Phase-Band Analysis—a Tool for Particle Reactions at High Energies*

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(Received 7 August 1968)

A modification of the traditional phase-shift analysis is suggested which is aimed at greatly reducing the number of parameters needed for the phenomenological description, and thereby making it possible to analyze a complete set of scattering data at energies which previously would have proven impractically high. Although devised with elementary-particle physics in mind, the new method might also find applications in nuclear physics.

ALTHOUGH in high-energy reactions many angular-momentum states are involved, and each of these must be parametrized by two real numbers, the phase-shift analysis as a phenomenological tool in the interpretation of such reactions has continued to be popular. The reason for this is simple: Such analyses have persisted in supplying interesting information by establishing particle resonances at quite high energies. As long as these resonances continue to appear predominantly in single isotopic and angular momentum states, the concept of partial-wave decomposition will remain useful.

At the same time, the difficulty of carrying out such analyses increases very fast as the energy rises, since the number of parameters to be determined from the data grows rapidly with the energy. This fact has discouraged extensive and complete experiments at higher energies, since it appeared doubtful that enough experimental information can be accumulated within a finite amount of time to yield a determination of such a large number of parameters.

The purpose of this paper is to suggest a new modification of the traditional phase-shift analysis which can drastically reduce the number of parameters, and might thereby make feasible an analysis of reactions at an energy which previously was felt to be impracticable.

The new scheme is based on some simple and plausible remarks. It appears both from general knowledge we have on particle reactions, as well as from specific analysis of certain reactions that if we follow a given partial-wave amplitude as a function of energy, its phase will first rise and contain no absorption. Then it will pass through an energy region where the amplitude might resonate, perhaps even repeatedly, while at the same time picking up more and more absorption. Finally it will "die down" again, becoming strongly absorptive and its real phase will change only slowly. The higher the angular momentum is, the larger the energy will be at which the "period of resonant glory" has passed and where the amplitude begins to settle down to a slow absorptive death.

Furthermore, it seems that the main interest in the phase shift parametrization of a reaction lies in the study of resonances. Thus, we are primarily interested in determining those phases which are in the position to resonate, and the search for the values of the other phases is, at the present stage of our knowledge, in most part just a tool to accomplish this aim. If, therefore, it turns out to be possible to get information on the "interesting" phases without having to face the arduous task of determining also all the "uninteresting" phases, the analysis is drastically simplified without losing any essential information.

I would like to suggest, therefore, a modified version of the usual phase-shift analysis, in which the partial waves from $j=0$ to $j=j_{\max}$ are divided into two groups. The first group, from $j=0$ to $j=j_0$, contains those partial waves whose amplitude has entered the highly absorptive and "uninteresting" region, while the top band of partial waves, from $j=j_0$ to $j=j_{\max}$, contains the "interesting," potentially resonating, and not too absorptive states. In this modified version of the phase-shift analysis (the "phase-band" analysis) the second group will be treated as usual, but the combined contribution of the first group will be calculated "collectively," assuming, for instance, a smooth rise in absorption with decreasing j, and a random distribution of the real part of the phase shifts. This collective description could itself contain some parameters, if needed, but many fewer than the total number of phases one would have to determine in the first group when using a conventional phase-shift analysis.

To give the above considerations a bit more quantitative basis, let us consider the scattering of spinless particles at such an energy that the conventional phase-shift analysis would be based on the expression

$$f(\theta,E)=\sum_{l=0}^{20}(2l+1)(2ik)^{-1}(\eta_l e^{2i\delta_l}-1)P_l(\cos\theta). \quad (1)$$

In this analysis one would have to determine 42 real numbers. In the phase-band analysis, this expression would be replaced by, say,

$$f(\theta,E)=\sum_{l=0}^{15}(2l+1)(2ik)^{-1}(\eta_l e^{2i\delta_l}-1)P_l(\cos\theta)+\sum_{l=16}^{20}(2l+1)(2ik)^{-1}(\eta_l e^{2i\delta_l}-1)P_l(\cos\theta) \quad (2)$$

in which the first sum on the right-hand side would be calculated collectively, containing perhaps 3–4 param-

* Research supported by the U. S. Atomic Energy Commission.

eters, while the second term would contain 10 real numbers to be determined from the data. Thus the total number of unknown parameters is reduced from 42 to 13 or 14. Such a reduction could easily represent the difference between what is practically feasible and what is not.

