# Algebraic Realizations of Chiral Symmetry*†

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The sum of tree graphs for forward pion scattering, generated by any chiral-invariant Lagrangian, is required to grow no faster at high energies than the actual scattering amplitude. In consequence, algebraic restrictions must be imposed on the axial-vector coupling matrix **X** and the mass matrix $m^2$: For each helicity, **X** must, together with the isospin **T**, form a representation of $SU(2)\otimes SU(2)$, and $m^2$ must behave with respect to commutation with **T** and **X** as the sum of a chiral scalar and the fourth component of a chiral four-vector. If it is further assumed that the contribution of tree graphs to inelastic forward pion scattering vanishes at high energy, the two parts of the mass matrix must commute; this fixes various mixing angles, and leads to predictions like $m_\sigma = m_\rho$, $m_{A_1}=\sqrt{2}m_\rho$, $\Gamma_\rho = 135$ MeV, etc. If all pion transitions involved only $p$-wave pions, then **X** would form part of the algebra of $SU(4)$, and the mass matrix would behave as the sum of a 1- and a 20-dimensional representation of $SU(4)$; if $s$-wave transitions are allowed, then the algebra must be enlarged to at least $SO(7)$.

## I. CHIRAL DYNAMICS AND CHIRAL ALGEBRA

NATURE exhibits symmetries of two different sorts.[1] On one hand, there are the *algebraic* symmetries, like Lorentz invariance, ordinary gauge invariance, and the charge independence of strong interactions. These symmetries yield conservation laws, as for energy, momentum and angular momentum, for charge, and for isotopic spin, and they predict ratios among $S$-matrix elements for processes involving a fixed number of particles. Algebraic symmetries are realized in field theories by homogeneous linear transformation laws, such as

$$\psi(x) \to S^{-1}(\Lambda)\psi(\Lambda x + a)\,, \qquad \text{(Lorentz invariance)}$$

$$\psi(x) \to e^{ie\theta}\psi(x)\,, \quad A^\mu(x) \to A^\mu(x)\,, \qquad \text{(ordinary gauge invariance)}$$

$$\psi(x) \to \exp(i\boldsymbol{\theta}\cdot\mathbf{t})\,\psi(x) \qquad \text{(charge independence)}.$$

The other symmetries are *dynamic*, like general covariance and local gauge invariance.[1] These do not yield conservation laws (aside from those already implied by their algebraic subgroups) and do not predict relations among processes involving fixed numbers of particles. Instead, the dynamic symmetries yield low-energy theorems for soft gravitons and photons, or to put it classically, they impose restrictions (such as Einstein's principle of equivalence) on the response of physical systems to slowly varying gravitational or electromagnetic fields. Dynamic symmetries are realized inhomogeneously on the fields of the massless bosons, i.e., on the affine connection

$$\Gamma_{\mu\nu}{}^{\lambda} \to \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma_{\tau\sigma}{}^{\rho} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}$$

and on the vector potential

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x)\,.$$

We now know that the strong interactions obey an approximate dynamic symmetry,[2] chiral $SU(2)\times SU(2)$. Chirality is realized through inhomogeneous transformations on the pion field,[3] i.e.,

$$\delta\boldsymbol{\pi} = F_\pi\{\tfrac{1}{2}\boldsymbol{\varepsilon} + F_\pi^{-2}[\boldsymbol{\pi}(\boldsymbol{\pi}\cdot\boldsymbol{\varepsilon}) - \tfrac{1}{2}\boldsymbol{\pi}^2\boldsymbol{\varepsilon}]\}\,, \qquad (1.1)$$

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[1] The distinction made here between algebraic and dynamic symmetries is essentially the same as drawn between "geometric" and dynamic symmetries by E. P. Wigner, *Symmetries and Reflections* (Indiana University Press, Bloomington, Indiana, 1967), except that we separate *ordinary* gauge invariance, by which we mean invariance under gauge transformations with constant phase, from *local* gauge invariance, for which the gauge transformations depend upon the space-time coordinates. (Wigner characterizes dynamic symmetries as those which characterize *theories* while the "geometric" symmetries govern *events* directly. The low-energy theorems arising from dynamic symmetries somewhat vitiate this distinction.)

[2] This interpretation of chiral symmetry goes back to the work of Nambu and his collaborators; Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961); Y. Nambu and D. Lurié, *ibid.* **125**, 1429 (1962); Y. Nambu and E. Shrauner, *ibid.* **128**, 862 (1962); etc.

[3] J. Schwinger, Phys. Letters **24B**, 473 (1967); J. W. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967). The relation between invariance of the Lagrangian under such transformations and the soft-pion theorems of current algebra had been pointed out by S. Weinberg, Phys. Rev. Letters **18**, 507 (1967). The essential uniqueness of these transformation rules was proved by W. A. Bardeen and B. W. Lee, in *Nuclear and Particle Physics*, edited by B. Margolis and C. Lam (W. A. Benjamin, Inc., New York, 1968), p. 273; L. S. Brown, Phys. Rev. **163**, 1802 (1967); S. Weinberg, *ibid.* **166**, 1568 (1968). This uniqueness theorem has been extended to general compact Lie groups by S. Coleman, J. W. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1968). Lagrangians invariant under nonlinear chiral transformations were considered before the advent of current algebra by F. Gürsey, Nuovo Cimento **16**, 705 (1960); in *Proceedings of the 1960 Rochester Conference on High Energy Physics* (Wiley-Interscience Publishers, Inc., New York, 1960), p. 572; and Ann. Phys. (N. Y.) **12**, 91 (1961); M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); F. Gürsey and B. Zumino (unpublished); and J. A. Cronin (unpublished); and Phys. Rev. **161**, 1483 (1967). For an assortment of recent related work, see H. S. Mani, Y. Tomozawa, and Y. P. Yao, Phys. Rev. Letters **18**, 1084 (1967); P. Chang and F. Gürsey, Phys. Rev. **164**, 1752 (1967); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968); etc.

where $\boldsymbol{\varepsilon}$ is an infinitesimal chiral "boost" vector, and $F_\pi \simeq 190$ MeV is the pion decay amplitude. In consequence chiral symmetry yields low-energy theorems for soft pions, which would presumably be exact in a better world with vanishing pion mass. Further, the chiral transformation properties of general fields (and hence the scattering of soft pions by the corresponding particles) are determined by their isospin,[3] i.e.,

$$\delta\psi = iF_\pi^{-1}(\mathbf{t}\times\boldsymbol{\pi})\cdot\boldsymbol{\varepsilon}\psi\,, \tag{1.2}$$

where $\mathbf{t}$ is the isospin matrix appropriate to $\psi$. Since the pion carries isospin, chiral-invariant Lagrangians must be highly nonlinear, just as the Einstein equations are nonlinear because gravitations carry energy and momentum, while Maxwell's equations are linear because photons do not carry charge. Even the detailed rules for constructing chiral-invariant Lagrangians are suggestively reminiscent of the rules for general covariance in general relativity: derivatives of general fields must appear in "covariant derivatives"[4]

$$D_\mu\psi = \partial_\mu\psi + 2iF_\pi^{-2}(1+F_\pi^{-2}\boldsymbol{\pi}^2)^{-1}\mathbf{t}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\psi \tag{1.3}$$

and the pion field may only appear in $D_\mu\psi$ and its own "covariant derivative"[4]

$$D_\mu\boldsymbol{\pi} = (1+F_\pi^{-2}\boldsymbol{\pi}^2)^{-1}\partial_\mu\boldsymbol{\pi}. \tag{1.4}$$

There are obvious differences between chirality and the other dynamical symmetries. First, pions are not massless, so predictions for soft-pion processes can never be trusted to better than a few per cent. This very important problem will not be discussed at all in this paper; we will take the pion mass as zero throughout.[5] Second, local guage invariance and general covariance can be derived[6] as requirements imposed on photon and graviton interactions by Lorentz invariance and quantum mechanics. No similar derivation exists for chirality nor does it seem likely that one will be found, since we can easily write Lagrangians for massless pions which seem physically satisfactory and yet are not chiral invariant. It has been suggested[7] that chirality is required, not by the spin of the pion, but by its membership in a family of Regge trajectories. This, also, will not be pursued further here.

The problem before us in this article arises from another difference between chirality and its older cousins. Gauge invariance and general covariance are purely dynamic symmetries, whose consequences are completely different in character from those of an algebraic symmetry like isotopic spin. Also, the symmetry groups they form are infinite-dimensional, so it is not clear how they could possibly have algebraic consequences in any case. In contrast, chiral $SU(2)\times SU(2)$ is an ordinary Lie group, and chiral transformations were until recently thought of as ordinary algebraic symmetry transformations, which happen to be spontaneously broken[8] by a large vacuum expectation value of the chiral partner $\sigma$ of the pion field. Indeed, the modern development of current algebra arose from the attempt to derive algebraic results from the chiral commutation relations, by saturating them with single-particle states.[9] It was soon realized that this saturation program could only be successful at infinite momentum,[10] where hopefully the disconnected three-particle and one-pion states would not contribute, and in this way there have been derived many interesting results of a purely algebraic character.

Our problem is: *How is it possible for a dynamic symmetry like chirality to have any algebraic consequences*? At first sight this seems a paradox, because chiral-invariant Lagrangians can be constructed with any values of the axial-vector coupling constants, using only ordinary isospin multiplets like $N$ and $\boldsymbol{\pi}$, with no need to add other particles like $\sigma$ which complete chiral multiplets. It is only necessary that a Lagrangian conserve isotopic spin and be constructed out of $\psi$, $D_\mu\psi$, and $D_\mu\boldsymbol{\pi}$ for it to be chiral-invariant. Clearly, chirality alone does *not* have algebraic consequences.

We can solve this paradox by taking another look at our formulas [Eqs. (1.3) and (1.4)] for the covariant derivatives $D_\mu\psi$ and $D_\mu\boldsymbol{\pi}$. Any Lagrangian built from such ingredients will be highly nonlinear and replete with derivative interactions. In consequence, the $S$-matrix elements, at least in lowest order, will have bad asymptotic behavior at high energy, unless detailed cancellations intervene. *The algebraic aspects of chiral symmetry arise from the need for cancellations which insure reasonable asymptotic behavior at high energy.*

How are we to implement this idea? Certainly we are not going to calculate the high-energy behavior of the exact $S$-matrix elements arising from a general chiral-invariant Lagrangian.[11] The best use to which

[4] S. Weinberg, Phys. Rev. **166**, 1568 (1968). For the extension to general compact Lie groups, see C. G. Callan, S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1968).

[5] For a recent and very thorough treatment of this problem, see S. Fubini and G. Furlan (to be published). This paper shows how the dynamic interpretation of chiral symmetry in a process like $\pi$-$N$ scattering depends upon the smallness of $m_\pi^2$ compared with $4m_N^2$ and $(m_N-m_{N'})^2$, where $N'$ is the $S_{11}$ resonance at 1550 MeV.

[6] S. Weinberg, Phys. Rev. **135**, B1049 (1964); **138**, B988 (1965). Also see D. Zwanziger, *ibid.* **133**, B1036 (1964).

[7] S. Mandelstam, Phys. Rev. **168**, 1884 (1968).

[8] See, for example, M. Gell-Mann and M. Lévy, Ref. 3. An algebraic symmetry broken by the vacuum is from our present point of view just a particular species of dynamic symmetry.

[9] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964); R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964); B. W. Lee, *ibid.* **14**, 271 (1965); R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 275 (1965); **17**, 279 (1965).

[10] S. Fubini and G. Furlan, Physics **1**, 229 (1965); R. F. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 371 (1966); and in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1966).

[11] E. S. Fradkin (to be published) and H. M. Fried (to be published) have argued that the nonlinearities of the chiral-invariant Lagrangian give the exact $S$ matrix a bad asymptotic behavior at high energy. It would be very interesting to know whether this really happens, but for our present purposes it does not matter, since we are using the Lagrangian only as a device for generating trees which satisfy the soft-pion theorems of current algebra.

we (or at any rate, I) can put such Lagrangians is to use them in the "tree" approximation, keeping only diagrams without loops. This approximation is guaranteed[12] to give the right answer for low-energy pion processes, because soft-pion matrix elements are predicted by current algebra proper, and predicted furthermore to be of the "tree" form.[13] But we do *not* expect and shall not assume the tree approximation, with any finite number of resonances, to give good results at very high energy. Rather, we shall impose in our trees only the weak requirement, that their sum *not grow faster* at high energy than the actual scattering amplitude. That is, since individual tree graphs do have a worse high-energy behavior than that expected for the actual scattering amplitude, we shall require that *the rapidly growing terms contributed by the tree graphs shall cancel among themselves*, and not with the "continuum" which is left out of the tree approximation.

Perhaps the most convincing argument I can give for this assumption is to point out that the sum of tree graphs obviously obeys the usual dispersion relations with the "correct" number of subtractions, if and only if it behaves no worse at high energy than the actual physical amplitude. The exclusion of all diagrams but the trees then just corresponds to the saturation of the dispersion integrals by single-particle states. Hence our assumption about cancellation of trees at high energy is just what is needed to ensure that the scattering amplitude calculated in the tree approximation should agree with that calculated by single-particle saturation of dispersion relations. The latter is known to give a good approximation in the low- and medium-energy regions (as for instance in $\pi$-$N$ forward scattering) so the effect of our assumption is not to extend the tree approximation into the region of high energies, where it has no place, but is rather to make it valid in the resonance-dominated region of medium energies, where it operates as a Lorentz- and crossing-invariant form of the Breit-Wigner approximation.

Why then (asks the unkind reader) do I bother with Lagrangians and trees, and not work directly with dispersion relations and the sum rules derived from them? I find it more illuminating to think about Lagrangian field theories than dispersion relations, but that is a matter of taste, and certainly in the simple processes where convenient dispersion sum rules are available, we get the same results by saturating them with single-particle states as by requiring the high-energy cancellation of chiral trees. (See Appendix A.) The real answer is that dispersion relations are simply not available for multiparticle processes like $\pi+N\to 2\pi+N$ or $\pi+N+N\to\pi+N+N$, while there is no difficulty in calculating them using chiral-invariant tree graphs. Thus the present work should be regarded, not so much as an attempt to derive new predictions, but as a first step towards a general understanding of those algebraic aspects of chiral symmetry which make possible chiral-invariant theories of general low- and medium-energy pion processes.

