Lower Bounds on Phase Shifts for Three-Body Systems: *n-d* Quartet Scattering*†

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The Faddeev equations are used to provide a variational-bound formulation of the three-body scattering problem. The present method has the distinct advantage that the Feshbach projection operators, which enter into previous formulations and which are generally difficult to construct, do not appear. The method requires the calculation of a variational approximation to the exact effective potential for the scattering of a particle by a bound two-body system. A reaction matrix is determined by using this effective potential as input to a two-body Lippmann-Schwinger equation which is easily solved numerically. The eigenphase shifts thus obtained provide lower bounds on the true eigenphases for energies below the three-body breakup threshold. To test its practicability, the method is applied to the problem of neutron-deuteron scattering in the quartet state. The results are in agreement with previous calculations and with experiment.

1. INTRODUCTION

THERE has been a considerable amount of research done in recent years[1–4] in an effort to develop for three-body scattering problems a calculational method analogous to the very powerful Rayleigh-Ritz method which is available for bound-state problems. That is, one seeks a variational method which provides a rigorous bound on the scattering parameters. One method, which involves the use of the projection operators introduced by Feshbach[5] in his development of an effective potential formalism, has been given by Hahn, O'Malley, and Spruch.[2] For those three-body systems in which the mass ratios are finite and in which rearrangement collisions can occur, the projection operators are difficult to construct in practice, and other methods must be sought. The approach we describe here is based on the fact that with the aid of the Faddeev equations[6] an effective-potential formalism can be derived without the use of projection operators.[7] The effective potential is defined in terms of the solution to a set of Faddeev integral equations having as input two-body subsystem scattering amplitudes whose bound-state pole contributions have been subtracted out. It was shown in Ref. 7 that a variational principle of the Schwinger type (i.e., based directly on the integral equations) can be set up which is in fact a minimum principle under certain well-defined circumstances. A variational lower bound on the elastic scattering phase shift then follows from the well-known monotonicity theorem which is proved, in a form particularly convenient for our purposes, in Ref. 7. In Sec. 2 of the present paper we develop the minimum principle for the effective potential in a different, more practical form. This method requires that we construct expectation values of a Hamiltonian which is defined in terms of energy-independent two-body potentials. The essential idea of this method appeared previously in Ref. 4 but was worked out there only for the special case of three identical particles.[8] A numerical application of the minimum principle to the problem of elastic scattering of neutrons by deuterons in the quartet state, with the assumption of local, central two-body potentials, is described in Sec. 3.

2. MINIMUM PRINCIPLE

A. Two-Body Preliminaries

In the derivation of the effective-potential formalism[7] it is necessary to express the scattering operator for a two-body subsystem as the sum of two parts; one part must be free of pole singularities arising from two-body bound states, while the other part, which contains these singularities, must be a sum of terms, each separable in momentum space. Of course, this decomposition of the scattering operator is not unique. In this subsection we define a particular decomposition; the reason for this choice will appear in the subsequent application. For simplicity we assume that only one two-body bound state exists. The generalization to the case where a finite number of bound states exist is straightforward.

According to the familiar "two-potential" theorem,[9] the scattering operator t associated with the potential $v=v_A+v_B$ can be written as $t=t_A+t_B$. The operator t_A

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[1] L. Rosenberg and L. Spruch, Phys. Rev. **125**, 1407 (1962), and references cited therein.
[2] Y. Hahn, T. F. O'Malley, and L. Spruch, Phys. Rev. **134**, B911 (1964).
[3] R. Sugar and R. Blankenbecler, Phys. Rev. **136**, B472 (1964).
[4] L. Rosenberg, Phys. Rev. **138**, B1343 (1965).
[5] H. Feshbach, Ann. Phys. (N. Y.) **5**, 357 (1958); **19**, 287 (1962).
[6] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[7] L. Rosenberg, Phys. Rev. **168**, 1756 (1968).
[8] See the discussion following Eq. (3.24) of Ref. 4. A simpler method, described in Sec. 3A of Ref. 4, is unfortunately restricted to single-channel scattering.
[9] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966), p. 194.