In order to explore some of the features of the method in the above example, a set of data was simulated by calculating the real and imaginary parts of the scattering amplitude from a set of tabulated "*Gedanken*" parameters δ_l and η_l, and adding to it an appropriate "experimental dispersion" to simulate experimental errors. The method was then applied to this set of amplitude "data" to recover the parameters which generated the "data" in the first place.

The results of these runs are given in Table I. The runs themselves are described in Table II.

As it is evident from the description of the runs, they were aimed at testing the following features (for nomenclature, see the caption of Table II):

(a) Dependence of the analysis on l_1. (Runs No. 1, 2, and 3.)

(b) Dependence of the analysis on the presence or absence of the small-scattering-angle region. (Runs No. 1 and 4.)

(c) Dependence on the input form of the δ_l's. (Runs No. 1, 2, 5, and 6.)

(d) Does the success of the phase-band analysis depend on our singling ot the *highest* angular momenta for individual treatment? (Runs No. 1, 7, 8, 9, and 10.)

The comparison of the relevant runs yields the following conclusions:

(1) The separation into an upper, individual band and a lower, collective remainder can be done successfully, and the values of the parameters of the upper band can be recovered to a quite satisfactory degree of accuracy, particularly if one remembers that what in a conventional analysis would have been described by 32 parameters is here described by only four.

(2) The procedure appears fairly stable with respect to small changes of l_1. The broader the phase band is within the small range tested, the better the output parameters in it approximate the input. There is also a tendency for the lower edge of the band to be less well determined than the rest.

(3) The functional form of the output δ in the collective range does not seem to matter very much, except that when the output and input functional forms are the same, the determination of the parameters is more accurate. This point, however, was not tested thoroughly.

(4) The presence or absence of data at small angles does not seem to have a serious effect on the accuracy of the fit.

(5) The success of the analysis does depend on the fact that we singled out the *highest* angular-momentum states. The two properties of these states which play a decisive role in the success are the weighting factors of $2l+1$ and the η_l values which are large compared to those of low angular-momentum states.

The above runs also showed that partial-wave-by-partial-wave comparison of the output with the input of the *collective* part does *not* show good agreement. This makes the success of the procedure in a sense more striking.

If looked at from a different point of view, the feasibility of this analysis is quite plausible. If we had infinitely accurate data in the whole, continuous range of angles, and these were decomposed in terms of an infinite number of angular momentum states, it would be possible to obtain each partial wave amplitude with infinite accuracy, *independently of the other partial-wave amplitudes*. In practice, however, the discrete set of data with errors, plus the truncated partial wave series, introduce complicated couplings among the amplitudes. What we have shown is that under certain circumstances, general physical considerations can partially undo these couplings.

Since this proposed scheme depends on certain assumptions outlined above, it might be useful to discuss them in a little more detail. We saw that the success of the method depends mainly on the fact that we separate out the high-angular-momentum states (i.e., the $2l+1$ factor), and that the low partial waves are fairly absorptive. The assumption that there be no resonances in the low partial waves is not essential for success. Thus, if Regge daughters or any other cause give rise to low-angular-momentum resonances at high energies, the scheme will still work as long as these resonances are quite absorptive.

The assumption that at high energies the low-angular-momentum states are, *on the average*, absorptive can be considered a model, although there is no known counterexample to date. Whether this model will persist to be valid in the future can be decided by the success or failure of this scheme. It seems unlikely that, using this scheme, one could get a statistically excellent fit by a collusion of false parameters and invalid assumptions. But, as in ordinary phase-shift analysis, certainty can never be guaranteed.

There are a number of internal checks that can be used within the scheme. Independent analyses at neighboring energies should "connect" fairly smoothly. The analysis should be stable under a variation of the boundary between the collective and individual bands. Also, eventually, there might be an experimental set of data on which both the conventional and this type of phase analysis can be tried out, thus offering a direct comparison between the two methods.

There are a number of ways the present scheme can be used as the initial stage of an approximation method to obtain the precise phases. For example, once the top

TABLE I. Comparison of input (in parenthesis) and output for $A_l \equiv \eta_l \sin\delta_l$ and $B_l \equiv \eta_l \cos\delta_l$ for the ten runs described in Table II. For run No. 3, the values for $l=14$ are $-0.03(0.04)$ and $-0.61(-0.73)$; and for $l=15$ are $0.68(0.61)$ and $0.59(0.51)$. For run No. 6, the values for $l=15$ are $0.71(0.71)$ and $0.36(0.36)$.