In this paper we will be applying these ideas to only one sort of process, the forward scattering (*elastic or inelastic*) of massless pions on an arbitrary target particle. Our results for this case may be expressed in terms of an axial-vector coupling matrix $X_a(\lambda)$, defined by giving the matrix elements at zero invariant momentum transfer of the axial-vector current between states with collinear momenta as

$$\langle \mathbf{p}'\lambda'\beta|(A_a{}^3+A_a{}^0)|\mathbf{p}\lambda\alpha\rangle=(2\pi)^{-3}(4p'^0p^0)^{-1/2}\times 4\mathcal{E}\delta_{\lambda'\lambda}[X_a(\lambda)]_{\beta\alpha}, \quad (1.5)$$

where $\alpha$ and $\beta$ are stable or unstable one-particle states with momenta $\mathbf{p}$ and $\mathbf{p}'$ in the 3 direction; $\lambda$ and $\lambda'$ are their helicities; $a$ is the isovector index; and the quantity $\mathcal{E}$ is defined by writing the condition of zero invariant momentum transfer as

$$|\mathbf{p}|+(\mathbf{p}^2+m_\alpha{}^2)^{1/2}=|\mathbf{p}|'+(\mathbf{p}'^2+m_\beta{}^2)^{1/2}\equiv\mathcal{E}. \quad (1.6)$$

This definition of the axial-vector couplings is particularly convenient as it can be used for particles of arbitrary spin, and in arbitrary collinear reference frames, including both the frames in which $\alpha$ is at rest and in which it moves with infinite momentum. (The matrix $\mathbf{X}(\lambda)$ is independent of the reference frame.) To make our normalization perfectly clear, we should add that $A_a{}^\mu$ is defined as in earlier papers,[4] and that the rate for a decay process $\alpha\to\beta+\pi$, with $\alpha$ unpolarized and at rest, is given by

$$\Gamma(\alpha\to\beta+\pi;\lambda)=\frac{(m_\alpha{}^2-m_\beta{}^2)^3}{4\pi F_\pi{}^2m_\alpha{}^3(2J_\alpha+1)}|[X_a(\lambda)]_{\beta\alpha}|^2, \quad (1.7)$$

where $\lambda$ is the helicity of $\beta$, and $a$ is the pion isovector index. (This formula is for zero pion mass; effects of the finite pion mass are discussed in Sec. V.) Also, the matrix elements of $X_a(\lambda)$ between one-nucleon states are given by the substitution

$$X_a(\lambda)=(g_A/g_V)\lambda\tau_a, \quad (1.8)$$

where $\tau_a$ is the Pauli matrix, and $|g_A/g_V|$ is experimentally about 1.2.

After kinematic preliminaries in Sec. II, we show in Sec. III that there are certain terms in a high-energy asymptotic expansion of the tree graphs for forward pion scattering which may, in a completely general chiral-invariant theory, be expressed entirely in terms of $X_a(\lambda)$, the isospin matrix $T_a$, and the mass matrix $m_{\beta\alpha}{}^2\equiv m_\alpha{}^2\delta_{\beta\alpha}$. These terms are proportional to a power of the energy higher than allowed by Regge-pole theory, and therefore should be assumed to cancel. We show in Sec. IV that the cancellation of the terms with $T=1$

[12] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[13] See, e.g., H. D. I. Abarbanel and S. Nussinov, Ann. Phys. (N. Y.) 42, 467 (1967).

in the $t$ channel requires that

$$[X_a(\lambda),X_b(\lambda)]=i\epsilon_{abc}T_c \tag{1.9}$$

while the cancellation of the terms with $T=2$ in the $t$ channel requires that[14]

$$[X_a(\lambda),[X_b(\lambda),m^2]]\propto\delta_{ab}. \tag{1.10}$$

In the language of group theory, Eq. (1.9) tells us that *the one-particle states of any given helicity must be assembled into representations of chiral $SU(2)\times SU(2)$, and that these representations (including their mixing angles) determine the measurable axial-vector coupling matrix* $X_a(\lambda)$. Also, we show in Sec. IV that (1.10) implies that *$m^2$ may be written as the sum of a chiral-scalar $m_0^2$ and the fourth component $m_4^2$ of a chiral four-vector*. That is, for each $\lambda$ we may write

$$m^2=m_0^2(\lambda)+m_4^2(\lambda), \tag{1.11}$$

where

$$[X_a(\lambda),m_0^2(\lambda)]=0 \tag{1.12}$$

and

$$[X_a(\lambda),m_4^2(\lambda)]=im_a^2(\lambda), \tag{1.13}$$

$$[X_a(\lambda),m_b^2(\lambda)]=-i\delta_{ab}m_4^2(\lambda). \tag{1.14}$$

If there were no Regge trajectories with the quantum numbers of the vacuum and with $\alpha(0)\geq 0$ then we would further assume the cancellation of terms with $T=0$ in the $t$ channel, and we would then find that $m_4^2(\lambda)=0$. Of course such vacuum trajectories do exist, so $m^2$ does *not* commute with $X_a(\lambda)$, and therefore particle states of definite mass generally belong to *reducible* representations of $SU(2)\otimes SU(2)$. In this sense, then, we can still say that algebraic chiral symmetry is broken by the vacuum. We may, however, conjecture that these vacuum trajectories do not have appreciable residues[15] for inelastic processes like $\pi+N\to\pi+N'(1470)$ or $\pi+\pi\to\pi+A_1$, in which case we find that $m_4^2(\lambda)$ is diagonal, i.e.,

$$[m^2,m_4^2(\lambda)]=[m_0^2(\lambda),m_4^2(\lambda)]=0. \tag{1.15}$$

These general results are applied in Sec. VI to the $\lambda=0$ states of $\pi$, $\rho$, $\sigma$, and $A_1$. We obtain the results already found[16] from saturation of elastic Adler-Weisberger relations, i.e., that the observed $\rho$ width determines the $\pi$-$A_1$ mixing angle $\psi$ to be 45°–50°, and that with $\psi=45°$ we have $m_{A_1}=\sqrt{2}m_\rho$, $m_\sigma=m_\rho$, $\Gamma_\sigma\approx 600$ MeV, etc. In addition we now have information on the relative *sign* of the $A_1\to\sigma+\pi\to 3\pi$ and $A_1\to\rho+\pi\to 3\pi$ decay amplitudes, equivalent to that which would have been obtained by saturation of inelastic Adler-Weisberger sum rules. But more important, we now have a clue as to *why* $\psi$ should be 45°; it turns out that the communtator (1.15) is proportional to $\cos 2\psi$, and therefore $\psi=45°$ is just the mixing angle required if tree contributions to the amplitude for $\pi+\pi\to\pi+A_1$ cancel at high energy.

A different technique is used in Sec. VII to apply our general results to the $\lambda=\frac{1}{2}$ states of nonstrange baryons. We include in our calculation only the nucleon, the $\Delta(1238)$, and a $T=\frac{1}{2}$ resonance $N'$. Our results are again just those obtained[16] from the Adler-Weisberger sum rules, giving $g_A/g_V$, $\Gamma_\Delta$, $m_{N'}$, etc., in terms of a single mixing angle $\theta$. If we further conjecture that the tree contributions to $\pi+N\to\pi+N'$ cancel at high energy then (1.15) applies, and tells us that $\theta=45°$, giving $|g_A/g_V|=\frac{4}{3}$ and $m_{N'}^2=2m_\Delta^2-m_N^2$.

In Sec. VIII we begin an attack on a more ambitious problem, of finding solutions to the chiral commutation relations (1.9) which satisfy the general requirement that the amplitude for pion transitions $\alpha\to\beta+\pi$ with orbital angular momentum $l$ has the helicity dependence $C_{J_\alpha l}(J_\beta\lambda;\lambda 0)$. We approach this problem by enlarging the algebra to include an angular-momentum vector $J_i$ which acts in the usual way on helicity indices, and by imposing certain *a priori* restrictions on the algebra generated by $T_a$, $J_i$, and $X_a$. We find that if this algebra contains no elements with $T=2$, and if all pion transitions have $l=1$, then the algebra generated is that of $SU(4)$. Further, (1.11)–(1.14) then require that the chiral-scalar and chiral four-vector parts of the mass matrix belong respectively to 1- and 20-dimensional representations of $SU(4)$. The algebra can be further enlarged to allow $s$-wave as well as $p$-wave pion transitions; the simplest possibility is then that $T_a$, $J_i$, and the $l=0$ and $l=1$ parts of $X_a(\lambda)$ generate the algebra of $SO(7)$.

This long introduction would not be complete without a comparison of our work with the saturation of commutation relations at infinite momentum. Certainly the commutation relation (1.9) was derived[17] some years ago by the $p=\infty$ method, though I am not sure whether the simple connection between $X_a(\lambda)$ and the center-of-mass pion transition amplitudes is generally realized. (It is this connection that leads to our speculations on helicity dependence and supermultiplet structure in Sec. VIII.) To my knowledge, the commutation property of $X_a(\lambda)$ with the mass matrix expressed in Eqs. (1.10) or (1.11)–(1.14), has not been derived for zero pion mass by the $p=\infty$ method, nor do I see how it could have been. Also, I see no way that the $p=\infty$

[14] It is assumed here that there are no Regge trajectories with $T=2$ and $\alpha(0)\geq 0$; see R. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Rev. Letters **21**, 576 (1968); F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968). The mass formulas derived by Gilman and Harari are special cases of the general formula (1.10).

[15] There is experimental evidence that the differential cross sections $d\sigma/dt$ for the inelastic processes $\pi+N\to\pi+N'$ and $\pi+N\to A_1+N$ approach constants as $s\to\infty$ with $t$ fixed and small, but that these constant cross sections are appreciably smaller than the corresponding limits of the cross sections for the elastic process $\pi+N\to\pi+N$. For a summary, see L. Van Hove, paper SMR 5/2, International Symposium on Contemporary Physics at Trieste, 1968 (to be published). Our conjectures with regard to the process $\pi+\pi\to\pi+A_1$ then follow via the factorization of the Pomeron residue. I wish to thank P. G. Freund for an informative discussion of this point.

[16] See particularly F. J. Gilman and H. Harari, Ref. 14.

[17] R. F. Dashen and M. Gell-Mann, Ref. 10.

method can be used to treat multiparticle processes like $\pi+N \to 2\pi+N$. On the other hand, the $p=\infty$ method allows us to make use of detailed properties of the currents (such as space-space commutators, etc.) which do not arise directly from the underlying chiral symmetry of the strong interactions, and which therefore can play no role in the approach used here. Apart from these differences in output, there is also a difference in the theoretical input; the saturation of commutation relations at infinite momentum can only work if the "Z graphs" or "pair states" do not contribute at $p=\infty$.[18] It is believed that this assumption is equivalent to the assumed absence of subtractions in dispersion relations, but I do not know of any *direct* connection between the two assumptions, aside of course from the equivalence of some of their consequences. Lacking such a connection, it seems safter to rely only upon those high-energy assumptions which arise directly from Regge-pole theory and which can be verified experimentally.

Possible extensions of our approach to other physical problems are discussed briefly in Sec. IX.

## II. KINEMATIC PRELIMINARIES

We consider the forward scattering process

$$\pi(q,a)+\alpha(p,\lambda) \to \pi(q',b)+\beta(p',\lambda'). \tag{2.1}$$

Here $\alpha$ and $\beta$ label the type and isospin of the initial and final target particles; $\lambda$ and $\lambda'$ are their helicities; $a$ and $b$ are pion isovector indices running over 1, 2, 3; and $q$, $p$, $q'$, and $p'$ are the respective four-momenta. We will adhere throughout to the approximation of neglecting the pion mass, so

$$q^2=q'^2=0. \tag{2.2}$$

By "forward" we mean only that

$$t\equiv-(p-p')^2=-(q-q')^2=0. \tag{2.3}$$

If in addition $m_\alpha=m_\beta$ then $p^\mu=p'^\mu$ and $q^\mu=q'^\mu$. The Feynman amplitude $M$ for this process is defined by the formula

$$\langle p'\lambda'\beta, q'b|S-1|p\lambda\alpha, qa\rangle=-i(2\pi)^4\delta^4(p'+q'-p-q) \times(2\pi)^{-6}(16p^{0\prime}p^0q^{0\prime}q^0)^{-1/2}M_{\lambda'\beta b,\lambda\alpha a}(p'q',pq).$$

In order to simplify our calculations, we will adopt a coordinate system in which all momenta are collinear:

$$q^\mu=n^\mu\omega\,,\qquad q'^\mu=n^\mu\omega'\,, \tag{2.5}$$

$$|\mathbf{n}|\equiv n^0\equiv 1\,, \tag{2.6}$$

$$\mathbf{p}=-\mathbf{n}|\mathbf{p}|\,,\quad p^0=(|\mathbf{p}|^2+m_\alpha^2)^{1/2}, \tag{2.7}$$

$$\mathbf{p}'=-\mathbf{n}|\mathbf{p}'|\,,\quad p^{0\prime}=(|\mathbf{p}'|^2+m_\beta^2)^{1/2}. \tag{2.8}$$

[18] S. Fubini and G. Furlan, Ref. 10. For a general discussion, see S. L. Adler and R. F. Dashen, *Current Algebra* (W. A. Benjamin, Inc., New York, 1968), Chaps. 4 and 5. The method of "lightlike" changes of J. Jersak and J. Stern (to be published) avoids this problem but requires that we know commutators on the light cone.

Energy and momentum conservation then yield the relations

$$|\mathbf{p}|+p^0=|\mathbf{p}'|+p^{0\prime}\equiv\mathcal{E}\,, \tag{2.9}$$

$$\omega'=\omega+(m_\alpha^2-m_\beta^2)/2\mathcal{E}\,, \tag{2.10}$$

$$s\equiv-(p+q)^2=m_\alpha^2+2\mathcal{E}\omega\,, \tag{2.11}$$

$$u\equiv-(p'-q)^2=m_\beta^2-2\mathcal{E}\omega\,, \tag{2.12}$$

and angular-momentum conservation tells us that the helicity is conserved. We will be concerned with the behavior of $M$ as a function of $\omega$ and $\lambda$, with the target four-momenta $p^\mu$, $p'^\mu$ held fixed, and so we shall write

$$M_{\lambda'\beta b,\lambda\alpha a}(p',q';p,q)=\delta_{\lambda'\lambda}M_{\beta b,\alpha a}(\omega,\lambda)\,, \tag{2.13}$$

the dependence of $M$ on $p$, $p'$, and $\mathbf{n}$ being suppressed throughout.

The crossing symmetry between $s$ and $u$ channels imposes on $M$ the restrictions

$$M_{\beta b,\alpha a}(\omega,\lambda)=M_{\beta a,\alpha b}(-\omega',\lambda). \tag{2.14}$$

It will be convenient to divide $M$ into parts symmetric and antisymmetric in the pion isovector indices $a$, $b$:

$$M_{\beta b,\alpha a}^{(+)}(\omega,\lambda)\equiv\tfrac{1}{2}[M_{\beta b,\alpha a}(\omega,\lambda)+M_{\beta a,\alpha b}(\omega,\lambda)]\,, \tag{2.15}$$

$$M_{\beta b,\alpha a}^{(-)}(\omega,\lambda)\equiv(\omega+\omega')^{-1}[M_{\beta b,\alpha a}(\omega,\lambda)-M_{\beta a,\alpha b}(\omega,\lambda)]\,, \tag{2.16}$$

so that

$$M_{\beta b,\alpha a}(\omega,\lambda)=M_{\beta b,\alpha a}^{(+)}(\omega,\lambda)+\tfrac{1}{2}(\omega+\omega')\times M_{\beta b,\alpha a}^{(-)}(\omega,\lambda). \tag{2.17}$$

Equation (2.14) lets us write (2.15) and (2.16) as

$$M_{\beta b,\alpha a}^{(+)}(\omega,\lambda)=\tfrac{1}{2}[M_{\beta b,\alpha a}(\omega,\lambda)+M_{\beta b,\alpha a}(-\omega',\lambda)]\,, \tag{2.18}$$

$$M_{\beta b,\alpha a}^{(-)}(\omega,\lambda)=(\omega+\omega')^{-1}[M_{\beta b,\alpha a}(\omega,\lambda)-M_{\beta b,\alpha a}(-\omega',\lambda)]. \tag{2.19}$$

Both $M^{(+)}$ and $M^{(-)}$ are then even under interchange of $\omega$ with $-\omega'$:

$$M_{\beta b,\alpha a}^{(\pm)}(\omega,\lambda)=M_{\beta b,\alpha a}^{(\pm)}(-\omega',\lambda). \tag{2.20}$$

## III. ASYMPTOTIC BEHAVIOR OF TREE GRAPHS

We shall now calculate the asymptotic behavior of the forward-scattering amplitudes $M_{\beta b,\alpha a}^{(\pm)}(\omega,\lambda)$ as $\omega\to\infty$. We use for this purpose a general chiral-invariant Lagrangian, and retain only tree graphs, involving a finite (though perhaps large) set of virtual particles. Of course, we do not in this way expect to reproduce the true asymptotic behavior of the forward-scattering amplitude. Rather, as discussed in Sec. I, we will be interested precisely in those contributions of the tree graphs to $M^{(\pm)}$ which behave *worse* at high energy than would be expected from Regge-pole theory; all such contributions must cancel, and it is from this cancellation that we expect to derive the algebraic consequences of chiral symmetry.

