Current Commutators and Electron Scattering at High Momentum Transfer*

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Sum rules for electron-proton total cross sections are deduced from the vanishing of various equal-time commutators, and it is shown how these cross sections determine the (spin-averaged) proton expectation values of all equal-time commutators involving components of the electric current and their time derivatives. Mild restrictions on the asymptotic behavior of electromagnetic form factors are also obtained.

ELECTRON scattering on a proton at rest is described by the cross section (electron mass=0)

$$\frac{d^2\sigma^{ep}}{dq^2d\nu}=\frac{\alpha^2}{M^2E^2}\frac{1}{q^2}[(2M^2EE'-\tfrac{1}{2}q^2M^2)A_1(q^2,\nu)+\nu^2A_1(q^2,\nu)-q^2A_2(q^2,\nu)],\quad(1)$$

where $\alpha=e^2/4\pi=(137)^{-1}$, $E(E')$ is the initial (final) electron energy, $q_\mu=e_\mu-e_\mu'$=momentum imparted to proton, $\nu=(E-E')M=-q\cdot p$ (p=proton momentum), and the A_i are the absorptive parts of the forward, shell Compton amplitudes F_i defined by

$$T_{\mu\nu}(q,p)=i\int d4x\, e^{-i\mathbf{q}\cdot x}\langle p|T(j_\mu(x)j_\nu(0))|p\rangle_c,\quad(2a)$$

$$T_{\mu\nu}(q,p)=[q^2p_\mu p_\nu+\nu(q_\mu p_\nu+q_\nu p_\mu)+\nu^2\delta_{\mu\nu}]F_1+(q_\mu q_\nu-q^2\delta_{\mu\nu})F_2,\quad(2b)$$

where j_μ is the electric current, and a spin average is implicit. The subscript c indicates the covariant time-ordered product; thus $T_{\mu\nu}$ differs from the ordinary time-ordered product by a polynomial in q if the equal-time commutator of j_0 and j_i has a connected matrix element.

Bjorken[1] has pointed out that for large q_0 and fixed $\mathbf{q}$, the coefficient of q_0^{-l-1} ($l\geq 0$) in an expansion of $T_{\mu\nu}$ gives the matrix element of the equal-time commutator of the electric current and its lth time derivative; in particular,

$$T_{\mu\nu}\xrightarrow{q_0\to\infty}\text{polynomial in }q_0-\sum_{0=l}^{\infty}\frac{(-i)^l}{q_0^{l+1}}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\times\langle p|[\partial_0{}^lj_\mu(\mathbf{x},0),j_\nu(0)]|p\rangle.\quad(3)$$

Thus, all of these commutators are determined by the coefficients $C_i{}^l(\mathbf{q}\cdot\mathbf{p},\mathbf{q}^2,p_0)$ occurring in

$$F_i\xrightarrow{q_0\to\infty}\text{polynomial}+\sum_{l=0}^{\infty}C_i{}^lq_0{}^{-l-1}.\quad(4)$$

We wish to show how the $C_i{}^l$ can be constructed from the A_i, and thus from the electron-proton scattering data.

We assume that each of the $F_i(q^2,\nu)$ satisfy the DGS representation[2,3]

$$F_i=\sum_{m=0}^{Mi}F_i{}^m=\frac{1}{\pi}\sum_{m=0}^{Mi}\int_0^\infty d\sigma\int_{-1}^{1}d\beta\frac{\nu^mh_i{}^m(\sigma,\beta)}{q^2+2\beta\nu+\sigma},\quad(5)$$

where, by crossing symmetry,

$$h_i{}^m(\sigma,-\beta)=(-1)^mh_i{}^m(\sigma,\beta).\quad(6)$$

Expanding this form of F_i as in Eq. (4), we obtain after some combinatorial calculation

$$C_i{}^l=\sum_{n,s,t}(-\mathbf{q}\cdot\mathbf{p})^{l-1-2s+2n}(\mathbf{q}^2)^{s-2n-t}(p_0)^{2s-l+1}\times\frac{s!(2n)!}{(2s-l+1)!(l-1-2s+2n)!(s-2n-t)!}K_i{}^{nt},\quad(7a)$$

where

$$K_i{}^{nt}=\sum_{m=0}^{Mi}\frac{1}{(2n-m)!(t+m)!}\times\int_0^\infty d\sigma\int_{-1}^{1}d\beta(2\beta)^{2n-m}\sigma^{t+m}h_i{}^m(\sigma,\beta),\quad(7b)$$

and where the sum over n, s, and t is restricted by $n\geq 0$, $s\geq 0$, and the requirement that the arguments of all the factorials are non-negative. Thus, for example, $t\geq$ maximum $(-m,-2n)$.