satisfies the Lippmann-Schwinger equation

$$t_A = v_A + v_A g_0 t_A , \tag{2.1}$$

where g_0 is the Green's operator for the unperturbed system. The operator t_B is given by

$$t_B = (1+t_A g_0) t_{AB} (1+g_0 t_A) , \tag{2.2}$$

where t_{AB} satisfies the integral equation

$$t_{AB} = v_B + v_B g_A t_{AB} , \tag{2.3}$$

with g_A defined as

$$g_A = g_0 + g_0 t_A g_0 . \tag{2.4}$$

Now suppose that the potential v supports a single bound state with state vector $|\chi\rangle$ and energy $-\epsilon$. We choose

$$v_B = \frac{v|\chi\rangle\langle\chi|v}{\langle\chi|v|\chi\rangle} . \tag{2.5}$$

It follows from the relation

$$v_A|\chi\rangle \equiv (v-v_B)|\chi\rangle = 0 \tag{2.6}$$

that v_B supports the same bound state $|\chi\rangle$. Since v_B is separable, Eq. (2.3) can be solved algebraically, with the result

$$t_{AB}(E) = \frac{v|\chi\rangle S(E)\langle\chi|v}{E+\epsilon} . \tag{2.7}$$

Here, E is the total energy and $S(E)$ is given by

$$S^{-1}(E) = \langle\chi|\tilde{\chi}(E)\rangle , \tag{2.8}$$

where we have defined

$$|\tilde{\chi}(E)\rangle = g_A(E) v|\chi\rangle . \tag{2.9}$$

The relation

$$t_A|\chi\rangle = 0 \tag{2.10}$$

[which follows from Eq. (2.6)] and the eigenvalue condition

$$g_0(-\epsilon)v|\chi\rangle = |\chi\rangle \tag{2.11}$$

imply that

$$|\tilde{\chi}(-\epsilon)\rangle = |\chi\rangle , \tag{2.12}$$

and, since $|\chi\rangle$ is normalized to unity, $S(-\epsilon)=1$. Thus t_B has the correct residue at the pole.

We remark that the significant feature of the above decomposition of the operator t is that the component t_A which is free of the bound-state pole is obtained from a potential v_A which is energy-independent. The advantage gained will be apparent when we set up a variational principle for the effective potential in the three-body problem. In the following we will assume that the potential v_A is too weak to support a bound state. This assumption is easily checked by direct computation. It turns out to be valid in the particular application we have made (see Sec. 3).

B. Effective-Potential Formalism

In order to state our definition of the effective potential, we must first introduce some notations. Consider a three-body system with total energy E. Let $T(\alpha; E)$ represent the scattering operator for subsystem α ($\alpha=1, 2, 3$) in the presence of the third noninteracting particle. We use the upper-case symbol to distinguish between the three-body operator T and the two-body operator t. Matrix elements of T and t differ in that the former contains an additional δ function expressing momentum conservation of the noninteracting particle. We consider a decomposition

$$T(\alpha; E) = T_A(\alpha; E) + T_B(\alpha; E) \tag{2.13}$$

such that T_A is free of bound-state poles. The coupled Faddeev equations with the $T_A(\alpha; E)$ as the input give rise to nine operators ${}^{(\alpha)}T_A{}^{(\beta)}$ ($\alpha, \beta = 1, 2, 3$) corresponding to the nine distinct combinations of initial- and final-state pair interactions. Matrix elements of these operators are used in the construction of the effective potential. We wish to state these results explicitly for the case where $T_A(\alpha; E)$ is the three-body counterpart of the operator $t_A(\alpha; E)$ defined in the previous subsection.

Let $|\chi_\alpha; \mathbf{k}_\alpha\rangle$ represent a state in which the bound pair α and the third particle move freely with relative momentum $\mathbf{k}_\alpha$. The state $|\tilde{\chi}_\alpha(E-\hbar^2\mathbf{k}_\alpha{}^2/2\mu_\alpha); \mathbf{k}_\alpha\rangle$ is now defined as that obtained from $|\chi_\alpha; \mathbf{k}_\alpha\rangle$ (in any given representation) by the replacement $|\chi_\alpha\rangle \to |\tilde{\chi}_\alpha(E-\hbar^2 k_\alpha{}^2/2\mu_\alpha)\rangle$ [see Eq. (2.9)]. The state

$$\langle\tilde{\chi}_\alpha{}^{(-)}(E-\hbar^2 k_\alpha{}^2/2\mu_\alpha); \mathbf{k}_\alpha|$$

is obtained from $\langle\chi_\alpha; \mathbf{k}_\alpha|$ in a similar way; the superscript $(-)$ indicates, as usual, that the energy is to approach the real axis from below rather than from above. With $T_A(\alpha; E)$ defined as above, the form taken by $T_B(\alpha; E)$ is most conveniently expressed as