Run\	$l=$	16	17	18	19	20
1	A_l	0.70 (0.79)	0.23 (0.13)	0.27 (0.30)	0.67 (0.61)	0.55 (0.57)
	B_l	0.38 (0.30)	0.95 (0.90)	0.82 (0.77)	0.67 (0.63)	0.77 (0.73)
2	A_l	...	0.25 (0.13)	0.25 (0.30)	0.68 (0.61)	0.54 (0.57)
	B_l	...	1.07 (0.90)	0.87 (0.77)	0.71 (0.63)	0.80 (0.73)
3	A_l	0.78 (0.79)	0.15 (0.13)	0.29 (0.30)	0.63 (0.61)	0.56 (0.57)
	B_l	0.34 (0.30)	0.94 (0.90)	0.80 (0.77)	0.67 (0.63)	0.75 (0.73)
4	A_l	0.74 (0.79)	0.05 (0.13)	0.22 (0.30)	0.50 (0.61)	0.49 (0.57)
	B_l	0.23 (0.30)	0.85 (0.90)	0.74 (0.77)	0.59 (0.63)	0.70 (0.73)
5	A_l	0.74 (0.74)	0.79 (0.79)	0.72 (0.72)	0.75 (0.75)	0.80 (0.80)
	B_l	0.41 (0.42)	0.48 (0.46)	0.41 (0.41)	0.44 (0.44)	0.45 (0.46)
6	A_l	0.78 (0.78)	0.86 (0.86)	0.80 (0.80)	0.85 (0.85)	0.91 (0.91)
	B_l	0.33 (0.33)	0.30 (0.30)	0.22 (0.22)	0.17 (0.17)	0.11 (0.11)
Run\	$l=$	0	1	2	3	4
7	A_l	−0.04 (0.002)	0.17 (0.12)	0.11 (0.16)	0.16 (0.05)	−0.01 (0.11)
	B_l	−0.21 (0.10)	0.35 (−0.03)	−0.28 (0.82)	0.81 (0.19)	−0.47 (0.20)
8	A_l	−0.03 (0.02)	0.92 (0.85)	0.68 (0.74)	0.41 (0.24)	0.26 (0.41)
	B_l	0.82 (0.92)	−0.08 (−0.19)	−0.27 (0.38)	1.03 (0.88)	0.03 (0.74)
9	A_l	−0.002 (0.001)	0.026 (0.023)	0.020(0.023)	0.012(0.006)	0.004(0.011)
	B_l	0.026 (0.033)	0.004(−0.005)	0.003(0.012)	0.036(0.022)	0.004(0.018)
10	A_l	0.58 (0.66)	11.2 (11.0)	5.36 (5.47)	1.53 (1.18)	1.17 (1.51)
	B_l	37.8 (37.8)	−2.45 (−2.54)	2.84 (2.79)	4.49 (4.40)	2.88 (2.72)

TABLE II. Description of the various runs whose results are tabulated in Table I. Nomenclature: l_{max} = the highest angular momentum considered; l_1 = the lowest angular momentum treated noncollectively; l_2 = the highest angular momentum treated noncollectively; "sc incr" denotes scattering around a curve increasing from about 0.1 at $l=0$ to about 0.95 at $l=20$; "sc decr" denotes scattering around a curve decreasing from about 0.95 at $l=0$ to about 0.1 at $l=20$. All collective η_l outputs were of the form $a+bl+cl^2$, and all δ_l of the form nl. In all cases l_{max} was 20. "Rand" denotes random. Unless otherwise indicated, the full angular range 0°–180° was used.

\Run No.	1	2	3	4	5	6	7	8	9	10
l_1	15	16	13	15	16	14	0	0	0	0
l_2	20	20	20	20	20	20	4	4	4	4
η_l input	sc incr	sc incr	sc incr	sc incr	sc incr	sc incr	sc incr	sc decr	$(2l+3)^{-1}$ ×sc incr	$[(43-2l)/(2l-1)]$ ×sc decr
δ_l input	rand	rand	rand	rand	$=20°$, $l=0-9$ $=60°$, $l=10-20$	$(4l+3)°$	rand	rand	rand	rand
Angular range	full	full	full	12°–180°	full	full	full	full	full	full

band of phases are determined by a phase-band analysis, they could be held fixed in a second phase-band analysis in which the next, somewhat lower band of phases are determined, etc. Again, internal checks are available by releasing the top phase-band after the second analysis to see if they change much. In general, even if the phase-band idea does not furnish very accurate information, it might be used to obtain plausible initial values for a complete phase-shift analysis, thus saving much computing time that would be spent on random searches in a complete phase-shift analysis.