The tree graphs that contribute to the Feynman amplitude are of two different kinds. First, there are diagrams arising from a direct interaction of both pions with the target, or with a virtual meson exchanged in the $t$ channel. (See Fig. 1.) Second, there are diagrams with a virtual particle $\gamma$ exchanged in the $s$ or $u$ channels. (See Fig. 2.) The contribution to the Feynman amplitude of all graphs of both kinds is then

$$M_{\beta b,\alpha a}(\omega,\lambda)=P_{\beta b,\alpha a}(\omega,\lambda)+\sum_\gamma (m_\alpha^2-m_\gamma^2+2\omega\mathcal{E})^{-1}Q_{\beta b,\alpha a}{}^{(\gamma)}(\omega,\lambda)+\sum_\gamma (m_\alpha^2-m_\gamma^2-2\omega'\mathcal{E})^{-1}Q_{\beta a,\alpha b}{}^{(\gamma)}(-\omega',\lambda). \quad (3.1)$$

The first term arises from the graphs of Fig. 1, and is just a polynomial in $\omega$. The second and third terms arise from the graphs of Fig. 2; the denominators are respectively just $s-m_\gamma^2$ and $u-m_\gamma^2$, and the numerator $Q$ is a polynomial arising from the numerators of the $\gamma$ propagator and from any derivatives in the pion-target interaction. [Note that (3.1) satisfies the crossing-symmetry requirement (2.14) if $P$ does.]

What does chiral symmetry tell us about the polynomials $P$ and $Q$? First, the direct pion-pair interactions of Fig. 1 must involve one or more covariant derivatives acting on *each* pion field, except that we must also include in the Lagrangian the interaction

$$2F_\pi^{-2}\mathbf{V}^\mu\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}), \quad (3.2)$$

where $\mathbf{V}^\mu$ is the conserved phenomenological vector current, normalized so that $\int\mathbf{V}^0d^3x$ is twice the isospin $\mathbf{T}$. [See Eq. (1.3).] Hence the contribution to $M$ of the graphs of Fig. 1 is proportional to

$$q^\mu q'^\nu=\omega\omega' n^\mu n^\nu \quad (3.3)$$

except that (3.2) gives a term proportional to $q^\mu+q'^\mu=(\omega+\omega')n^\mu$. The contribution of the Fig. 1 graphs is thus of the form

$$P_{\beta b,\alpha a}(\omega,\lambda)=\omega\omega'\tilde{P}_{\beta b,\alpha a}(\omega,\lambda)+4iF_\pi^{-2}\mathcal{E}(\omega+\omega')\epsilon_{abc}(T_c)_{\beta\alpha}, \quad (3.4)$$

where $\tilde{P}$ is an unknown polynomial, possibly just a constant. Second, the interactions of single pions with the target in the graphs of Fig. 2 must be of the form[19]

$$-F_\pi^{-1}\mathbf{A}^\mu D_\mu\boldsymbol{\pi}, \quad (3.5)$$

where $\mathbf{A}^\mu$ is a phenomenological axial-vector current [a sum of terms like $(g_A/g_V)\bar{N}\boldsymbol{\tau}\gamma^\mu\gamma_5 N$]. The polynomial $Q$ in (3.1) must then also contain the factor (3.3), i.e.,

$$Q_{\beta b,\alpha a}{}^{(\gamma)}(\omega,\lambda)=\omega\omega'\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma)}(\omega,\lambda), \quad (3.6)$$

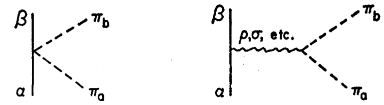


FIG. 1. Tree graphs which contribute polynomials to the Feynman amplitude for the forward scattering $\pi+\alpha\to\pi+\beta$.

where $\tilde{Q}$ is again an unknown polynomial in $\omega$. The forward-scattering amplitude is thus of the form

$$M_{\beta b,\alpha a}(\omega,\lambda)=\omega\omega'\tilde{P}_{\beta b,\alpha a}(\omega,\lambda)+4iF_\pi^{-2}\mathcal{E}(\omega+\omega')\epsilon_{abc}(T_c)_{\beta\alpha}+\omega\omega'\sum_\gamma[(m_\alpha^2-m_\gamma^2+2\omega\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma)}(\omega,\lambda)+(m_\alpha^2-m_\gamma^2-2\omega'\mathcal{E})^{-1}\tilde{Q}_{\beta a,\alpha b}{}^{(\gamma)}(-\omega',\lambda)]. \quad (3.7)$$

This result for the forward-scattering amplitude displays all the information we can hope to derive from the chiral symmetry of the phenomenological Lagrangian. Evidently chiral symmetry does not put very strong constraints on the asymptotic behavior of $M$ as $\omega\to\infty$. The polynomials $\tilde{P}$ and $\tilde{Q}$ can (and for high spins generally will) contain high powers of $\omega$ with unknown coefficients, so nothing very much is learned by demanding that these rapidly growing terms cancel. However, Eq. (3.7) does put strong constraints on the *lowest* powers of $\omega$ in an asymptotic expansion of $M$, so it is to these lowest terms that we must look for our algebraic results.[20]

To make this more specific, it will be convenient to decompose $M$ into parts symmetric or antisymmetric in the pion isovector indices $a$, $b$. Using (3.7), (2.18), and (2.19), these are

$$M_{\beta b,\alpha a}{}^{(+)}(\omega,\lambda)=\tfrac{1}{2}\omega\omega'[\tilde{P}_{\beta b,\alpha a}(\omega,\lambda)+\tilde{P}_{\beta b,\alpha a}(-\omega',\lambda)+\sum_\gamma(m_\alpha^2-m_\gamma^2+2\omega\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,+)}(\omega,\lambda)+\sum_\gamma(m_\alpha^2-m_\gamma^2-2\omega'\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,+)}(-\omega',\lambda)], \quad (3.8)$$

$$M_{\beta b,\alpha a}{}^{(-)}(\omega,\lambda)=8iF_\pi^{-2}\mathcal{E}\epsilon_{abc}(T_c)_{\beta\alpha}+[\omega\omega'/(\omega+\omega')]\times[\tilde{P}_{\beta b,\alpha a}(\omega,\lambda)-\tilde{P}_{\beta b,\alpha a}(-\omega',\lambda)+\sum_\gamma(m_\alpha^2-m_\gamma^2+2\omega\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,-)}(\omega,\lambda)-\sum_\gamma(m_\alpha^2-m_\gamma^2-2\omega'\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,-)}(-\omega',\lambda)], \quad (3.9)$$

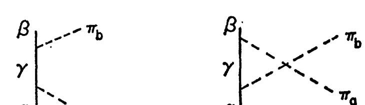


FIG. 2. Tree graphs which contribute pole terms to the Feynman amplitude for the forward scattering $\pi+\alpha\to\pi+\beta$.

[19] The exact axial-vector current $\mathbf{A}^\mu$ is equal to $-F_\pi\partial\mathcal{L}/\partial(\partial_\mu\boldsymbol{\pi})$ plus terms of first or higher order in the pion field. This connection between single-pion couplings and the axial-vector current is of course just the Goldberger-Treiman relation written in a Lagrangian language.

[20] This conclusion has been reached independently by J. W. Wess and B. Zumino (private communication). They have used this method to derive algebraic relations which are special cases of our general results.

where

$$\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,\pm)}(\omega,\lambda)=\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma)}(\omega,\lambda)\pm Q_{\beta a,\alpha b}{}^{(\gamma)}(\omega,\lambda). \quad (3.10)$$

By expanding the denominators in (3.8) and (3.9) in inverse powers of $\omega$, we may split $M^{(\pm)}$ into a polynomial $A^{(\pm)}$ plus a remainder $R^{(\pm)}$ which vanishes as $\omega\to\infty$:

$$M_{\beta b,\alpha a}{}^{(\pm)}(\omega,\lambda)=A_{\beta b,\alpha a}{}^{(\pm)}(\omega,\lambda)+R_{\beta b,\alpha a}{}^{(\pm)}(\omega,\lambda), \quad (3.11)$$

$$R_{\beta b,\alpha a}{}^{(\pm)}(\omega,\lambda)=O(\omega^{-1}) \quad \text{for } \omega\to\infty. \quad (3.12)$$

The term in $A^{(\pm)}$ of *zeroth* order in $\omega$ may then be picked out by a contour integral

$$A_{\beta b,\alpha a}{}^{(\pm)}(0,\lambda)=\frac{1}{2\pi i}\oint M_{\beta b,\alpha a}{}^{(\pm)}(\omega,\lambda)\omega^{-1}d\omega, \quad (3.13)$$

the contour of integration being taken about a very large circle in the complex $\omega$ plane. For $A^{(+)}$ the pole at $\omega=0$ is killed by the factor $\omega$ in (3.8), and the integral is given by the residues of the poles at $s=m_\gamma{}^2$ and $u=m_\gamma{}^2$:

$$A_{\beta b,\alpha a}{}^{(+)}(0,\lambda)=\frac{1}{4\mathcal{E}^2}\sum_\gamma(2m_\gamma{}^2-m_\alpha{}^2-m_\beta{}^2)\times\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,+)}\left(\frac{m_\gamma{}^2-m_\alpha{}^2}{2\mathcal{E}},\lambda\right). \quad (3.14)$$

For $A^{(-)}$ the pole at $\omega=0$ survives only in the first term of (3.8). The pole from the factor $(\omega+\omega')^{-1}$ is killed by the antisymmetry of its coefficient under the interchange of $\omega$ with $-\omega'$, so the second term contributes only its residue at the pole $s=m_\gamma{}^2$ and $u=m_\gamma{}^2$, and we find

$$A_{\beta b,\alpha a}{}^{(-)}(0,\lambda)=8iF_\pi{}^{-2}\mathcal{E}\epsilon_{abc}(T_c)_{\beta\alpha}+\frac{1}{2\mathcal{E}}\sum_\gamma\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma,-)}\left(\frac{m_\gamma{}^2-m_\alpha{}^2}{2\mathcal{E}},\lambda\right). \quad (3.15)$$

Terms in $A^{(\pm)}$ of first and higher order in $\omega$ are of less interest, both because the unknown polynomial $\tilde{P}$ does contribute to these terms, and because they depend on values of the polynomial $\tilde{Q}$ off the $\gamma$-mass shell.

By inspection of Fig. 2 and the interaction (3.5), we find that the value of the polynomial $\tilde{Q}$ on the $\gamma$-mass shell is

$$\tilde{Q}_{\beta b,\alpha a}{}^{(\gamma)}\left(\frac{m_\gamma{}^2-m_\alpha{}^2}{2\mathcal{E}},\lambda\right)=16F_\pi{}^{-2}\mathcal{E}^2[X_b(\lambda)]_{\beta\gamma}\times[X_a(\lambda)]_{\gamma\alpha}, \quad (3.16)$$

where $\mathbf{X}(\lambda)$ is the Hermitian axial-vector coupling matrix for helicity $\lambda$, defined (see Sec. I) by the statement that for $\mathbf{p}'$ and $\mathbf{p}$ antiparallel to $\mathbf{n}$:

$$n_\mu\langle\mathbf{p}',\lambda',\beta|\mathbf{A}^\mu|\mathbf{p},\lambda,\alpha\rangle\equiv\frac{4\mathcal{E}[X(\lambda)]_{\beta\alpha}\delta_{\lambda'\lambda}}{(2\pi)^3(4p^0p'^0)^{1/2}}. \quad (3.17)$$

[Recall that a factor $n^\mu n^\nu$ is contributed by (3.3).] By studying the behavior of this matrix element with respect to boosts along the direction of $\mathbf{n}$, we can easily show that $\mathbf{X}(\lambda)$ is independent of $\mathcal{E}$; the conservation of helicity follows of course from invariance with respect to rotations about the $\mathbf{n}$ axis. Using (3.16) and (3.10) in (3.14) and (3.15), we see finally that the terms in $M^{(\pm)}$ that go like $\omega^0$ as $\omega\to\infty$ are

$$A_{\beta b,\alpha a}{}^{(+)}(0,\lambda)=4F_\pi{}^{-2}\{[X_b(\lambda),[m^2,X_a(\lambda)]]+[X_a(\lambda),[m^2,X_b(\lambda)]]\}_{\beta\alpha}, \quad (3.18)$$

$$A_{\beta b,\alpha a}{}^{(-)}(0,\lambda)=8F_\pi{}^{-2}\mathcal{E}\{i\epsilon_{abc}T_c-[X_a(\lambda),X_b(\lambda)]\}_{\beta\alpha}, \quad (3.19)$$

where $m^2$ is the mass matrix

$$m_{\beta\alpha}{}^2\equiv m_\alpha{}^2\delta_{\beta\alpha}. \quad (3.20)$$

## IV. CHIRAL RESULTS FROM TREE GRAPHS

We now demand that the asymptotic behavior of $M$ that we have calculated for the tree graphs of Figs. 1 and 2 not be worse than would be expected from Regge-pole theory.