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[1] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); see also D. G. Boulware and S. Deser, Phys. Rev. Letters **20**, 1399 (1968); Phys. Rev. **175**, 1912 (1968); H. Epstein and R. Jackiw, Harvard University Report, 1968 (unpublished); R. Brandt and J. Sucher, Phys. Rev. Letters **20**, 1131 (1968); P. Olesen, Phys. Rev. **172**, 1461 (1968); J. M. Cornwall and R. E. Norton, *ibid.* **173**, 1637 (1968).

[2] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959); M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1151 (1960); N. Nakanishi, *ibid.* **26**, 337 (1961); Suppl. **18**, 70 (1961).

[3] There also could be a polynomial in q^2 in the numerator of Eq. (5), but it could be eliminated in favor of polynomials in ν and in $q^2+2\beta\nu+\sigma$. The latter contribute neither to the commutators nor to the absorptive parts and hence are ignored.

The absorptive parts A_i of the F_i can be read off from Eq. (5):

$$A_i=\sum_{m=0}^{Mi} A_i{}^m=\sum_{m=0}^{Mi}\int d\sigma d\beta\, \nu^m h_i{}^m \delta(q^2+2\beta\nu+\sigma)\,, \quad (8)$$

and hence, for $n=1, 2, 3, \cdots$,

$$\int_0^\infty \frac{d\nu}{\nu^{2n+1}}A_i=\sum_{\substack{m=0\\2n-m\geq 0}}^{Mi}(-1)^m \times\int_0^\infty d\sigma\int_{-1}^0 d\beta\frac{(2\beta)^{2n-m}}{(q^2+\sigma)^{2n+1-m}}h_i{}^m. \quad (9)$$

As indicated, the sum on m includes only terms for which $2n-m\geq 0$.[4] We will show in a moment that the right-hand side of Eq. (9) can be expanded for large q^2 in terms of the $K_i{}^{nt}$ occurring in Eqs. (7). However, to complete this connection, we first must extend Eq. (9) to $n=0$.

Assuming Regge asymptotic behavior of the A_i for large ν, we guess that[5]

$$A_1(q^2,\nu)\xrightarrow{\nu\to\infty}0\,, \quad (10a)$$

$$A_2(q^2,\nu)\xrightarrow{\nu\to\infty}f_p(q^2)\nu+\sum_\alpha f_\alpha(q^2)\nu^\alpha\,, \quad (10b)$$

where, in addition to the Pomeranchon, the sum on α includes whatever other trajectories contribute with $0<\alpha<1$; for example, the A_2 and f_0. Because of Eq. (10a), the integral on the left-hand side of Eq. (9) exists for A_1 when $n=0$,[6] and Eq. (9) holds for this case. However, for A_2 the left-hand side of Eq. (9) must be modified for $n=0$.

As suggested by the form of Eq. (5), we assume that the Regge limit in Eq. (10b) arises only from the terms with $m\geq 1$ in Eq. (8). That is, we assume that (10b) is satisfied with A_2 replaced by $A_2-A_2{}^0$. Since $F_2-F_2{}^0$ vanishes at $\nu=0$, a subtracted dispersion relation for this difference reads

$$F_2-F_2{}^0=\frac{2\nu^2}{\pi}\int_0^\infty\frac{d\nu'}{\nu'(\nu'^2-\nu^2)} \times[A_2(q^2,\nu')-A_2{}^0(q^2,\nu')], \quad (11)$$

and if we add and subtract the right-hand side of (10b) to the integrand of this expression, and in the first case do the integral explicitly, we obtain

$$F_2-F_2{}^0=if_p\nu-\sum_\alpha\frac{(1+e^{-i\pi\alpha})}{\sin\pi\alpha}f_\alpha\nu^\alpha+\frac{2\nu^2}{\pi}\int_0^\infty\frac{d\nu'}{\nu'(\nu'^2-\nu^2)} \times[A_2-A_2{}^0-f_p\nu'-\sum_\alpha f_\alpha\nu'^\alpha]. \quad (12)$$