The success of the method also depends on the assumption that the collective band contains enough partial waves to be able to be treated statistically. What "enough" means in this context must also be determined by actual analysis of real data. I would guess, however, that the collective band would have to contain at least of the order of ten values of l before the method will work. Thus, for instance, pion-nucleon scattering would be relevant roughly around 10 BeV, although perhaps one could try it even at, say, 7 BeV. The method could be applied to the scattering of pions on nuclei. For example, it should work for medium light nuclei even below 1 BeV, especially for zero spin where only differential cross section would have to be measured. There are a host of other nuclear reactions at medium energies where this method might come into consideration.

To the best of my knowledge, there is no elementary-particle reaction at present for which a sufficient amount of data is available at a high enough energy so that it would be possible to try out a phase-band analysis. High-energy measurements have been aimed primarily at checking special models for some aspects of high-energy reactions. Perhaps the method of analysis proposed in this note will contribute to providing an

incentive for more extensive measurements in pion-nucleon, kaon-nucleon, and nucleon-nucleon reactions, in high-energy photoproduction, as well as in a number of interesting nuclear reactions belonging to what has come to be called intermediate-energy physics.

The idea of a phase-band analysis occurred to me while attending the Conference on Pion-Nucleon Scattering at Irvine. I am grateful to the University of California at Irvine, and to Gordon Shaw in particular, for organizing this stimulating meeting. I had a number of very valuable discussions about this problem with Paul Csonka. I am also grateful to David L. Bridges for a rapid and imaginative performance of the numerical computations, and to the Statistical Laboratory and Computing Center of the University of Oregon for cooperation.

PHYSICAL REVIEW VOLUME 177, NUMBER 5 25 JANUARY 1969

Self-Consistent Multiple-Quark-Scattering Analysis and Its Application to Elastic pp Scattering*

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We discuss a viewpoint for interpreting elastic scattering of hadrons in terms of a picture in which the hadrons behave as if they are comprised effectively of A distinct subparticles Q which contribute essentially individually to multiple-internal-scattering processes within the hadrons. The formalism of this multiple-internal-scattering picture is developed and applied to the analysis of elastic pp scattering. The discontinuities in the slope of the pp differential cross section at large momentum transfers are interpreted as transitions between domains of momentum transfer that are dominated by successively higher-order multiple-scattering contributions. The structure of the pp cross section is fitted in good detail with a self-consistent analysis that circumvents the necessity for conjectures about the wave functions of internal motion of the subparticles Q within p by exploiting simpler and more direct conditions and conjectures on the effective generalized form factors and scattering amplitudes. The higher-order multiple-scattering contributions are self-consistently calculated in terms of the effective QQ scattering amplitude determined from the region of the first slope. The analysis distinguishes among subparticle models to yield an essentially exclusive fit to the experimental data with the value of the quark number of the proton $A=3$, thereby affording corroborative evidence in favor of the SU_3 quark model from a non-group-theoretical, dynamical basis. Our results are compatible with quarks of very small, or even pointlike, spatial extension as compared to the effective electromagnetic radius of the proton.

I. INTRODUCTION

ELASTIC pp (proton-proton) scattering experiments indicate that the differential cross section at sufficiently high energies tends toward a characteristic structure that is more or less energy-independent, suggestive of a semiclassical regime where diffractive effects are perhaps dominant. The structure of the differential cross section shows, in addition to a narrow forward diffraction peak, several succeeding domains at larger momentum transfers in each of which the cross section decreases at a successively slower rate.[1] The transitions between these successive domains are rather abrupt, suggestive of contributions from different orders of physical processes. All hadron elastic-scattering cross sections share these principal characteristics to some degree and our analysis should apply generally as well as to the pp case. We discuss here a viewpoint for interpreting these elastic scattering data in terms of the proton, or other hadron, behaving as if it were comprised effectively of A internal subnucleonic particles Q which contribute effectively individually to multiple-scattering processes within the proton. The successive domains of the differential cross section are to be identified as contributions of successive orders of multiple internal diffraction scattering of the Q's.

Antecedents of the Multiple-Internal-Scattering Picture

Several antecedent analyses have contributed to the suggestion of the multiple-internal-scattering picture: (i) The differential cross sections for elastic scattering of high-energy protons from light nuclei also exhibit breaks in the angular distributions that are similar to those in pp scattering, and recently these data have been successfully analyzed in terms of multiple internal nucleon-nucleon diffraction scattering within the nucleus.[2] (ii) Analysis of several hadronic total cross sec-

* Supported in part by National Science Foundation, Grant No. GP-8924. Computer facilities supported by National Science Foundation, Grant No. G-22294.
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