The amplitude $M^{(-)}$ has pure isospin $T=1$ exchanged in the $t$ channel, and is normalized so that

$$M_{\beta b,\alpha a}{}^{(-)}(\omega,\lambda)\propto\omega^{\alpha_1(0)-1} \text{ for } \omega\to\infty, \quad (4.1)$$

where $\alpha_1(0)$ is the value of the dominant $T=1$ trajectory at $t=0$. [The extra factor $\omega^{-1}$ arises because we divided by $\omega+\omega'$ in defining $M^{(-)}$, see Eq. (2.16).] Presumably $\alpha_1(0)=\alpha_\rho(0)\approx0.5$, so (4.1) shows that $M^{(-)}$ vanishes as $\omega\to\infty$. Hence we shall demand that the term (3.19) in $M^{(-)}$ which behaves like $\omega^0$ as $\omega\to\infty$ must be zero, i.e.,

$$[X_a(\lambda),X_b(\lambda)]=i\epsilon_{abc}T_c. \quad (4.2)$$

(Of course we must also demand that terms in $M^{(-)}$ which behave like higher powers of $\omega$ must also vanish, but chiral symmetry does not relate such terms to interesting mass-shell matrix elements.) The application of isospin conservation to the defining formula (3.17) tells us further that

$$[T_a,X_b(\lambda)]=i\epsilon_{abc}X_c(\lambda) \quad (4.3)$$

and of course $T_a$ obeys the usual commutation relations for the isospin matrix

$$[T_a,T_b]=i\epsilon_{abc}T_c. \quad (4.4)$$

Taken together, (4.2)–(4.4) tell us that *the particles we include in our tree graphs must for each helicity furnish an irreducible or reducible representation of $SU(2)\times SU(2)$.*

The amplitude $M^{(+)}$ has both isospins $T=0$ and $T=2$ in the $t$ channel; the part with $T=2$ may be isolated as

$$M_{\beta b,\alpha a}{}^{(T=2)}(\omega,\lambda)\equiv M_{\beta b,\alpha a}{}^{(+)}(\omega,\lambda)-\tfrac{1}{3}\delta_{ba}M_{\beta c,\alpha c}{}^{(+)}(\omega,\lambda). \quad (4.5)$$

It is normalized so that

$$M_{\beta b,\alpha a}{}^{(T=2)}(\omega,\lambda)\propto\omega^{\alpha_2(0)} \text{ for } \omega\to\infty\,, \quad (4.6)$$

where $\alpha_2(0)$ is the value of the dominant $T=2$ trajectory at $t=0$. There are reasons to believe[14] that $\alpha_2(0)<0$; if this is true then we must require that the term (3.18) in $M^{(+)}$ which behaves like $\omega^0$ as $\omega\to\infty$ must not have any $T=2$ exchange part, i.e.,

$$A_{\beta b,\alpha a}{}^{(+)}(0,\lambda)\propto\delta_{ba}\,. \quad (4.7)$$

If (4.2) is valid then we can use Jacobi's identity, and the commutativity of $m^2$ with $T_c$, to write (3.18) as

$$A_{\beta b,\alpha a}{}^{(+)}(0,\lambda)=8F_\pi{}^{-2}[X_b(\lambda),[m^2,X_a(\lambda)]]_{\beta\alpha} \quad (4.8)$$

and (4.7) may then be written as

$$[X_b(\lambda),[m^2,X_a(\lambda)]]=-m_4{}^2(\lambda)\delta_{ab}\,. \quad (4.9)$$

In order to put this in group-theoretical terms, we may define an isovector matrix $\mathbf{m}^2(\lambda)$ by

$$[X_a(\lambda),m^2]\equiv im_a{}^2(\lambda) \quad (4.10)$$

and write (4.9) as

$$[X_b(\lambda),m_a{}^2(\lambda)]=-im_4{}^2(\lambda)\delta_{ab}\,. \quad (4.11)$$

Using (4.11) and Jacobi's identity, we have then

$$\begin{aligned}[]
[X_a(\lambda),m_4{}^2(\lambda)]\delta_{bc}-[X_b(\lambda),m_4{}^2(\lambda)]\delta_{ac}
&=-\epsilon_{abd}[T_d,m_c{}^2(\lambda)]=-i\epsilon_{abd}\epsilon_{dce}m_e{}^2(\lambda)\\
&=-i\delta_{ac}m_b{}^2(\lambda)+i\delta_{bc}m_a{}^2(\lambda)
\end{aligned}$$

and therefore

$$[X_a(\lambda),m_4{}^2(\lambda)]=im_a{}^2(\lambda)\,. \quad (4.12)$$

Equations (4.11) and (4.12) tell us that $\mathbf{m}^2(\lambda)$ and $m_4{}^2(\lambda)$ together form a chiral four-vector, with regard to their commutation relations with $\mathbf{X}(\lambda)$. Finally, note from (4.10) and (4.12) that

$$m^2=m_0{}^2(\lambda)+m_4{}^2(\lambda)\,, \quad (4.13)$$

where $m_0{}^2(\lambda)$ is chiral-invariant, i.e.,

$$[X_a(\lambda),m_0{}^2(\lambda)]=0\,. \quad (4.14)$$

Hence *the mass matrix $m^2$ behaves as the sum of a chiral scalar $m_0{}^2(\lambda)$ and the fourth-component $m_4{}^2(\lambda)$ of a chiral four-vector*. We emphasize that (4.13) is *not* an approximation based on some assumption of weak chiral-symmetry breaking; rather, it is an *exact* consequence of our assumptions about the asymptotic behavior of the tree graphs, and we do not in fact expect that $m_4{}^2$ is smaller than $m_0{}^2$.

We may also entertain conjectures about the structure of the remaining $T=0$ exchange term which goes as $\omega^0$ for $\omega\to\infty$, given by (4.8) and (4.9) as

$$A_{\beta b,\alpha a}{}^{(+)}(0,\lambda)=-8F_\pi{}^{-2}[m_4{}^2(\lambda)]_{\beta\alpha}\delta_{ab}\,. \quad (4.15)$$

If for example we believe that differences of the forward-scattering amplitude for different helicities vanish as $\omega\to\infty$, then $m_4{}^2(\lambda)$ and hence $m_0{}^2(\lambda)$ will be helicity-independent. If we believe[15] that $M^{(+)}$ vanishes as $\omega\to\infty$ for all inelastic processes, such as $\pi+\pi\to\pi+A_1$, then (4.15) must commute with the mass matrix $m^2$, and therefore

$$0=[m_4{}^2(\lambda),m^2]=[m_4{}^2(\lambda),m_0{}^2(\lambda)]\,. \quad (4.16)$$

We shall see that this does seem to agree with observed masses.

The value of our results lies in the algebraic constraints they impose on *observable* parameters, the axial-vector coupling matrix $\mathbf{X}(\lambda)$ [which as shown in Sec. V determines all one-pion transition rates] and the mass matrix $m^2$. These constraints may be obtained either of two ways:

(1) We may write each physical particle state of definite helicity as a sum of irreducible representations of $SU(2)\otimes SU(2)$ [for instance, $N(\lambda=\frac{1}{2})=(\frac{1}{2},1)\oplus(0,\frac{1}{2})\oplus\cdots$]. The matrix elements of $X(\lambda)$ are then entirely determined by the mixing angles which define the coefficients of the representations in this sum, while the physical masses are determined by these mixing angles and by mass parameters which specify the elements of $m_0{}^2$ and $m_4{}^2$ between the irreducible representations in the sum. This approach will be applied in Sec. VI to the $\lambda=0$ states of $\pi$, $\rho$, $\sigma$, and $A_1$.

(2) Alternatively, we may apply our results directly to the physical particle states. The Wigner-Eckart theorem allows us to define $T_z$-independent reduced matrix elements $\langle\beta\|X(\lambda)\|\alpha\rangle$ by the formula

$$\langle\beta t''|X^{(t)}(\lambda)|\alpha t'\rangle=C_{T_\alpha 1}(T_\beta t'';t't)\langle\beta\|X(\lambda)\|\alpha\rangle\,, \quad (4.17)$$

where $t''$ and $t'$ are the $T_z$ values of $\beta$ and $\alpha$, and

$$X^{(\pm1)}\equiv\mp(\sqrt{2})^{-1}(X_1\pm iX_2)\,,\quad X^{(0)}\equiv X_3\,. \quad (4.18)$$

The Hermiticity condition on $X_a(\lambda)$ now reads

$$\langle\beta\|X(\lambda)\|\alpha\rangle^*=(-)^{T_\beta-T_\alpha}\left(\frac{2T_\alpha+1}{2T_\beta+1}\right)^{1/2}\langle\alpha\|X(\lambda)\|\beta\rangle\,, \quad (4.19)$$

while (4.2) becomes, for $T_\beta=T_\alpha\neq0$,

$$\begin{aligned}
\delta_{\beta\alpha}=\sum_\gamma\langle\beta\|X(\lambda)\|\gamma\rangle\langle\gamma\|X(\lambda)\|\alpha\rangle
&\times\left[-\frac{1}{T_\alpha}\left(\frac{2T_\alpha-1}{2T_\alpha+1}\right)^{1/2}\delta_{T_\gamma,T_\alpha-1}+\frac{1}{T_\alpha(T_\alpha+1)}\delta_{T_\gamma,T_\alpha}\right.\\
&\left.+\frac{1}{T_\alpha+1}\left(\frac{2T_\alpha+3}{2T_\alpha+1}\right)^{1/2}\delta_{T_\gamma,T_\alpha-1}\right] \quad (4.20)
\end{aligned}$$

and, for $T_\beta=T_\alpha+1$,

$$0=\sum_\gamma\langle\beta\|X(\lambda)\|\gamma\rangle\langle\gamma\|X(\lambda)\|\alpha\rangle\times[(T_\alpha)^{1/2}\delta_{T_\gamma,T_\alpha}-(T_\alpha+2)^{1/2}\delta_{T_\gamma,T_\beta}]\,. \quad (4.21)$$

The condition that $[X_a(\lambda),[X_b(\lambda),m^2]]$ not have a $T=2$ part may be met by requiring that

$$[X^{(+)}(\lambda),[X^{(+)}(\lambda),m^2]]=0. \qquad (4.22)$$

This approach will be applied in Sec. VI to the $\lambda=\frac{1}{2}$ states of the nonstrange baryons. Needless to say, both approaches are entirely equivalent.

## V. PROPERTIES OF X(λ): PION DECAYS, SELECTION RULES

We have shown that elementary particle states and their masses must furnish representation of the chiral algebra formed by the isospin **T** and the axial-vector coupling matrix $\mathbf{X}(\lambda)$. Both in order to determine empirical values of the matrix $\mathbf{X}(\lambda)$ and to appreciate its general properties, it should be kept in mind that $[X_a(\lambda)]_{\beta\alpha}$ has a direct physical significance as the matrix element for the pion transition process

$$\alpha(p,\lambda)\rightarrow\beta(p',\lambda)+\pi(q,a) \qquad (5.1)$$

in any reference frame in which the decay is collinear, including not only the $\mathbf{p}=\infty$ frame *but also the center-of-mass* $\mathbf{p}=0$ *frame*. The Feynman amplitude $M$ for this process may in general be defined by

$$\langle p'\lambda'\beta,qa|S|p\lambda\alpha\rangle=(2\pi)^4\delta^4(p'+q-p)(2\pi)^{-9/2}\times(8q^0p^0p'^0)^{-1/2}M_a(p'\lambda'\beta',p\lambda\alpha). \qquad (5.2)$$

We recall that the one-pion interaction is given[19] by (3.5) as $-F_\pi^{-1}\mathbf{A}^\mu\partial_\mu\boldsymbol{\pi}$, so

$$(2\pi)^{-3}(4p^0p'^0)^{-1/2}M_a(p'\lambda'\beta,p\lambda\alpha)=F_\pi^{-1}(p-p')_\mu\times\langle p'\lambda'\beta|A_a^\mu(0)|p\lambda\alpha\rangle. \qquad (5.3)$$

In a general collinear frame we have

$$\mathbf{p}=-\mathbf{n}p\,,\quad p^0=(p^2+m_\alpha^2)^{1/2},$$
$$\mathbf{p}'=-\mathbf{n}p'\,,\quad p'^0=(p'^2+m_\beta^2)^{1/2},$$
$$p+(p^2+m_\alpha^2)^{1/2}=p'+(p'^2+m_\beta^2)^{1/2}\equiv\mathcal{E}\,,$$
$$\mathbf{n}^2=1\,,$$

and so

$$(p-p')^\mu=\left(\frac{m_\alpha^2-m_\beta^2}{2\mathcal{E}}\right)n^\mu\quad(n^0\equiv1). \qquad (5.4)$$

Thus using in (5.3) our definition (3.17) of $\mathbf{X}(\lambda)$, we find the Feynman amplitude for the process (5.1) in *any* collinear frame as

$$M_a(p'\lambda'\beta,p\lambda\alpha)=2F_\pi^{-1}(m_\alpha^2-m_\beta^2)[X_a(\lambda)]_{\beta\alpha}\delta_{\lambda'\lambda}. \qquad (5.5)$$

It is now a simple matter to show that the rate for the process (5.1) [with $\alpha$ at rest and unpolarized] is

$$\Gamma(\alpha\rightarrow\beta+\pi;\lambda)=\frac{(m_\alpha^2-m_\beta^2)^3}{4\pi m_\alpha^3F_\pi^2(2J_\alpha+1)}|[X_a(\lambda)]_{\beta\alpha}|^2. \qquad (5.6)$$

This is of course only true for zero pion mass. In the real world we can presumably use the $X$-matrix elements computed for zero pion mass, but we must take into account the effect of the finite pion mass on phase-space and centrifugal barrier factors. If the pion is emitted with orbital angular momentum $l$ then we should supply in (5.6) a correction factor

$$[p_\pi(\text{true})/p_\pi(m_\pi=0)]^{2l+1}, \qquad (5.7)$$

where $p_\pi$ is the center-of-mass pion momentum:

$$p_\pi=[(m_\alpha^2-m_\beta^2+m_\pi^2)^2-4m_\pi^2m_\alpha^2]^{1/2}/2m_\alpha. \qquad (5.8)$$

Thus the rate for $\alpha\rightarrow\beta+\pi$ when $m_\beta^2\gg m_\pi^2$ is given by

$$\Gamma(\alpha\rightarrow\beta+\pi;\lambda)=\frac{(m_\alpha^2-m_\beta^2)^3}{4\pi m_\alpha^3F_\pi^2(2J_\alpha+1)}|[X_a(\lambda)]_{\beta\alpha}|^2\times\left(1-\frac{2(m_\alpha^2+m_\beta^2)m_\pi^2}{(m_\alpha^2-m_\beta^2)^2}\right)^{l+1/2}. \qquad (5.9)$$

If $\beta$ is a pion then we should replace $(m_\alpha^2-m_\beta^2)^3$ in (5.6) with just $m_\alpha^6$, and the rate becomes

$$\Gamma(\alpha\rightarrow\pi+\pi)=\frac{m_\alpha^3}{4\pi F_\pi^2(2J_\alpha+1)}|[X_a(0)]_{\pi\alpha}|^2\times\left(1-\frac{4m_\pi^2}{m_\alpha^2}\right)^{l+1/2}. \qquad (5.10)$$

The identification of $X(\lambda)$ as the pion transition amplitude provides an easy path to the derivation of exact and approximate selection rules. If we apply to the process $\alpha\rightarrow\beta+\pi$ a space reflection times a rotation of 180° about some axis perpendicular to $\mathbf{n}$ we get back the same process, but with spins reversed, and with a sign factor coming from the rotation

$$(-)^{J_\alpha-\lambda}(-)^{\lambda-J_\beta}$$

and another sign factor from the intrinsic parities of $\pi$, $\alpha$, and $\beta$:

$$-\Pi_\alpha\Pi_\beta.$$

It follows then that $X(\lambda)$ is related to $X(-\lambda)$ by

$$[X_a(-\lambda)]_{\beta\alpha}=-\Pi_\alpha\Pi_\beta(-)^{J_\alpha-J_\beta}[X_a(\lambda)]_{\beta\alpha}. \qquad (5.11)$$

For $\lambda\neq0$ the significance of (5.11) is that it ensures that the fundamental commutation relation (4.2) will be satisfied by $X(-\lambda)$ if it is satisfied by $X(\lambda)$. For $\lambda=0$ Eq. (5.11) tells us that $[X_a(0)]_{\beta\alpha}$ must vanish unless $\alpha$ and $\beta$ satisfy the selection rule

$$\Pi_\alpha(-)^{J_\alpha}=-\Pi_\beta(-)^{J_\beta}. \qquad (5.12)$$

Thus in building each vector in an irreducible representation of $SU(2)\times SU(2)$, we never mix $\lambda=0$ particle states with different values of $\Pi(-)^J$.