$$G_0(E) T_B(\alpha; E) G_0(E) = \int d^3k_\alpha \left|\tilde{\chi}_\alpha\left(E-\frac{\hbar^2 k_\alpha{}^2}{2\mu_\alpha}\right); \mathbf{k}_\alpha\right\rangle \frac{S(E-\hbar^2 k_\alpha{}^2/2\mu_\alpha)}{E-\hbar^2 k_\alpha{}^2/2\mu_\alpha+\epsilon_\alpha} \times \left\langle\tilde{\chi}_\alpha{}^{(-)}\left(E-\frac{\hbar^2 k_\alpha{}^2}{2\mu_\alpha}\right); \mathbf{k}_\alpha\right| . \tag{2.14}$$

Here, $G_0(E)$ is the Green's operator for the unperturbed three-body system, i.e.,

$$G_0(E) = (E-H_0)^{-1} , \tag{2.15}$$

where H_0 is the total-kinetic-energy operator.

Let $T_{\alpha\beta}(\mathbf{k}_\alpha, \mathbf{k}_\beta; E)$ represent the amplitude for scattering from the initial state $|\chi_\beta; \mathbf{k}_\beta\rangle$ to the final state $|\chi_\alpha; \mathbf{k}_\alpha\rangle$. The physical amplitude is obtained when the

energy-shell conditions

$$E=\frac{\hbar^2k_\alpha^2}{2\mu_\alpha}-\epsilon_\alpha=\frac{\hbar^2k_\beta^2}{2\mu_\beta}-\epsilon_\beta \tag{2.16}$$

are satisfied. The basic result of this subsection is the statement that $T_{\alpha\beta}$ satisfies the integral equation

$$T_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)+\sum_\gamma\int d^3k_\gamma V_{\alpha\gamma}(\mathbf{k}_\alpha,\mathbf{k}_\gamma;E)\frac{S(E-\hbar^2k_\gamma^2/2\mu_\gamma)}{E-\hbar^2k_\gamma^2/2\mu_\gamma+\epsilon_\gamma+i\eta}\times T_{\gamma\beta}(\mathbf{k}_\gamma,\mathbf{k}_\beta;E), \tag{2.17}$$

where the effective-potential matrix is defined as

$$V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=\left\langle\tilde{\chi}_\alpha^{(-)}\left(E-\frac{\hbar^2k_\alpha^2}{2\mu_\alpha}\right);\mathbf{k}_\alpha\right|\times[(E-H_0)(1-\delta_{\alpha\beta})+\sum_{\rho\neq\alpha}\sum_{\sigma\neq\beta}{}^{(\rho)}T_A{}^{(\sigma)}(E)]\times\left|\tilde{\chi}_\beta\left(E-\frac{\hbar^2k_\beta^2}{2\mu_\beta}\right);\mathbf{k}_\beta\right\rangle. \tag{2.18}$$

C. Minimum Principle for the Effective Potential

In order to establish a Rayleigh-Ritz type of construction for the effective potential, we first observe that, since the operators ${}^{(\rho)}T_A{}^{(\sigma)}$ are solutions of the Faddeev equations, the sum which appears in Eq. (2.18) can be expressed in the form

$$\sum_{\rho\neq\alpha}\sum_{\sigma\neq\beta}{}^{(\rho)}T_A{}^{(\sigma)}(E)=\sum_{\gamma\neq\alpha,\beta}V_A(\gamma)+[V_A-V_A(\alpha)]G_A(E)[V_A-V_A(\beta)]. \tag{2.19}$$

Here, $V_A(\alpha)$ is the modified potential for subsystem α, and V_A is the sum of the three pair potentials. $G_A(E)$ is the resolvent operator for the Hamiltonian

$$H_A=H_0+V_A. \tag{2.20}$$

We observe that the continuous spectrum of H_A, corresponding to states in which all three particles are free asymptotically, begins at zero energy. Negative-energy scattering states, in which a pair is bound and the third particle is free asymptotically, do not exist for H_A as they do for the original Hamiltonian $H=H_0+V$, since the pair potentials $V_A(\alpha)$ are too weak to support a bound state. In the following, the energy E will always be taken to be negative. We shall assume that the resolvent operator $G_A(E)$ which appears in Eq. (2.19) can be obtained by analytic continuation from the region just above the real positive E axis to the negative E axis.[10] The resolvent operator is then a bounded two-sided inverse,[11] i.e.,

$$G_A(E)(H_A-E)=(H_A-E)G_A(E)=-1. \tag{2.21}$$

A variational principle for G_A may be obtained from the identity[7,12]