Let us now assume that the behavior of F_2 for large ν is given entirely by the Regge terms, namely, that there is no part of F_2 constant in ν in this limit. Then, since both $F_2{}^0$ and the brackets in the integrand of Eq. (12) vanish as $\nu\to\infty$, it follows that[7]

$$\int_0^\infty\frac{d\nu}{\nu}[A_2-A_2{}^0-f_p\nu-\sum_\alpha f_\alpha\nu^\alpha]=0\,, \quad (13)$$

which from (8) gives

$$\int_0^\infty\frac{d\nu}{\nu}[A_2-f_p\nu-\sum_\alpha f_\alpha\nu^\alpha]=\int_0^\infty d\sigma\int_{-1}^0 d\beta\frac{h_2{}^0(\sigma,\beta)}{q^2+\sigma}. \quad (14)$$

The right-hand side of this expression is the same as the right-hand side of (9) with $n=0$ and $i=2$. Thus Eq. (9) can be extended to $n=0$ with no change in form for $i=1$, and with (9) replaced by (14) for $i=2$. Summarizing this result, and expanding the right-hand side of (9) in a power series in $(q^2)^{-1}$, we obtain, for $n=0, 1, 2, \cdots$,

$$\int_0^\infty\frac{d\nu}{\nu^{2n+1}}[A_i-\delta_{i2}\delta_{n0}(f_p\nu+\sum_\alpha f_\alpha\nu^\alpha)] =\tfrac{1}{2}\sum_t(-1)^t\frac{(2n+t)!}{(q^2)^{2n+1+t}}K_i{}^{nt}, \quad (15)$$

with $K_i{}^{nt}$ given in (7b).

The connection between the commutators in Eq. (3) and the integrals for $n\geq 0$ on the left-hand side of (15) can be read off from Eqs. (2)–(4), (7), and (15). In principle, the integrals in Eq. (15), and thus the $K_i{}^{nt}$, can be determined from the electron scattering data, and from these results the $C_i{}^l$ and the matrix elements of all commutators in Eq. (3) can be constructed.

Rather than pursue the connection between commutators and the integrals in (15) in more detail, let us present some restrictions on the electron scattering cross sections which follow from the vanishing of various equal-time commutators. These restrictions can be derived straightforwardly from our previous results. Actually, there are many more restrictions [involving higher values of n in (15)] than the ones we list. However, all these others are implied from the relations that we write explicitly, because of the conditions

$$0\leq A_1\,, \quad (16a)$$

$$-M^2A_1\leq A_2\leq \nu^2/q^2A_1\,. \quad (16b)$$

These inequalities follow from the definitions in Eq. (2), or, equivalently, from the requirement of non-negative cross sections for both transverse and longitudinally

[4] To show this: (a) Write a twice-subtracted dispersion relation for $F_i{}^m$ $(m>2n\geq 2)$ noting that both subtraction constants vanish, since $F_i{}^m\to\nu^m$ for small ν; (b) expand the result for small ν and set the coefficient of ν^{2n} $(2n<m)$ equal to zero.

[5] H. Harari, Phys. Rev. Letters 17, 1303 (1966).

[6] For $A_1{}^m$, $n=0$, and $m=1$, the argument of Ref. 4 restricting $2n-m\geq 0$ is applicable if an unsubtracted dispersion relation is used.

[7] Similar relations are the basis of the finite-energy sum rules. See R. Dolen, D. Horn, and S. Schmid, Phys. Rev. Letters 19, 402 (1967); Phys. Rev. 166, 1765 (1967); L. A. P. Balázs and J. M. Cornwall, *ibid.* 160, 1313 (1967).

polarized photons.[8] Since $2\nu > q^2$ when $A_i \neq 0$ in the integrand of (15), these inequalities are useful in allowing us to conclude that if

$$(q^2)^p \int \frac{d\nu}{\nu^{2n+1}} A_1 \xrightarrow{q^2\to\infty} 0, \qquad (17a)$$

then

$$(q^2)^{p+2} \int \frac{d\nu}{\nu^{2n+3}} A_1 \xrightarrow{q^2\to\infty} 0, \qquad 17b)$$

$$(q^2)^{p+1} \int \frac{d\nu}{\nu^{2n+3}} A_2 \xrightarrow{q^2\to\infty} 0. \qquad (17c)$$

The restrictions listed below can be extended to larger values of n by Eqs. (17). They are similar in form, and also in their origin, to relations discussed recently by Bander and Bjorken.[9]

(i) If $\langle p|[j_0(\mathbf{x}), j_0(0)]|p\rangle = 0$, then for $i=1, 2$

$$q^2 \int \frac{d\nu}{\nu^3} A_i(q^2,\nu) \xrightarrow{q^2\to\infty} 0. \qquad (18)$$

(ii) If $\langle p|[j_0(\mathbf{x}), j_i(0)]|p\rangle = 0$, then for $i=1, 2$

$$q^2 \int_0^\infty \frac{d\nu}{\nu} [A_i - \delta_{i2}(f_p\nu + \sum_\alpha f_\alpha \nu^\alpha)] \xrightarrow{q^2\to\infty} 0. \quad (19a)$$

(iii) If $\langle p|[j_i(\mathbf{x}), j_k(0)]|p\rangle = 0$, then there are no further restrictions if (i) and (ii) hold.