Further information about the helicity dependence of $X(\lambda)$ may be obtained by performing a partial-wave decomposition of the Feynman amplitude for the process

$\alpha\rightarrow\beta+\pi$ with $\alpha$ at rest:

$$[X_a(\lambda)]_{\beta\alpha}=\sum_l C_{lJ_\beta}(J_\alpha\lambda;0\lambda)(X_{al})_{\beta\alpha}. \tag{5.13}$$

The parity rule (5.11) is then satisfied if $(X_{al})_{\beta\alpha}$ obeys the familiar selection rule

$$\Pi_\beta\Pi_\alpha=-(-)^l. \tag{5.14}$$

In itself, (5.13) tells us nothing, but if $\alpha$ and $\beta$ are nearly degenerate then the contribution to $M$ of the $l$th partial wave will be of the order $(m_\alpha^2-m_\beta^2)^l$, and hence (5.5) gives

$$(X_{al})_{\beta\alpha}\propto(m_\alpha^2-m_\beta^2)^{l-1}. \tag{5.15}$$

This allows us to discard high partial waves, and (5.13) then does provide useful information about the helicity dependence of matrix elements of $\mathbf{X}(\lambda)$ between close-lying states. In particular, if $\alpha$ and $\beta$ are states of equal mass and equal parity, then only the $p$-wave term can contribute in (5.13), and the helicity dependence of $\mathbf{X}$ is given exactly by

$$\begin{aligned}[\mathbf{X}_a(\lambda)]_{\beta\alpha}&\propto C_{1J_\beta}(J_\alpha\lambda;0\lambda)\\ &\propto\lambda, &&\text{for}\quad J_\alpha=J_\beta\\ &\propto(J_\alpha^2-\lambda^2)^{1/2}, &&\text{for}\quad J_\alpha=J_\beta+1\\ &\propto(J_\beta^2-\lambda^2)^{1/2}, &&\text{for}\quad J_\alpha=J_\beta-1\\ &\propto 0, &&\text{for}\quad |J_\alpha-J_\beta|>1.\end{aligned} \tag{5.16}$$

## VI. EXAMPLE: THE PION AND ITS CHIRAL PARTNERS

To see how we can use the algebraic results derived in Sec. IV, let us apply them to the pion and its chiral partners. According to the selection rules discussed in Sec. V, the pion will in general belong to a reducible representation of the chiral algebra containing the $\lambda=0$ states of all nonstrange mesons with $G\Pi(-)^J=+1$, i.e., the $\rho$, $\sigma$, $A_1$, $f^0$, etc. None of these have isospin $T=2$, so this big reducible representation must consist of those irreducible representations of $SU(2)\times SU(2)$, or equivalently of $SO(4)$, containing $T=0$ and $T=1$ only. There are just four of these, the scalar $s$, the four-vector $v_L$, and the self-chiral and anti-self-chiral parts of the antysymmetric tensor $t_{LM}=-t_{ML}$. We want the meson masses to contain a chiral-symmetry-breaking term $m_4^2$, but the representation $v$ to which it belongs occurs only in $s\otimes v$ and $v\otimes t$, not $s\otimes s$, $v\otimes v$, $t\otimes t$, or $s\otimes t$, so we had better include at least one $v$ and one $t$ or one $s$. The simplest reducible representation with mass splitting is therefore $s\oplus v$, but this only includes the pion and a pair of $\sigma$ mesons, and is therefore too simple for our purposes. We shall explore the possibility that the pion and its chiral partners can be joined in the second-simplest representation with mass splitting, which is $v\oplus t$.

The matrix elements of $X_a(0)$ in this reducible representation may be obtained by using ordinary $SO(4)$ tensor analysis to compute the effects of infinitesimal chiral "boosts" on $v_L$ and $t_{LM}$. In this way we find the matrix elements

$$\begin{aligned}&(v_b|X_a|v_4)=-(v_4|X_a|v_b)=i\delta_{ab},\\ &(v_b|X_a|v_c)=(v_4|X_a|v_4)=0,\\ &(t_{bc}|X_a|t_{4d})=-(t_{4d}|X_a|t_{bc})=i(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{db}),\\ &(t_{bc}|X_a|t_{de})=(t_{4b}|X_a|t_{4c})=0,\\ &(v_L|X_a|t_{MN})=(t_{MN}|X_a|v_L)=0.\end{aligned} \tag{6.1}$$

(We omit the helicity label $\lambda=0$.) The reader can easily verify the commutation relations (4.2).

As discussed in Sec. V, the matrix $X_a$ anticommutes with the $G$ parity, so the states $v_4$ and $v_a$ must have opposite $G$ parity, and the states $t_{4a}$ and $t_{ab}$ must have opposite $G$ parity. There is no loss of generality in assuming that $t_{4a}$ has the same $G$ parity as $v_a$; for if this is not the case then we can just replace $t_{LM}$ with its dual $\epsilon_{LMNP}t_{NP}$. Hence our representation $v\oplus t$ contains two isotriplets with the same $G$ parity, described by linear combinations of $v_a$ and $t_{4a}$, one isosinglet with opposite $G$ parity, described by $v_4$, and one isotriplet with the same $G$ parity as the isosinglet, described by $t_{ab}$. We therefore make the tentative identifications

$$\begin{aligned}&\pi_a=t_{4a}\cos\psi+v_a\sin\psi,\\ &A_{1a}=-t_{4a}\sin\psi+v_a\cos\psi,\\ &\sigma=v_4,\quad\rho_a=\tfrac{1}{2}\epsilon_{abc}t_{bc}.\end{aligned} \tag{6.2}$$

Using (6.1) and (6.2), we may calculate the pion-decay matrix elements

$$\begin{aligned}&(\pi_b|X_a|\rho_c)=-i\epsilon_{abc}\cos\psi,\\ &(\pi_b|X_a|\sigma)=i\delta_{ab}\sin\psi,\\ &(\rho_b|X_a|A_{1c})=-i\epsilon_{abc}\sin\psi,\\ &(\sigma|X_a|A_{1b})=-i\delta_{ab}\cos\psi,\end{aligned} \tag{6.3}$$

and the decay widths are then given by (5.10) and (5.9) as

$$\Gamma(\rho\rightarrow\pi+\pi)=\frac{m_\rho^3}{12\pi F_\pi^2}\left(1-\frac{4m_\pi^2}{m_\rho^2}\right)^{3/2}\cos^2\psi, \tag{6.4}$$

$$\Gamma(\sigma\rightarrow\pi+\pi)=\frac{3m_\sigma^3}{8\pi F_\pi^2}\left(1-\frac{4m_\pi^2}{m_0^2}\right)^{1/2}\sin^2\psi, \tag{6.5}$$

$$\Gamma(A_1\rightarrow\sigma+\pi)=\frac{(m_{A_1}{}^3-m_\sigma^2)^3}{12\pi F_\pi^2 m_{a_1}{}^3}\times\left(1-\frac{2m_\pi^2(m_{A_1}{}^2+m_\rho^2)}{(m_{A_1}{}^2-m_\rho^2)^2}\right)^{3/2}\cos^2\psi. \tag{6.6}$$

(We cannot calculate the rate for $A_1\rightarrow\rho+\pi$ without knowing $X_a$ for $\lambda=\pm1$ as well as for $\lambda=0$.) Using $F_\pi=190$ MeV and $m_\rho=760$ MeV, we find from (6.4) that $\psi=50°$ if $\Gamma_\rho=100$ MeV and $\psi=45°$ if $\Gamma_\rho=135$ MeV. The rates for $\sigma\rightarrow\pi+\pi$ and $A_1\rightarrow\sigma+\pi$ could thus be calculated if we knew the $\sigma$ mass.

Let us now compute the mass matrix. Its chiral-scalar part has matrix elements of the form

$$\begin{aligned}(v_L|m_0{}^2|v_M)&=\delta_{LM}m_v{}^2\,,\\ (t_{LM}|m_0{}^2|t_{NP})&=(\delta_{LN}\delta_{MP}-\delta_{LP}\delta_{MP})m_t{}^2\,,\\ (t_{LM}|m_0{}^2|v_N)&=(v_N|m_0{}^2|t_{LM})=0\,,\end{aligned}\tag{6.7}$$

while the chiral four-vector $m_L{}^2$ has the matrix elements

$$\begin{aligned}(v_M|m_L{}^2|v_N)&=(t_{MN}|m_L{}^2|t_{PQ})=0\,,\\ (v_M|m_L{}^2|t_{NP})&=(t_{NP}|m_L{}^2|v_M)\\ &=(\delta_{LN}\delta_{MP}-\delta_{LP}\delta_{MN})m'^2\,.\end{aligned}\tag{6.8}$$

Note that a possible term in $(v_L|m_M{}^2|t_{NP})$ proportional to $\epsilon_{LMNP}$ is excluded here by $G$ parity. From Eqs. (6.2), (6.7), and (6.8) we see that the nonzero elements of $m_0{}^2$ and $m_4{}^2$ for physical particle states are

$$\begin{aligned}(\pi_a|m_0{}^2|\pi_b)&=(m_t{}^2\cos^2\psi+m_v{}^2\sin^2\psi)\delta_{ab}\,,\\ (A_{1a}|m_0{}^2|A_{1b})&=(m_t{}^2\sin^2\psi+m_v{}^2\cos^2\psi)\delta_{ab}\,,\\ (A_{1a}|m_0{}^2|\pi_b)&=(m_v{}^2-m_t{}^2)\cos\psi\sin\psi\,\delta_{ab}\,,\\ (\sigma|m_0{}^2|\sigma)&=m_v{}^2\,,\\ (\rho_a|m_0{}^2|\rho_b)&=m_t{}^2\delta_{ab}\,,\\ (\pi_a|m_4{}^2|\pi_b)&=2m'^2\cos\psi\sin\psi\,\delta_{ab}\,,\\ (A_{1a}|m_4{}^2|A_{1b})&=-2m'^2\cos\psi\sin\psi\,\delta_{ab}\,,\\ (A_{1a}|m_4{}^2|\pi_b)&=m'^2(\cos^2\psi-\sin^2\psi)\delta_{ab}.\end{aligned}\tag{6.9}$$

In order that $m_0{}^2+m_4{}^2$ be diagonal, we must have

$$(m_v{}^2-m_t{}^2)\sin2\psi=-2m'^2\cos2\psi\tag{6.10}$$

and in order that the pion mass shall vanish, we also must have

$$(m_t{}^2+m_v{}^2)+(m_t{}^2-m_v{}^2)\cos2\psi=-2m'^2\sin2\psi.\tag{6.11}$$

The physical masses are

$$m_\rho{}^2=m_t{}^2\,,\quad m_\sigma{}^2=m_v{}^2\,,$$
$$m_{A_1}{}^2=\tfrac{1}{2}(m_t{}^2+m_v{}^2)-\tfrac{1}{2}(m_t{}^2-m_v{}^2)\cos2\psi-m'^2\sin2\psi.$$

Equation (6.11) then gives, independently of the value of $\psi$,

$$m_{A_1}{}^2=m_\rho{}^2+m_\sigma{}^2\tag{6.12}$$

and (6.10) gives $m_\rho{}^2/m_\sigma{}^2$ in terms of $\psi$:

$$m_\rho{}^2/m_\sigma{}^2=\tan^2\psi.\tag{6.13}$$

As already remarked, the $\rho$ width gives a value of $\psi$ in the range 45°–50°. If we adopt the value $\psi=45°$ then (6.13) gives

$$m_\sigma=m_\rho\tag{6.14}$$

in apparent agreement with experiment.[21] Using (6.14) in (6.12), we further find

$$m_{A_1}{}^2=2m_\rho{}^2\tag{6.15}$$

in agreement with experiment (if the $A_1$ is real) and with the spectral-function sum rules.[22] Except for some signs, all these results have already been obtained by Gilman and Harari.[14]

However, we still do not know *why* $\psi$ should have the delightful value 45°. This can be understood if we are willing to assume that there are no $T=0$ Regge trajectories with $\alpha(0)\geq0$ that can contribute to the forward inelastic scattering process[15]

$$\pi+\pi\to\pi+A_1.$$

If this is the case then, as discussed in Sec. IV, the matrix element $(\pi_a|m_4{}^2|A_{1b})$ must vanish. But $m'$ cannot vanish if we are to avoid complete degeneracy, and so according to the last line of Eq. (6.9) we must have

$$\tan^2\psi=1.\tag{6.16}$$

With $m_\sigma=m_\rho$ and $\tan^2\psi=1$ the ratio of the $\sigma$ and $\rho$ widths is given by (6.4) and (6.5) as

$$\Gamma_\sigma/\Gamma_\rho=\tfrac{9}{2}(1-4m_\pi{}^2/m_\rho{}^2)^{-1}\tag{6.17}$$

so $\Gamma_\sigma$ should be of order 600 MeV. With the $\sigma$ this broad, our treatment of it as a narrow resonance must be regarded as at best a rough approximation. In particular, it does not really make sense to compute the widths for the processes $A_1\to\rho+\pi$ and $A_1\to\sigma+\pi$; rather, we should take our formulas (6.3) for $(\rho|X|A_1)$ and $(\sigma|X|A_1)$ as inputs to a dynamical calculation of the decay $A_1\to3\pi$ actually observed.

Unfortunately the result given by Eq. (6.3) for the $A_1\to\rho+\pi$ amplitude with $\lambda=0$ does not agree with that given by the "hard-pion" method[23] or by an equivalent gauge-invariant phenomenological Lagrangian.[24] (The same disagreement occurs again for $\lambda=\pm1$.) There are many possible ways to resolve this discrepancy, but at present I do not know which is the right way.

## VII. EXAMPLE: NONSTRANGE BARYONS WITH $\lambda=\frac{1}{2}$

We shall now apply the algebraic results derived in Sec. IV to the nucleon and its chiral partners, using for this purpose the direct approach described at the end of Sec. IV. A glance at the compilation of Rosenfeld

[21] A general analysis of all peripheral pion-production data has been performed by L. J. Gutay, D. D. Carmony, P. L. Csonka, F. J. Loeffler, and F. T. Meire (to be published). (I am grateful to R. Arnowitt for informing me of this work.) They find three solutions for the $T=J=0$ phase shift, one going through 90° at about 700 MeV, one going through 90° at about 900 MeV, and one remaining negative. Of these three solutions only the first, with $m_\sigma\approx700$ MeV, is in agreement with a phase-shift solution found in the analysis of backward $\pi N$ scattering of C. Lovelace, Conference on $\pi$-$N$ Scattering at the University of California at Irvine, 1967 (to be published).