$$G_A=G_{At}+G_A[1+(H_A-E)G_{At}], \tag{2.22}$$

where G_{At} is some trial resolvent operator. If we introduce

$$G_A=G_{At}+\Delta G_A \tag{2.23}$$

on the right-hand side of Eq. (2.22), the identity takes the form

$$G_A=G_{Av}+\text{error},$$

where the variational estimate is

$$G_{Av}=G_{At}+G_{At}[1+(H_A-E)G_{At}]. \tag{2.24}$$

The error is expressed explicitly as

$$G_A-G_{Av}=-\Delta G_A(H_A-E)\Delta G_A. \tag{2.25}$$

These results, when combined with Eq. (2.18), lead to a variational principle for the effective potential. The expressions take on a convenient form when written in terms of the trial functions

$$|\psi_{\beta t}\rangle=G_{At}(E)[V_A-V_A(\beta)]\times\left|\tilde{\chi}_\beta\left(E-\frac{\hbar^2k_\beta^2}{2\mu_\beta}\right);\mathbf{k}_\beta\right\rangle, \tag{2.26}$$

$$\langle\psi_{\alpha t}|=\left\langle\tilde{\chi}_\alpha\left(E-\frac{\hbar^2k_\alpha^2}{2\mu_\alpha}\right);\mathbf{k}_\alpha\right|\times[V_A-V_A(\alpha)]G_{At}(E). \tag{2.27}$$

In using this notation we have assumed that $G_{At}{}^\dagger=G_{At}$, a relation satisfied by G_A and one which guarantees that the variational expression for the effective potential and its error are separately Hermitian matrices. Note that the distinction between $|\tilde{\chi}^{(+)}\rangle$ and $|\tilde{\chi}^{(-)}\rangle$ vanishes for $E<0$; we have simplified the notation accordingly in Eq. (2.27). We find that

$$V_{\alpha\beta}=V_{\alpha\beta v}+\Delta V_{\alpha\beta}, \tag{2.28}$$

where

$$V_{\alpha\beta v}=\langle\tilde{\chi}_\alpha|(E-H_0)|\tilde{\chi}_\beta\rangle(1-\delta_{\alpha\beta})+\langle\tilde{\chi}_\alpha|\sum_{\gamma\neq\alpha,\beta}V_A(\gamma)|\tilde{\chi}_\beta\rangle+\langle\tilde{\chi}_\alpha|V_A-V_A(\alpha)|\psi_{\beta t}\rangle+\langle\psi_{\alpha t}|V_A-V_A(\beta)|\tilde{\chi}_\beta\rangle+\langle\psi_{\alpha t}|H_A-E|\psi_{\beta t}\rangle. \tag{2.29}$$

[10] R. G. Newton, Ref. 9, p. 207. There will of course be simple poles in $G_A(E)$ at the bound-state eigenvalues of H_A.

[11] R. G. Newton, Ref. 9, Sec. 7.3.1. With regard to boundary conditions in a configuration-space representation of G_A, we would interpret the above statements as implying that the outgoing waves for $E>0$ become exponentially decaying waves, for any interparticle distance extending to infinity, in the negative-energy region.

[12] See also S. Aranoff and J. Percus, Phys. Rev. **166**, 1255 (1968).

If we define $|\Delta\psi_\beta\rangle$ and $\langle\Delta\psi_\alpha|$ by the replacement $G_{At}\rightarrow\Delta G_A$ in Eqs. (2.26) and (2.27), the error term can be written

$$\Delta V_{\alpha\beta}=-\langle\Delta\psi_\alpha|H_A-E|\Delta\psi_\beta\rangle. \quad (2.30)$$

Let us suppose that there is no bound state of the Hamiltonian H_A with energy below E. Then diagonal elements of the error matrix, given by Eq. (2.30) with $\alpha=\beta$, will be negative. This result, which holds for arbitrary (normalizable) functions $|\Delta\psi_\alpha\rangle$, is a direct consequence of the Rayleigh-Ritz principle. We now invoke a monotonicity theorem which states that, if the error in the effective potential matrix is negative (i.e., diagonal elements, in any representation, are negative), the eigenphase shifts, obtained by solving Eq. (2.17) with $V_{\alpha\beta}$ replaced by $V_{\alpha\beta v}$, will lie below the true eigenphases.[13] The variational calculation can then be supplemented by a rigorous validity criterion: The optimum choice of variational parameters in the trial function is the one which maximizes the eigenphases.

