions in L_D/λ and L_D/L correct to second order in L_D/λ.

If we restrict the above equations to the case where $k_3 = E_3 = 0$, take the divergence of (17), and set $\vec{E} = -\nabla\Phi$, we obtain the Buchsbaum-Hasegawa equation

$$\nabla^2\Phi = (\epsilon - \Lambda\nabla^2)\nabla\cdot g\nabla\Phi + i(\omega_c/\omega)(\epsilon - 2\Lambda\nabla^2)\hat{z}\cdot\nabla\times g\nabla\Phi, \tag{19}$$

where ϵ and Λ were given in connection with Eqs. (13) and (14).

Care must be exercised in deriving Eqs. (17) and (18) from known expressions[12] for the dielectric tensor as some of them have the wave vector in one of the coordinate planes, and we must rotate it off them and next substitute $i\nabla$ for $\vec{k}$.

Buchsbaum and Hasegawa theory is therefore seen to be valid only to zero order in r_c/L for arbitrary λ, and the fact that it fitted very well the experimental data is probably due both to the rotational symmetry of the longitudinal modes under study and to the smallness of the density gradients in the core of the plasma column. This can be seen by use of Eq. (13) under these restrictions.

III. WAVES ALONG THE POSITIVE-COLUMN

We now work out an example of the above theory and derive dispersion relations ($\omega - k$ curves) for the first and second bands of dipole propagation for waves along the column axis first measured by O'Brien, Gould, and Parker.[9] These waves have the Tonks-Datner[13,14] resonances as cut-off conditions; we set $\vec{E} = -\nabla\Phi$, use cylindrical coordinates, and separate the z, ϕ dependence in the form

$$\exp[i(\omega t - \phi - k_3 z)],$$

where ϕ is the polar angle. The only magnetic field present is the Earth's, and it is important for the correct analysis[9] of the experiment. We set it equal to zero as a first approximation, and use (17) and (18) for the current density. Taking account of the fact that $-(4\pi/i\omega)\nabla\cdot\vec{J} = \nabla\cdot\vec{E}$ and making use of the above prescriptions together with the assumption of small density gradients, we obtain, by neglecting the coupling between transverse and plasma waves, the following equation:

$$(\Lambda_4 + \Lambda_6)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)g\Phi + [1/g - \epsilon_3 - (\Lambda_4 + \Lambda_6)/r^2]g\Phi = 0\,. \tag{20}$$

We consider only the real parts of the plasma parameters in (12b). The density profiles can be characterized[14,15] by the parameter $r_w^2/\overline{L_D^2}$, where r_w is the radius of the plasma column and $\overline{L_D^2}$ is an average over the plasma cross section of the Debye length squared. For the experiment in question this parameter is of the order of 500, in which case the density can be expressed with good accuracy by

$$g = [1 - (r/r_w)^2]/[1 + \tfrac{1}{2}(r/r_w)^2]\,. \tag{21}$$

Setting $z = (\epsilon_3 + \frac{1}{2})(r/r_w)^2$, $R = rg\Phi$, and $\epsilon_3 = c - 1$ in Eq. (20), we obtain

$$\left[\frac{d^2}{dz^2} + \left(\frac{r_w^2}{\Lambda_4 + \Lambda_6}\right)\frac{z - c}{z(c - \frac{1}{2} - z)}\right]R = 0. \tag{22}$$

We can solve the above equation by the WKB approximation by restricting to $\alpha^2 < 1$; matching the two solutions at the turning point $z = c$ we obtain

$$\int_{c-\frac{1}{2}}^{c} dz\left(\frac{z - c}{z(c - \frac{1}{2} - z)}\right)^{1/2} = \pi(N + m)\frac{(\Lambda_4 + \Lambda_6)^{1/2}}{r_w}\,,$$

$$N = 0, 1, 2, \ldots, \tag{23}$$

where m is also a constant to be adjusted by the knowledge of one experimental point. Solving the above equation and neglecting terms of smaller order, we obtain

$$-\tfrac{1}{4}[(\Lambda_4 + \Lambda_6)(c - \tfrac{1}{2})]^{1/2} = N + m \tag{24}$$