[22] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[23] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967); R. Arnowitt, M. H. Friedman, and P. Nath, *ibid.* **19**, 1085 (1967); S. G. Brown and G. W. West, *ibid.* **19**, 812 (1967); S. L. Glashow and S. Weinberg, *ibid.* **20**, 224 (1968); L. Gerstein and H. Schnitzer, Phys. Rev. **175**, 1876 (1968); I. Gerstein, H. Schnitzer, and S. Weinberg, *ibid.* **175**, 1873 (1968); etc.

[24] J. Schwinger, Ref. 3; J. W. Wess and B. Zumino, Ref. 3; B. W. Lee and H. T. Nieh, Ref. 3.

*et al.*[25] shows that there are no known nucleon isobars with isospins higher than $\frac{3}{2}$, so the whole axial-vector coupling matrix may be specified in terms of the reduced matrix elements

$$\begin{aligned}\langle\beta\|X\|\alpha\rangle &\equiv A_{\beta\alpha}, \quad \text{for} \quad T_\beta=T_\alpha=\tfrac{1}{2}\\ &\equiv B_{\beta\alpha}, \quad \text{for} \quad T_\beta=T_\alpha=\tfrac{3}{2}\\ &\equiv C_{\beta\alpha}, \quad \text{for} \quad T_\beta=\tfrac{3}{2}, \quad T_\alpha=\tfrac{1}{2}.\end{aligned} \tag{7.1}$$

The Hermiticity requirement (4.19) tells us that $A$ and $B$ are Hermitian, and that

$$\langle\beta\|X\|\alpha\rangle=-\sqrt{2}C_{\beta\alpha}{}^\dagger \quad \text{for} \quad T_\beta=\tfrac{1}{2}, \quad T_\alpha=\tfrac{3}{2}. \tag{7.2}$$

In this notation, (4.20) gives for $T_\alpha=T_\beta=\frac{1}{2}$ and for $T_\alpha=T_\beta=\frac{3}{2}$ the two relations

$$1=\tfrac{4}{3}A^2-\tfrac{4}{3}C^\dagger C, \tag{7.3}$$

$$1=(4/15)B^2+\tfrac{2}{3}CC^\dagger, \tag{7.4}$$

while (4.21) gives for $T_\alpha=\frac{1}{2}$, $T_\beta=\frac{3}{2}$ the relation

$$CA=(\sqrt{5})BC. \tag{7.5}$$

For the purposes of orientation we note that the axial-vector coupling $g_A/g_V\simeq 1.25$ of the nucleon is given here by

$$g_A/g_V\equiv\langle p|X_1(\tfrac{1}{2})+iX_2(\tfrac{1}{2})|n\rangle=-(\sqrt{\tfrac{4}{3}})A_{NN}, \tag{7.6}$$

while the amplitude for $\Delta^0\to n+\pi^0$ is

$$\langle n|X^0(\tfrac{1}{2})|\Delta^0\rangle=-(1/\sqrt{2})C_{\Delta n}{}^*. \tag{7.7}$$

The mass-matrix constraints (4.22) here read

$$\begin{aligned}2Bm_3{}^2C-m_2{}^3BC-BCm_1{}^2\\ =-(\sqrt{5})(2Cm_1{}^2A-m_3{}^2CA-CAm_1{}^2),\end{aligned} \tag{7.8}$$

$$\begin{aligned}2Bm_3{}^2B-m_3{}^2B^2-B^2m_3{}^2\\ =(5/4)(2Cm_1{}^2C^\dagger-m_3{}^2CC^\dagger-CC^\dagger m_1{}^2),\end{aligned} \tag{7.9}$$

where $m_1{}^2$ and $m_3{}^2$ are the diagonal mass matrices, respectively, of the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states.

Equations (7.3)–(7.5) may be solved with $C=0$ and

$$A^2=\tfrac{3}{4}, \quad B^2=15/4. \tag{7.10}$$

[For instance, if the nucleon is the only $T=\frac{1}{2}$ state then (7.10) and (7.6) give $|g_A/g_V|=1$, as in the $\sigma$ model.] Here (7.8) is empty, while (7.9) tells us that only $B$ is diagonal, i.e., that there are no pion transitions among $T=\frac{3}{2}$ baryons. This solution corresponds to putting all $T=\frac{1}{2}$ baryons in $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ representations and all $T=\frac{3}{2}$ ($\lambda=\frac{1}{2}$) baryons in $(\frac{3}{2},0)$ or $(0,\frac{3}{2})$ representations.

In the real world there *are* one-pion transitions between $T=\frac{1}{2}$ and $T=\frac{3}{2}$ baryons, so we must look for solutions with $C\neq 0$. For instance, if there is one $T=\frac{1}{2}$ particle $N$ and one $T=\frac{3}{2}$ particle $\Delta$ then $A$, $B$, and $C$ are simply numbers, and we can find a solution of Eqs. (7.3)–(7.5) with

$$A=\pm\sqrt{(25/12)}, \; B=\pm\sqrt{(5/12)}, \; |C|=\sqrt{(4/3)}, \tag{7.11}$$

so that $g_A/g_V=\mp\frac{5}{3}$. The mass-matrix constraints (7.8) and (7.9) now just give $m_N=m_\Delta$. This solution corresponds to putting the $\lambda=\frac{1}{2}$ $N$ and $\Delta$ in a $(1,\frac{1}{2})$ or $(\frac{1}{2},1)$ representation.

Finally, we can begin to get an interesting model if we suppose that there is one $T=\frac{3}{2}$ state $\Delta$ and *two* $T=\frac{1}{2}$ states $N$ and $N'$. Then $B$ is a number, $C$ is a 2-vector and $A$ is a $2\times 2$ matrix. There is now a family of solutions with $C\neq 0$, defined by

$$A_{\beta\alpha}=\pm(\sqrt{\tfrac{3}{4}})[\delta_{\beta\alpha}+\tfrac{1}{2}C_{\Delta\beta}{}^*C_{\Delta\alpha}], \tag{7.12}$$

$$B=\pm\sqrt{(5/12)}, \tag{7.13}$$

$$|C_{\Delta N}|^2+|C_{\Delta N'}|^2=\tfrac{4}{3}. \tag{7.14}$$

The $N$, $N'$, and $\Delta$ now form a reducible $(\frac{1}{2},1)\oplus(0,\frac{1}{2})$ or $(1,\frac{1}{2})\oplus(\frac{1}{2},0)$ representation, with a mixing angle $\theta$ defined by

$$|C_{\Delta N}|=(\sqrt{\tfrac{4}{3}})\cos\theta, \quad |C_{\Delta N'}|=(\sqrt{\tfrac{4}{3}})\sin\theta, \tag{7.15}$$

and (8.6) and (8.7) now give[26]

$$|g_A/g_V|=1+\tfrac{2}{3}\cos^2\theta, \tag{7.16}$$

$$|\langle n|X^0(\tfrac{1}{2})|\Delta^0\rangle|=(1/\sqrt{2})\cos\theta. \tag{7.17}$$

With $g_A/g_V=1.18$ we would get $\theta=59°$, and with the newer value $g_A/g_V=1.25$ we have $\theta=52°$. (Both results yield adequate results for the $\Delta$ width.) The mass-matrix constraints (8.8) and (8.9) now tell us that

$$(m_{N'}{}^2-m_\Delta{}^2)/(m_\Delta{}^2-m_N{}^2)=\tan^2\theta. \tag{7.18}$$

With $\theta=52°$ this gives $m_{N'}=1600$ MeV, which is in a mass region thickly populated with $T=\frac{1}{2}$ states.

We can use this model to test our idea that the sum of tree graphs for an off-diagonal process $\pi+\alpha\to\pi+\beta$ should vanish as $\omega\to\infty$. According to Eqs. (4.5) and (4.9) this requires that

$$(\beta|[X_-(\tfrac{1}{2}),[X_+(\tfrac{1}{2}),m^2]]|\alpha)=0 \quad \text{for} \quad \alpha\neq\beta.$$

In general this gives for $T=\frac{1}{2}$ states the requirement that

$$\begin{aligned}[\tfrac{2}{3}(Am_1{}^2A-A^2m_1{}^2)+\tfrac{1}{3}(C^\dagger m_1{}^2C-C^\dagger Cm_3{}^2)\\ +(C^\dagger m_3{}^2C-m_1{}^2C^\dagger C)]_{\beta\alpha}=0 \quad \text{for} \quad \beta\neq\alpha.\end{aligned} \tag{7.19}$$

Thus in our model the requirement that the sum of the tree graphs for $\pi+N\to\pi+N'$ should vanish as $\omega\to\infty$ gives

$$m_N{}^2+m_{N'}{}^2=2m_\Delta{}^2 \tag{7.20}$$

and therefore

$$\theta=45°, \quad |g_A/g_V|=\tfrac{4}{3}. \tag{7.21}$$

The agreement between theory and experiment is again encouraging, if not precisely spectacular.

[25] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[26] These relations were obtained by Gilman and Harari, Ref. 12, and under somewhat different assumptions by I. Gerstein and B. W. Lee, Phys. Rev. Letters **16**, 1060 (1966).

## VIII. HIGHER SYMMETRIES: $SU(4)$ AND $SO(7)$

We have already remarked in Sec. V that the helicity dependence of the matrix $\mathbf{X}(\lambda)$ can be determined if we assume that only a few partial waves enter the important pion decay processes $\alpha \to \beta+\pi$. In particular, transitions between states of equal mass and parity involve $p$-wave pions only, and the corresponding matrix elements $[\mathbf{X}(\lambda)]_{\beta\alpha}$ must have the well-defined helicity dependence (5.16). How is it possible to construct matrices $\mathbf{X}(\lambda)$ with a given helicity dependence so that they satisfy the fundamental commutation relations (4.2) for each helicity? This problem is difficult to attack directly, and we shall instead approach it by finding an enlarged simple Lie algebra which contains the $X_a(\lambda)$ for all helicities.

Just to get started, let us see what Lie algebra is implied if we assume that all pion decay processes involve $p$-wave pions alone. We define an angular-momentum matrix $\mathbf{J}$ which acts on helicity indices alone:

$$(\beta\lambda'|J_3|\alpha\lambda)\equiv\delta_{\beta\alpha}\delta_{\lambda'\lambda}, \tag{8.1}$$

$$(\beta\lambda'|J_1\pm iJ_2|\alpha\lambda) \equiv\delta_{\beta\alpha}\delta_{\lambda',\lambda\pm1}[(J_\alpha\mp\lambda)(J_\alpha\pm\lambda+1)]^{1/2},$$

and write the pion-decay matrix as

$$(\beta\lambda'|X_a|\alpha\lambda)\equiv\delta_{\lambda'\lambda}[X_a(\lambda)]_{\beta\alpha}. \tag{8.2}$$

Our assumption that (8.2) involves $p$ waves only can be formulated by requiring $X_a$ to transform as the third component of a 3-vector isovector:

$$X_a\equiv D_{a3}, \tag{8.3}$$

$$[J_i,D_{aj}]=i\epsilon_{ijk}D_{ak}, \tag{8.4}$$

$$[T_a,D_{bi}]=i\epsilon_{abc}D_{ci}, \tag{8.5}$$

and the fundamental commutation rule (4.2) now reads

$$[D_{a3},D_{b3}]=i\epsilon_{abc}T_c. \tag{8.6}$$

We shall also assume that the commutators $[D_{ai},D_{bj}]$ do not contain any terms carrying isospin $T=2$, so as not to be forced to introduce particles of high isospin. We show in Appendix B that these assumptions imply that

$$[D_{ai},D_{bj}]=i\delta_{ij}\epsilon_{abc}T_c+i\delta_{ab}\epsilon_{ijk}J_k. \tag{8.7}$$

Therefore $D_{ai}$, $T_a$, *and* $J_i$ *form the algebra of* $SU(4)$.

To see the algebraic structure of these commutation relations more clearly, it is very convenient to introduce a six-dimensional notation, with ordinary vector indices $i$, $j$, $\cdots$ running over the values 1, 2, 3, and isovector indices $a$, $b$, $\cdots$ running over the value 4, 5, 6. We define an antisymmetric $6\times6$ array of matrices $J_{LM}$ by

$$J_{ij}\equiv\epsilon_{ijk}J_k,\quad J_{ia}\equiv-J_{ai}\equiv D_{ai},\quad J_{ab}\equiv\epsilon_{abc}T_c. \tag{8.8}$$

Our commutation relations (8.4), (8.5), and (8.7), plus the usual commutation relations for $J_i$ and $T_a$, may then all be summarized in the single formula

$$[J_{LM},J_{NP}]=i\delta_{LN}J_{MP}-i\delta_{LP}J_{MN} -i\delta_{MN}J_{LP}+i\delta_{MP}J_{LN}, \tag{8.9}$$

where $L$, $M$, $N$, $P$ run over 1, 2, $\cdots$, 6. We instantly recognize this as the algebra of $SO(6)$, which of course is the same as that of $SU(4)$.

We further show in Appendix C that the mass matrix has the form

$$m^2=m_0^2+m_4^2, \tag{8.10}$$

where $m_0^2$ behaves under commutation with $D_{ai}$, $J_i$, and $T_a$ as an $SU(4)$ singlet and $m_4^2$ behaves like an isoscalar 3-scalar member of a 20-dimensional representation, i.e., the totally antisymmetric $SO(6)$ tensor of rank 3. Note that we *prove* here what was conjectured in Sec. IV, that both the chiral-scalar $m_0^2$ and the chiral-vector $m_4^2$ are separately independent of helicity, so that the tree graphs give a forward-scattering amplitude which becomes helicity-independent at high energy.

It follows from the above that if the commutation relations can be saturated with a set of particles, such as $N$ and $\Delta$ or $\rho$, $\omega$, and $\pi$, between which only $p$-wave pion transitions are allowed by parity and angular-momentum conservation, then these particles *must* furnish a representation, usually reducible, of $SU(4)$. However, we certainly wish to include other particles, such as $\sigma$, $A_1$, and negative-parity baryons, which decay by $s$, $d$, $\cdots$ wave pions, so we must enlarge our algebra even more. For instance, suppose we wish to allow $s$-wave as well as $p$-wave pion transitions. We can include an $l=0$ term in $X_a$, without losing the $SU(4)$ structure of the $l=1$ term, by writing

$$X_a=\sin\theta\, S_a+\cos\theta\, D_{a3}, \tag{8.11}$$

where $S_a$ is an isovector 3-scalar

$$[T_a,S_b]=i\epsilon_{abc}S_c, \tag{8.12}$$

$$[J_i,S_a]=0, \tag{8.13}$$

and $D_{ai}$ is as before. We also again assume that no $T=2$ terms appear in the algebra generated by $S_a$ and $D_b$: With these assumptions, we show in Appendix $D$ that $S_a$, $D_{ai}$, $T_a$, *and* $J_i$ *all belong to the algebra of* $SO(7)$. That is, we have the commutation relations

$$[J_{LM},J_{NP}]=i\delta_{LN}J_{MP}-i\delta_{LP}J_{MN} -i\delta_{MN}J_{LP}+i\delta_{MP}J_{LN}, \tag{8.14}$$

where $L$, $M$, $N$, $P$ now run over 1, 2, $\cdots$, 7, with

$$J_{ij}\equiv\epsilon_{ijk}J_k, \tag{8.15}$$

$$J_{ia}\equiv-J_{ai}\equiv D_{ai}, \tag{8.16}$$

$$J_{ab}\equiv\epsilon_{bc}T_c, \tag{8.17}$$

$$J_{7a}\equiv-J_{a7}=S_a, \tag{8.18}$$

and $J_{7i}\equiv -J_{i7}$ is an auxiliary element of no known physical significance. (We are again letting ordinary space indices $i, j, \cdots$ run over 1, 2, 3 and isospin indices $a, b, \cdots$ run over 4, 5, 6.)