With the aid of a theorem derived previously,[14] the above minimum principle for the effective potential (or maximum principle for the eigenphase shifts) may be generalized to the case where H_A has M eigenvalues below the energy E. For completeness we restate the theorem as follows. Suppose we have constructed M linearly independent trial bound-state functions $|\phi_{it}\rangle$, $i=1, 2, \cdots M$, which are accurate enough so that the matrix $\mathbf{D}$, with elements

$$D_{ij}=\langle\phi_{it}|H_A-E|\phi_{jt}\rangle, \quad (2.31)$$

is negative. Then

$$\hat{H}_A-E\equiv H_A-E-\sum_{i,j=1}^{M}|(H_A-E)\phi_{it}\rangle(\mathbf{D}^{-1})_{ij}\times\langle\phi_{jt}(H_A-E)| \quad (2.32)$$

is a positive operator in the space of square integrable functions. It follows that if we adopt as our modified variational expression the matrix

$$V_{\alpha\beta v}'=V_{\alpha\beta v}-\sum_{i,j=1}^{M}\langle\Delta\psi_\alpha|(H_A-E)\phi_{it}\rangle(\mathbf{D}^{-1})_{ij}\times\langle\phi_{jt}(H_A-E)|\Delta\psi_\beta\rangle \quad (2.33)$$

then the modified error matrix takes the form

$$V_{\alpha\beta}-V_{\alpha\beta v}'=-\langle\Delta\psi_\alpha|\hat{H}_A-E|\Delta\psi_\beta\rangle. \quad (2.34)$$

Since $\hat{H}_A-E$ is positive, the minimum principle is established, just as in the previous case $M=0$. Note that while $\Delta\psi_\beta=\psi_\beta-\psi_{\beta t}$ in Eq. (2.33) is unknown, the variational expression is calculable, since the unknown function ψ_β appears in the form

$$(H_A-E)\psi_\beta=(H_A-E)G_A[V_A-V_A(\beta)]|\tilde{\chi}_\beta\rangle=-[V_A-V_A(\beta)]|\tilde{\chi}_\beta\rangle, \quad (2.35)$$

which *is* known.

In the *n-d* quartet calculation described in Sec. 3, the modified Hamiltonian H_A has no negative eigenvalues. This follows from the fact that H_A is constructed by removing purely attractive potential operators from the Hamiltonian H which supports no bound quartet states.

The short range of the two-body interaction has been an implicit assumption in the preceding discussion. We now wish to point out how (screened) Coulomb interactions can be handled in a fairly simple way.[15] Consider, for example, the problem of proton-deuteron scattering. If particles 2 and 3 are the protons, then, in the notation established above, the potential $V_A(1)$ contains a Coulomb contribution $V^c(1)$. According to Eq. (2.29) the symmetrized effective potential $V_{22}-V_{23}$ will contain the direct-scattering term $\langle\tilde{\chi}_2|V^c(1)|\tilde{\chi}_2\rangle$. We can extract the long-range Coulomb contribution to the effective potential from the above term by performing a multipole expansion of $V^c(1)$ [in configuration space] about the center of mass of the deuteron (particles 1 and 3) and isolate the monopole term. The symmetrized effective potential may then be thought of as the sum of two parts, the above monopole term and a remainder V_R arising from nuclear and shorter-ranged Coulomb forces. With the aid of the two-potential theorem (Sec. 2 A) we may express the scattering amplitude as the sum of a pure Coulomb amplitude plus a remainder which is obtained by solving a two-body Lippmann-Schwinger equation with V_R as input.[16] Such a calculation, while complicated by the distortion effects due to the Coulomb monopole potential, nevertheless still seems quite feasible.

3. APPLICATION TO *n-d* QUARTET SCATTERING

As a test of our formulation, we now consider elastic neutron-deuteron scattering in the quartet state. The two-body potentials are taken to be central, with a

[13] After an angular momentum decomposition Eq. (2.17) is replaced by a one-dimensional integral equation. It is more convenient to work with the reaction matrix rather than the T matrix. This amounts to replacing the $+i\eta$ prescription by the principal value prescription. [The function S in Eq. (2.17) is nonsingular for $E<0$]. The standard discussion of the reaction matrix formalism, as well as the proof of the monotonicity theorem, has been generalized in Ref. 7 to take into account the function S which modifies the kernel of the integral equation.

[14] L. Rosenberg, L. Spruch, and T. F. O'Malley, Phys. Rev. **118**, 184 (1960).