which can also be written in terms of the plasma parameters as

$$\left(\frac{1}{\alpha^2 - \gamma^2} + \frac{1}{2}\right)\left(\frac{2}{(\alpha^2 - \gamma^2)^2} + \frac{1}{\alpha^4}\right)$$

$$= (\tfrac{3}{4} r_w/L_D)^2 (N + m)^{-2}. \tag{25}$$

Despite the approximations made, we obtain good agreement with experiment. In fact for this case $\overline{L_D^2} \cong 5.5\, L_D^2$ and $\overline{\alpha^2} \cong 5.5\,\alpha^2$, so that we obtain from (25) the values $\overline{\alpha_1^2} \cong 0.36$ and $\overline{\alpha_2^2} \cong 0.85$. The dispersion relation (25) is plotted in Fig. 1, and the agreement is reasonable as seen from the experimental points. The solid curve refers to the profile (21), and we see that it does not give account of the observed[9] backward wave. This is due to the density profile used. In fact the density profile

$$g^{-1} = 1 + A(r/r_w)^2 \tag{26}$$

is also in good agreement with experimental data for $r_w^2/\overline{L_D^2} = 500$ if we set $A = 1.8$ and restrict r to $r < 0.58 r_w = r_0$. Setting

$$z = r_w\left(\frac{A}{\Lambda_4 + \Lambda_6}\right)^{1/2}\left(\frac{r}{r_w}\right)^2 \quad\text{and}\quad c = \frac{r_w(\epsilon_3 - 1)}{[A(\Lambda_4 + \Lambda_6)]^{1/2}}\,,$$

we write (20) in the form

$$d^2R/dz^2 + [1 - (c/z)]R = 0. \tag{27}$$

The solution of the above equation can be written in terms of Coulomb functions. We restrict to the case $z \gg c$ and look for the asymptotic solution of (27) to obtain the following expression, where we introduced a phase shift m in order to match the solution to the experimental data

$$R(z) \sim \sin[z - \tfrac{1}{2}c\ln 2z + \arg\Gamma(1 + ic/2) - m]\,. \tag{28}$$

As before we adjust m to one experimental point only. We set $R(r_0) = 0$ and obtain from (28) the dispersion relation

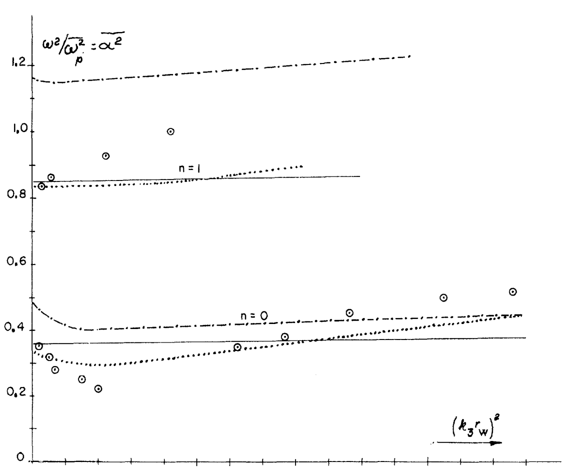

FIG. 1. Theoretical $w - k$ curves calculated for the first and second bands of dipole propagation. Solid lines are a plot of Eq. (25), dotted lines are a plot of Eq. (29), dashed-dotted lines refer to computer calculations by O'Brien, Gould, and Parker,[9] and circles represent the experimental results obtained by them.

$$\frac{r_0^2}{r_w}\left(\frac{A}{\Lambda_4+\Lambda_6}\right)^{1/2}-\frac{c}{2}\ln\frac{2r_0^2}{r_w}\left(\frac{A}{\Lambda_4+\Lambda_6}\right)^{1/2}$$

$$\left(+\arg\Gamma\ \ 1+i\frac{c}{2}\right)=N\pi+m\,. \tag{29}$$

We set $N=0$, and adjust m to the experimental data to obtain $m=0.13\pi$. Equation (29) is also plotted in the figure, and we see that the lower band exhibits a backward wave whereas the upper curve has none.

The results derived in Part III are semiquantitative. A detailed analysis should take into account the correct density profiles,[14, 15] the coupling between the waves, and the correct matching of boundary conditions. This is being carried by computer calculations and will be published elsewhere.

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