The algebra of $SO(7)$ has an irreducible representation, the antisymmetric tensor of rank 3, which seems to furnish a promising scheme for the nonstrange mesons; it consists of states with the quantum numbers (spin, parity, isospin, and $G$ parity) of the $\pi$, $\rho$, $\sigma$, $A_1$, $\omega$, and $B$ plus a state with the unobserved quantum numbers $J=1$, $\Pi=+$, $T=0$, $G=-1$. In particular, the states with $\lambda=0$ and $G\Pi(-)^J=+1$ are just those $(\pi,\sigma,\rho,A_1)$ considered in Sec. VI, so for these states $SO(7)$ will yield the same reasonably successful results as before [However, $SO(7)$ does *not* give $m_\omega \simeq m_\rho$.] Unfortunately, the lowest representation for the nonstrange baryons which contains the $\Delta(1236)$ is already probably too big, consisting of one $P_{33}$ and one $S_{33}$ resonances, plus *two* resonances each with the $S_{31}$, $P_{31}$, $S_{11}$, $P_{11}$, $P_{13}$, and $D_{13}$ quantum numbers. A few of these states seem to be missing, though it is not clear to me whether they would have been seen.

The lesson to be learned from all this is *not* that the axial-vector matrix $X_a$ does or does not belong to an $SU(4)$ or $SO(7)$ algebra. The point is, rather, that the difficult problem, of constructing matrices $X(\lambda)$ which satisfy chiral-commutation relations for all helicity states of a finite set of hadrons, can be solved by enlarging the algebra to include $J_i$ as well as $X_a$ and $T_a$. It hardly needs to be said that this mixture of spatial with internal symmetries is very limited, applying as it does only to axial couplings and masses, so that no conflict arises with various no-go theorems.[27] However, perhaps it should be stressed that the higher symmetries encountered here have not been hypothesized on the basis of a quark model or by a free act of intuition, but have been *derived* from chirality and from our assumptions about the partial waves which enter in pion transition amplitudes.

## IX. FURTHER APPLICATIONS

Aside from the determination of a few signs and mixing angles, our detailed predictions have generally not been new. Rather, as explained in Sec. I, we have aimed at establishing a point of view and a general algebraic formalism which can serve as a basis for future applications of chiral dynamics. We therefore close with a brief list of some of these further applications:

(1) *Other massless particles*. We may ask what algebraic restrictions follow from the assumption of good asymptotic behavior for the tree contributions to a forward scattering process $B+\alpha \to B'+\beta$, where $B$ and/or $B'$ are photons, gravitons, massless $K$ mesons, or massless pions. For pion photoproduction the result appears to be that

$$\mu'(\lambda)X_a(\lambda)=X_a(\lambda+1)\mu'(\lambda)\,, \qquad (9.1)$$

where $\mu'(\lambda)$ is the matrix element of the anomalous part of the magnetic moment $\mu_i+i\mu_2$, evaluated between states with helicity $\lambda+1$ and $\lambda$. This is just a generalized form of the Fubini-Furlan-Rossetti sum rules,[28] saturated with single-particle states. Note that (9.1) and Schur's lemma require that all elements of the matrix $\mu'(\lambda)$ vanish if $X_a(\lambda)$ and $X_a(\lambda+1)$ are irreducible and inequivalent.

(2) *Finite pion mass*. It is not easy to extend our approach to pions (or $K$ mesons) with nonzero mass, because the condition that $q^0=(\mathbf{q}^2+m_\pi{}^2)^{1/2}$ prevents our being able to count powers of $q^\mu$ as we did here in Sec. III. Perhaps it will be possible to solve this problem by expanding in powers of $m_\pi{}^2$.

(3) *Finite momentum transfer*. This problem is even harder. For instance in the elastic nonforward scattering of massless pions we have

$$\mathbf{q}=\mathbf{k}+\mathbf{\Delta}\,,\quad \mathbf{q}'=\mathbf{k}-\mathbf{\Delta}\,,\quad q^0=q'^0=(\mathbf{k}^2+\mathbf{\Delta}^2)^{1/2}\,,$$

with $\mathbf{\Delta}\equiv\frac{1}{2}(\mathbf{p}-\mathbf{p}')$ and $\mathbf{k}\cdot\mathbf{\Delta}=0$. It is difficult to see how to count powers of $\mathbf{k}$ as $\mathbf{k}\to\infty$ with $\mathbf{\Delta}$ fixed, as we did in Sec. III with $\mathbf{\Delta}=0$. Again, the solution is perhaps to be found by expanding in powers of $\mathbf{\Delta}$.

4 *Forward scattering on general states*. We also might consider the forward scattering of massless pions $\pi+\alpha\to\pi+\beta$ where $\alpha$ and/or $\beta$ are general several-particle states. The requirement of good asymptotic behavior for such processes will perhaps tell us about the commutation properties of $\mathbf{X}(\lambda)$ with the matrix element for $\alpha\to\beta$.

(5) *Multipion processes*. Possibly with greater promise of success, we may explore the consequences of the assumption of good asymptotic behavior of the tree contributions to the process $\pi+\alpha\to n\pi+\beta$, where $\alpha$ and $\beta$ are single-particle states with zero invariant momentum transfer. Presumably this will yield further algebraic restrictions on the axial-coupling matrix $\mathbf{X}(\lambda)$ and the mass matrix $m^2$.

(6) *Off-Mass-Shell Behavior*. Finally, what relation is there between the on-mass-shell asymptotic behavior discussed here and the assumptions of good asymptotic behavior off the mass shell, which are used in deriving spectral function sum rules[22] (and which underly the "smoothness" assumptions of Refs. 23, 24, and also Ref. 5)? Both approaches lead to $m_{A_1}/m_\rho=\sqrt{2}$, but they yield different results for the $A_1$ decay amplitudes. At present this peculiar blend of agreement and conflict appears entirely mysterious.

## ACKNOWLEDGMENTS

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[27] S. Coleman, Phys. Rev. **138**, B1262 (1965); M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 509 (1965); **14**, 577(E) (1965); S. Weinberg, Phys. Rev. **139**, B597 (1965).

[28] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965).

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## APPENDIX A

In this Appendix we shall rederive our chief results, Eqs. (4.2) and (4.9), by saturating the sum rules derived from soft-pion theorems and dispersion relations. Of these sum rules, some are already familiar as the general Adler-Weisberger relations,[29] (including "type-I" superconvergence relations,[16]) for elastic scattering, while the others are the corresponding sum rules for inelastic pion scattering.

Our first step is to write down the general low-energy theorems for $\pi+\alpha\to\pi+\beta$ with $m_\alpha=m_\beta$ *or* $m_\alpha>m_\beta$. These are

$$M_{\beta b,\alpha a}^{(-)}(0,\lambda)=8iF_\pi^{-2}\mathcal{E}\epsilon_{abc}(T_c)_{\beta\alpha}+8F_\pi^{-2}\mathcal{E}\sum_\gamma{}'\{[X_b(\lambda)]_{\beta\gamma}[X_a(\lambda)]_{\gamma\alpha}-[X_a(\lambda)]_{\beta\gamma}[X_b(\lambda)]_{\gamma\alpha}\}\quad\text{(A1)}$$

and

$$M_{\beta b,\alpha a}^{(+)}(0,\lambda)=2F_\pi^{-2}\sum_\gamma{}'(2m_\gamma^2-m_\alpha^2-m_\beta^2)\times\{[X_b(\lambda)]_{\beta\gamma}[X_a(\lambda)]_{\gamma\alpha}+[X_a(\lambda)]_{\beta\gamma}[X_b(\lambda)]_{\gamma\alpha}\},\quad\text{(A2)}$$

where $\mathbf{X}(\lambda)$ is the general axial-coupling matrix introduced in Sec. III, and the primed sums run over one-particle states $\gamma$ with $m_\gamma=m_\alpha$ or $m_\gamma=m_\beta$. (If $m_\alpha<m_\beta$ then these formulas hold for $\omega'=0$.) In the current-algebra derivation of (A1) and (A2), the first term in (A1) arises from the equal-time commutator, while the sum terms arise from the "gradient coupling" poles. In order to show that (A1) and (A2) give the correct values for these terms, it is convenient to recall the form given by Eq. (3.7) for the pion forward-scattering Feynman amplitude in any chiral-invariant theory:

$$M_{\beta b,\alpha a}(\omega,\lambda)=\omega\omega'\tilde{P}_{\beta b,\alpha a}(\omega,\lambda)+4iF_\pi^{-2}\mathcal{E}(\omega+\omega')\epsilon_{abc}(T_c)_{\beta\alpha}+\omega\omega'\sum_\gamma[(m_\alpha^2-m_\gamma^2+2\omega\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}^{(\gamma)}(\omega,\lambda)+(m_\alpha^2-m_\gamma^2-2\omega'\mathcal{E})^{-1}\tilde{Q}_{\beta b,\alpha a}^{(\gamma)}(-\omega',\lambda)].\quad\text{(A3)}$$

We are not using the tree approximation now, so $\tilde{P}$ and $\tilde{Q}$ will no longer be polynomials, but nevertheless $M$ will in general have the form (A3), with $\tilde{P}$ and $\tilde{Q}$ analytic at $\omega=0$. By inspection of (A3) we immediately obtain the low-energy theorem for *elastic* scattering,

$$M_{\beta b,\alpha a}(\omega,\lambda)\to 8iF_\pi^{-2}\mathcal{E}\omega\epsilon_{abc}(T_c)_{\beta\alpha}+\frac{\omega}{2\mathcal{E}}\sum_{\gamma,m_\gamma=m_\alpha}[\tilde{Q}_{\beta b,\alpha a}^{(\gamma)}(0,\lambda)-\tilde{Q}_{\beta a,\alpha b}^{(\gamma)}(0,\lambda)]+O(\omega^2),\ \text{for } \omega=\omega'\to 0\quad\text{(A4)}$$

and for exoenergetic scattering,

$$M_{\beta b,\alpha a}(\omega,\lambda)\to\left(\frac{m_\alpha^2-m_\beta^2}{4\mathcal{E}^2}\right)\left[\sum_{\gamma,m_\gamma=m_\alpha}\tilde{Q}_{\beta b,\alpha a}^{(\gamma)}(0,\lambda)-\sum_{\gamma,m_\gamma=m_\beta}\tilde{Q}_{\beta a,\alpha b}^{(\gamma)}\left(\frac{m_\beta^2-m_\alpha^2}{2\mathcal{E}},\lambda\right)\right]+O(\omega),$$
$$\text{for } \omega\to 0,\ \omega'\to(m_\alpha^2-m_\beta^2)/2\mathcal{E}.\quad\text{(A5)}$$

Furthermore, the pole residues $Q^{(\gamma)}$ are given on the $\gamma$ mass shell by Eq. (3.16):

$$\tilde{Q}_{\beta b,\alpha a}^{(\gamma)}\left(\frac{m_\gamma^2-m_\alpha^2}{2\mathcal{E}},\lambda\right)=16F_\pi^{-2}\mathcal{E}^2[X_b(\lambda)]_{\beta\gamma}\times[X_a(\lambda)]_{\gamma\alpha}.\quad\text{(A6)}$$

To compute $M^{(-)}$ we must antisymmetrize $M$ in $a$ and $b$ and divide by the quantity $\omega+\omega'$, which for elastic scattering is just $2\omega$ and for $m_\alpha>m_\beta$ approaches $\omega'=(m_\alpha^2-m_\beta^2)/2\mathcal{E}$. In both cases we get (A1). To compute $M^{(+)}$ we symmetrize in $a$ and $b$, and note that $2m_\gamma^2-m_\alpha^2-m_\beta^2$ equals $m_\alpha^2-m_\beta^2$ for $m_\gamma=m_\alpha$ and equals $-(m_\alpha^2-m_\beta^2)$ for $m_\gamma=m_\beta$. This gives (A2).

The next step is to invoke our assumption that $M^{(-)}$ and the $T=2$ part of $M^{(+)}$ vanish as $\omega\to\infty$. They then satisfy the unsubtracted dispersion relations

$$M_{\beta b,\alpha a}^{(-)}(\omega,\lambda)=\int d\mu^2\left(\frac{1}{\mu^2-s}-\frac{1}{\mu^2-u}\right)\times[\rho_{\beta b,\alpha a}(\mu^2,\lambda)-\rho_{\beta a,\alpha b}(\mu^2,\lambda)]\left(\frac{2\mathcal{E}}{2\mu^2-m_\alpha^2-m_\beta^2}\right),\quad\text{(A7)}$$

$$M_{\beta b,\alpha a}^{(T=2)}(\omega,\lambda)\equiv M_{\beta b,\alpha a}^{(+)}(\omega,\lambda)-\tfrac{1}{3}\delta_{ba}M_{\beta c,\alpha c}^{(+)}(\omega,\lambda)=\tfrac{1}{2}\int d\mu^2\left(\frac{1}{\mu^2-s}+\frac{1}{\mu^2-u}\right)[\rho_{\beta b,\alpha a}(\mu^2,\lambda)+\rho_{\beta a,\alpha a}(\mu^2,\lambda)+\tfrac{2}{3}\delta_{ab}\rho_{\beta c,\alpha c}(\mu^2,\lambda)],\quad\text{(A8)}$$

where $\rho_{\beta b,\alpha a}(\mu^2,\lambda)$ is the absorptive part of $M_{\beta b,\alpha a}(\omega,\lambda)$, evaluated at $s=\mu^2$, i.e., at $\omega=(\mu^2-m_\alpha^2)/2\mathcal{E}$. [The last factor in (A7) is just the quantity $(\omega+\omega')^{-1}$ appearing in the definition of $M^{(-)}$.] We can always write $\rho_{\beta b,\alpha a}(\mu^2,\lambda)$ as a formal sum over states:

$$\rho_{\beta b,\alpha a}(\mu^2,\lambda)=4F_\pi^{-2}\sum_\gamma(m_\gamma^2-m_\beta^2)(m_\alpha^2-m_\gamma^2)[X_b(\lambda)]_{\beta\gamma}\times[X_a(\lambda)]_{\gamma\alpha}\delta(\mu^2-m_\gamma^2),\quad\text{(A9)}$$

where $2F_\pi^{-1}(m_\alpha^2-m_\gamma^2)[X_a(\lambda)]_{\gamma\alpha}$ is the Feynman amplitude for the transition $\alpha\to\gamma+\pi$. Using this in (A7) and (A8) gives

$$M_{\beta b,\alpha a}^{(-)}(\omega,\lambda)=-8\mathcal{E}F_\pi^{-2}\sum_\gamma\frac{(m_\alpha^2-m_\gamma^2)(m_\beta^2-m_\gamma^2)}{(m_\alpha^2-m_\gamma^2-2\mathcal{E}\omega)(m_\beta^2-m_\gamma^2+2\mathcal{E}\omega)}\times\{[X_b(\lambda)]_{\beta\gamma}[X_a(\lambda)]_{\gamma\alpha}-[X_a(\lambda)]_{\beta\gamma}[X_b(\lambda)]_{\gamma\alpha}\},\quad\text{(A10)}$$