[15] One way to do this, which is different from the method we suggested above, is to replace the free Green's function which appears in the Faddeev equations by a modified Green's function which takes into account the Coulomb interactions. This method has the disadvantage that one must solve a three-body problem to determine the input to the Faddeev equations. Approximation methods for overcoming this difficulty are discussed by J. V. Noble, Phys. Rev. **161**, 945 (1967).

[16] While this statement is correct, there are some subtle points which arise due to the long range of the Coulomb potential. See G. B. West, J. Math. Phys. **8**, 942 (1967), especially the discussion leading to Eq. (C7).

symmetrical exchange mixture, and with the radial form

$$V(r)=V_0\exp(-r^2/b^2), \tag{3.1}$$

where

$$V_0=-86.4\text{ MeV}, \quad b=1.332\text{ F}.$$

We use the Christian-Gammel approximation[17] to the deuteron wave function for this interaction, which is in the form of a sum of three Gaussians. We shall confine ourselves to scattering in the $L=0$ state.

In order to construct the variational approximation $V_{\alpha\beta v}$ to the effective potential, we must first determine the vertex function $|\tilde{\chi}(z)\rangle$. It follows immediately from Eq. (2.9) that this function satisfies the integral equation

$$|\tilde{\chi}(z)\rangle=g_0(z)v|\chi\rangle+g_0(z)v_A|\tilde{\chi}(z)\rangle. \tag{3.2}$$

This equation was solved numerically. To allow an analytic evaluation of the integrations occurring in $V_{\alpha\beta v}$, the numerical solution for $|\tilde{\chi}\rangle$ was matched to a sum of three Gaussians. Since the energy of the two-particle subsystem described by the state $|\tilde{\chi}(E-\hbar^2k^2/2\mu);\mathbf{k}\rangle$ varies over the continuum $(-\infty, E)$ in intermediate states, Eq. (3.2) had to be solved for many values of z. The function S was then determined from Eq. (2.8).

The symmetrized effective potential $V_v=V_{22v}-V_{23v}$ is now specified by the choice of a trial Green's function G_{At}. We work in configuration space and take, for the present calculation, the simple separable form

$$G_{At}=c\exp[-(\sum_{i=1}^{3}\alpha_i(r_i^2+r_i'^2))X_Q^\dagger X_Q], \tag{3.3}$$

where r_i is the distance separating the ith pair, X_Q is the quartet spin function, and c and α_i are variational parameters which may be redetermined for each choice of the total energy E.

According to the minimum principle for the effective potential the optimum choice of linear variational parameters is that given by the usual Kohn[18] prescription, which in our case is just

$$\partial V_v/\partial c=0. \tag{3.4}$$

A more cumbersome procedure is required to determine the best set of nonlinear parameters. We have calculated the phase shift δ_v for a range of values of the parameters α_i and have searched for a maximum. (Recall that the variationally determined phase shift gives a lower bound on the true phase shift.)

The Born term, i.e., the first term on the right-hand side of Eq. (2.29), is the major contribution to the effective potential. The second term in Eq. (2.29) is ten percent of the Born term. The remaining variational contributions are one percent of the Born term. These variational contributions should contain the intrinsic three-body effects. It is not surprising that they turn out to be small, since the two neutrons are spatially well separated in the quartet state.

The reaction matrix satisfies an integral equation of the form shown in Eq. (2.17) where the effective potential is now V_v and where the principal-value prescription is used to avoid the pole in the propagator. This equation reduces, after an angular momentum decomposition, to a one-dimensional equation and is solved, in the usual manner, by replacing it with a set of algebraic equations.[19] Using a 50×50 Gaussian mesh, the solution was found to be stable to within 1% with respect to mesh variation.

In view of the dominance of the Born term we have confined our calculation to a single energy, $E=-1.2$ MeV, for which we obtain the result

$$k\cot\delta_v=-0.136,$$

in close agreement with previous calculations[20] and with experiment.[21] The doublet state is now under consideration, and the preliminary results indicate that the variational terms play a more important role here.

[17] R. S. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953).
[18] W. Kohn, Phys. Rev. **74**, 1763 (1948).
[19] To obtain the principal value, we use the method described by D. Y. Wong and G. Zambotti, Phys. Rev. **154**, 1540 (1967).
[20] For example, J. W. Humberston, Nuclear Phys. **69**, 291 (1965); R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **140**, B1291 (1965).
[21] W. T. H. Van Oers and J. D. Seagrave, Phys. Letters **24B**, 562 (1967).