[29] S. L. Adler, Phys. Rev. **140**, B736 (1965); W. I. Weisberger, *ibid.* **143**, 1302 (1966); and earlier letters quoted therein.

$$M_{\beta b,\alpha a}^{(T=2)}(\omega,\lambda)$$

$$=-2F_\pi^{-2}\sum_\gamma \frac{(m_\alpha^2-m_\gamma^2)(m_\beta^2-m_\gamma^2)(2m_\gamma^2-m_\alpha^2-m_\beta^2)}{(m_\alpha^2-m_\gamma^2-2\mathcal{E}\omega)(m_\beta^2-m_\gamma^2+2\mathcal{E}\omega)}$$
$$\times\{[X_b(\lambda)]_{\beta\gamma}[X_a(\lambda)]_{\gamma\alpha}+[X_a(\lambda)]_{\beta\gamma}[X_b(\lambda)]_{\gamma\alpha}$$
$$-\tfrac{2}{3}\delta_{ab}[X_c(\lambda)]_{\beta\gamma}[X_c(\lambda)]_{\gamma\alpha}\}. \quad \text{(A11)}$$

States $\gamma$ with $m_\gamma=m_\alpha$ or $m_\gamma=m_\beta$ do not contribute to these sums because of the factors $(m_\alpha^2-m_\gamma^2)$ and $(m_\beta^2-m_\gamma^2)$, even though these factors are cancelled by the denominators as $\omega\to 0$. However, the states $\gamma$ which are missing in (A10) and (A11) are just those which appear in the "gradient coupling terms" in (A1) and (A2). Hence the consistency of (A10) with (A1) and of (A11) with (A2) requires that

$$i\epsilon_{abc}(T_c)_{\beta\alpha}=\sum_\gamma \{[X_a(\lambda)]_{\beta\gamma}[X_b(\lambda)]_{\gamma\alpha}-[X_b(\lambda)]_{\beta\gamma}[X_a(\lambda)]_{\gamma\alpha}\}, \quad \text{(A12)}$$

$$0=\sum_\gamma (2m_\gamma^2-m_\alpha^2-m_\beta^2)\{[X_a(\lambda)]_{\beta\gamma}[X_b(\lambda)]_{\gamma\alpha}+[X_b(\lambda)]_{\beta\gamma}[X_a(\lambda)]_{\gamma\alpha}-\tfrac{2}{3}\delta_{ab}\times[X_c(\lambda)]_{\beta\gamma}[X_c(\lambda)]_{\gamma\alpha}\}, \quad \text{(A13)}$$

the sums now running over *all* states $\gamma$. These are the desired results.

## APPENDIX B: DERIVATION OF $SU(4)$

We have assumed that $D_{ai}$ is an isovector 3-vector, so the most general possible form for the $[D,D]$ commutators is

$$[D_{ai},D_{bj}]=i\epsilon_{abc}A_{cij}+i\epsilon_{ijk}B_{abk}, \quad \text{(B1)}$$

where $A$ is an isovector symmetric 3-tensor and $B$ is a symmetric isotensor 3-vector. Our assumption that this commutator not contain $T=2$ terms implies that

$$B_{abk}=\delta_{ab}B_k \quad \text{(B2)}$$

and the fundamental chiral commutation relation (4.2) or (8.6) now reads

$$A_{c33}=T_c. \quad \text{(B3)}$$

Since $T_c$ is a 3-scalar, it follows immediately from (B3) [formally, by taking repeated commutators with $J_i$] that $A_{cij}$ is also a 3-scalar, i.e.,

$$A_{cij}=T_c\delta_{ij}. \quad \text{(B4)}$$

Our remaining task is then to prove that $B_k=J_k$.

From (B1) and (B2) we have

$$B_j=-\tfrac{1}{6}i\epsilon_{jkl}[D_{bk},D_{bl}] \quad \text{(B5)}$$

and therefore

$$\begin{aligned}[D_{ai},B_j]&=-\tfrac{1}{6}i\epsilon_{jkl}[D_{ai},[D_{bk},D_{bl}]]\\&=\tfrac{1}{3}\epsilon_{jil}\epsilon_{abc}[T_c,D_{bl}]+\tfrac{1}{3}\epsilon_{jkl}\epsilon_{ikm}[J_m,D_{al}]\\&=\tfrac{2}{3}i\epsilon_{ijk}D_{ak}-\tfrac{1}{3}\delta_{ij}[D_{al},B_l]+\tfrac{1}{3}[D_{ai},B_j].\end{aligned}$$

By taking the trace on $i$ and $j$, we see that $[D_{al},B_l]$ vanishes, and hence

$$[D_{ai},B_j]=i\epsilon_{ijk}D_{ak}. \quad \text{(B6)}$$

This result may in turn be used to compute the last commutator we need:

$$\begin{aligned}[B_i,B_j]&=-\tfrac{1}{6}i\epsilon_{jkl}[B_i,[D_{bk},D_{bl}]]\\&=\tfrac{1}{3}\epsilon_{jkl}\epsilon_{ikm}[D_{bm},D_{bl}]\\&=-\tfrac{1}{3}[D_{bj},D_{bi}],\end{aligned}$$

and using (B1) again, this is

$$[B_i,B_j]=i\epsilon_{ijk}B_k. \quad \text{(B7)}$$

Equation (B1), (B2), (B4), (B6), and (B7), together with the commutators for $T_a$ and $J_i$ with $D_{ai}$, $B_j$, and each other, show that $D_{ai}$, $B_i$, $T_a$, and $J_i$ form a complete Lie algebra. However, this algebra will in general contain an invariant Abelian subalgebra spanned by $B_i-J_i$, for (B6) and (B7) and will show that

$$[B_i-J_i,\ D_{aj}]=[B_i-J_i,\ B_j-J_j]=0, \quad \text{(B8)}$$

while of course $B_i-J_i$ also commutes with $T_a$, and transforms into itself under $J_i$, and hence also under $B_i$:

$$[J_i,\ B_j-J_j]=i\epsilon_{ijk}(B_k-J_k), \quad \text{(B9)}$$

$$[B_i,\ B_j-J_j]=i\epsilon_{ijk}(B_k-J_k). \quad \text{(B10)}$$

But these are all supposed to be finite matrices, so we must require their algebra to be semisimple, and thus

$$B_i-J_i=0 \quad \text{(B11)}$$

as was to be proved.

## APPENDIX C: $SU(4)$ PROPERTIES OF THE MASS MATRIX

The result of Sec. IV tell us that

$$[D_{a3},[D_{b3},m^2]]\propto\delta_{ab}. \quad \text{(C1)}$$

Furthermore the mass matrix must obviously be helicity-independent and conserve isospin, so

$$[J_i,m^2]=[T_a,m^2]=0. \quad \text{(C2)}$$

We may therefore define an isovector 3-vector $m_{ai}^2$ by

$$[D_{ai},m^2]\equiv im_{ai}^2, \quad \text{(C3)}$$

with

$$[J_i,m_{aj}^2]=i\epsilon_{ijk}m_{ak}^2, \quad \text{(C4)}$$

$$[T_a,m_{bj}^2]=i\epsilon_{abc}m_{cj}^2. \quad \text{(C5)}$$

By applying Jacobi's identity to the double commutator $[D_{ai},[D_{bj},m^2]]$ and using (C2), we easily see that

$$[D_{ai},m_{bj}^2]=[D_{bj},m_{ai}^2]. \quad \text{(C6)}$$

Hence, since this commutator is an isotensor and a 3-tensor, (C1) and (C6) force it to have the form

$$[D_{ai},m_{bj}^2]=i\delta_{ab}\delta_{ij}m_4^2+i\epsilon_{abc}\epsilon_{ijk}\mu_{ck}^2, \quad \text{(C7)}$$

where $m_4^2$ is an isoscalar 3-scalar, and $\mu_{ck}^2$ is an isovector 3-vector. The Jacobi identity for $[D_{ai},[D_{bj},m_{ck}^2]]$ can then be used to show that

$$[D_{ai},m_4^2]=im_{ai}^2, \qquad \text{(C8)}$$

$$[D_{ai},\mu_{bj}^2]=i\delta_{ab}\delta_{ij}\mu_4^2-i\epsilon_{abc}\epsilon_{ijk}m_{ck}^2, \qquad \text{(C9)}$$

where $\mu_4^2$ is another isoscalar 3-scalar. Finally, the Jacobi identity for $[D_{ai},\mu_{ck}^2]]$ gives

$$[D_{ai},\mu_4^2]=-i\mu_{ck}^2. \qquad \text{(C10)}$$

Equations (C7)–(C10) show that $m_{ai}^2$, $m_4^2$, $\mu_{ai}^2$, and $\mu_4^2$ form a $9+1+9+1=20$-dimensional representation of $SU(4)$. Furthermore, since $\mu_4^2$ and $m_4^2$ are isoscalars and 3-scalars, this representation must be one of the ordinary tensor representations of the locally isomorphic group $SO(6)$. The only tensor representation of $SO(6)$ which is 20-dimensional, and contains two isoscalar 3-scalars and two isovector 3-vectors, is the totally antisymmetric tensor of third rank $T^{KLM}$, with

$$T^{abc}\propto m_4^2\epsilon^{abc}, \quad T^{ijk}\propto \mu_4^2\epsilon^{ijk},$$
$$T^{abi}\propto m_{ci}^2\epsilon^{abc}, \quad T^{ija}\propto \mu_{ak}^2\epsilon^{ijk}.$$

It is easy, by using the $SO(6)$ notation for $D_{ai}$, to verify that (C7)–(C10) are indeed satisfied in this way.

Just as in Appendix B, we now complete our argument by noting from (C3) and (C8) that $m^2-m_4^2$ commutes with $D_{ai}$, i.e.,

$$m^2=m_0^2+m_4^2, \qquad \text{(C10)}$$

where

$$[D_{ai},m_0^2]=0. \qquad \text{(C11)}$$

Further, $m^2$ and $m_4^2$ both commute with $T_a$ and $J_i$, and therefore so does $m_0^2$:

$$[J_i,m_0^2]=[T_a,m_0^2]=0. \qquad \text{(C12)}$$

Thus, (C10) gives $m^2$ as the sum of two $SU(4)$ representations

$$m^2=\mathbf{1}+\mathbf{20} \qquad \text{(C13)}$$

as was to be proved.

## APPENDIX D: DERIVATION OF $SO(7)$

With $X_a$ given by (8.11), the fundamental commutation relation (4.2) reads

$$\sin^2\theta\,[S_a,S_b]+\sin\theta\cos\theta\,\{[S_a,D_{b3}]-[S_b,D_{a3}]\} +\cos^2\theta\,[D_{a3},D_{b3}]=i\epsilon_{abc}T_c. \qquad \text{(D1)}$$

We are assuming that $D_{ai}$ still obeys the $SU(4)$ commutation rule (8.7), so the last term on the right is given by (8.6), and (D1) reads

$$\sin\theta\,[S_a,S_b]+\cos\theta\,\{[S_a,D_{b3}]-[S_b,D_{a3}]\} =i\sin\theta\,\epsilon_{abc}T_c. \qquad \text{(D2)}$$

The 3-scalar and 3-vector parts of this equation must be separately valid, so

$$[S_a,S_b]=i\epsilon_{abc}T_c, \qquad \text{(D3)}$$

$$[S_a,D_{bi}]=[S_b,D_{ai}]. \qquad \text{(D4)}$$

We again exclude the possibility of $T=2$ terms in our algebra, so the symmetric tensor ($D4$) must be a pure isoscalar, i.e.,

$$[S_a,D_{bi}]=i\delta_{ab}K_i, \qquad \text{(D5)}$$

with $K_i$ an isoscalar 3-vector. The Jacobi identity for $S_a$, $S_b$, and $D_{ci}$ gives

$$i\delta_{bc}[S_a,K_i]-i\delta_{ac}[S_b,K_i]=i\epsilon_{abd}[T_d,D_{ci}] =\delta_{bc}D_{ai}-\delta_{ac}D_{bi}$$

and therefore

$$[S_a,K_i]=iD_{ai}. \qquad \text{(D6)}$$

The Jacobi identity for $D_{ai}$, $D_{bj}$, and $S_c$ gives

$$-i\delta_{bc}[D_{ai},K_j]+i\delta_{ac}[D_{bj},K_i]=i\delta_{ij}\epsilon_{abd}[T_d,S_c] +i\delta_{ab}\epsilon_{ijk}[J_k,S_c]=\delta_{ij}[\delta_{bc}S_a-\delta_{ac}S_b]$$

and therefore

$$[D_{ai},K_j]=i\delta_{ij}S_a. \qquad \text{(D7)}$$

Finally, the Jacobi identity for $S_a$, $D_{bi}$, and $K_j$ gives

$$i\delta_{ab}[K_i,K_j]=i[D_{bi},D_{aj}]+i[S_a,S_b]\delta_{ij}=-\delta_{ab}\epsilon_{ijk}J_k$$

and therefore

$$[K_i,K_j]=i\epsilon_{ijk}J_k. \qquad \text{(D8)}$$

We identify the $SO(7)$ generators as

$$J_{ij}\equiv\epsilon_{ijk}J_k, \quad J_{ab}\equiv\epsilon_{abc}T_c,$$
$$J_{ia}\equiv-J_{ai}\equiv D_{ai},$$
$$J_{7a}\equiv-J_{a7}\equiv S_a,$$
$$J_{7i}\equiv-J_{i7}\equiv-K_i,$$

so that our results (D3), (D5)–(D8) verify the $SO(7)$ commutation rules (8.14) for $LMNP=7a7b$, $7aib$, $7a7i$, $7iaj$, $7i7j$. The commutators (8.14) with $LMNP=7abc$, $7aij$, $7iab$, $7ijk$ merely express the facts that $S_a$ is an isovector 3-scalar and $K_i$ is an isoscalar 3-vector, while the commutators (8.14) with none of $L$, $M$, $N$, $P$ equal to 7 are just the $SO(6)\equiv SU(4)$ commutation relations, which we assume are still valid. Thus we have verified that $J_{LM}$ obeys all the $SO(7)$ commutation relations.

*Note added in proof.* (1) Despite the remarks made in Sec. IX, there is in fact no difficulty in deriving dispersion relations for the forward scattering of a massless pion on general multiparticle states. (This work is being readied for publication.) (2) The elastic-scattering sum rules discussed by Gilman and Harari[14] are not actually sufficient to allow a derivation of all algebraic consequences of chirality, even apart from signs. For instance the study of elastic $\rho\pi$ and $D\pi$ scattering leaves undetermined one mixing angle $\chi$, which Gilman and Harari determine to be close to zero by comparison with experiment. By studying the inelastic process $\omega\pi\rightarrow A_1\pi$ one can derive the value $\chi=0$.