(iv) If $\langle p|[\partial_0 j_i(\mathbf{x}), j_k(0)]|p\rangle = 0$, as well as (i) and (ii), then for $i=1, 2$

$$(q^2)^2 \int_0^\infty \frac{d\nu}{\nu} [A_i - \delta_{i2}(f_p\nu + \sum_\alpha f_\alpha \nu^\alpha)] \xrightarrow{q^2\to\infty} 0, \quad (20)$$

and because of (16a) this has the amusing consequence that the nucleon charge and magnetic (Dirac) form factors—f_1 and f_2, respectively—must vanish for large q^2 according to

$$f_1^2(q^2) \xrightarrow{q^2\to\infty} 0, \qquad (20')$$

$$q^2 f_2^2(q^2) \xrightarrow{q^2\to\infty} 0. \qquad (20'')$$

Similarly, from (19a) it follows that

$$q^{-2} f_1^2(q^2) \xrightarrow{q^2\to\infty} 0, \qquad (19b)$$

$$f_2^2(q^2) \xrightarrow{q^2\to\infty} 0. \qquad (19c)$$

Thus, for example, the existence of a c-number Schwinger term implies that the magnetic form factor f_2 must vanish as q^2 becomes infinite. Although these restrictions on the form factors are rather mild, to our knowledge they are the first to be obtained from general arguments.

The conditions (18)–(20) for A_1 can be expressed simply in terms of high-energy high-momentum-transfer sum rules for the electron scattering cross section in Eq. (1). The amplitude A_2 makes its presence felt for backward scattering angles[10] and is relatively more difficult to extract from the data. For A_1,

$$\int \frac{d\nu}{\nu} \frac{d^2\sigma^{ep}}{dq^2 d\nu} \xrightarrow{E\to\infty} \frac{2\alpha^2}{q^2} \int_0^\infty \frac{d\nu}{\nu} A_1 + O\left(\frac{1}{E}\right). \qquad (21)$$

Thus, for example, if the commutators (i)–(iv) are all satisfied, as has been suggested,[11] then it follows from (20) and (21) that

$$\lim_{E\to\infty} (q^2)^3 \int \frac{d\nu}{\nu} \frac{d^2\sigma^{ep}}{dq^2 d\nu} \xrightarrow{q^2\to\infty} 0. \qquad (22)$$

If only (i)–(iii) are valid, we would expect the right-hand side of (22) to be a (nonzero) constant. In fact, for the Sugawara model,[12] this constant can (almost) be calculated exactly.[13]

To give some perspective to these results, let us compare them with the inequalities derived previously by Bjorken.[10,14] The first inequality, derived from isospin manipulations on Adler's sum rule,[15] reads (for all q^2)

$$q^2 \int_0^\infty d\nu\, A_{1\,(\text{isoscalar})} \geq \tfrac{1}{2}\pi. \qquad (23)$$

The second inequality, based upon quark commutation rules for the space components of the isospin current, is

$$q^2 \int_0^\infty d\nu \left(A_1 - \frac{q^2}{\nu^2} A_2\right)_{\text{isoscalar}} \xrightarrow{q^2\to\infty} \geq \tfrac{1}{2}\pi.$$

It is clear that we have been optimistic in assuming the existence of all moments of the h_i^m in Eq. (7b). It is a likely possibility that, for t greater than some value, the K_i^{nt} do not exist; that is, essentially, that for high-order derivatives, the commutators in Eq. (3) are not defined. An interesting possibility[16] is that the integrals in Eq. (15) $\sim\exp[-\sqrt{(q^2)}]$. The K_1^{nt} determined from scattering data according to Eq. (15) would then all be zero, and the connection given in (7) between the commutators and the K_i^{nt} would break down. Except for this kind of occurrence, conditions (i)–(-v) are reversible; that is, if they are satisfied, then the corresponding commutators vanish.

Finally, we remark that radiative corrections and/or multiple photon exchange would tend to decrease the significance of our results.

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[11] See, e.g., T. D. Lee, Columbia University Report, 1967 (unpublished).
[12] H. Sugawara, Phys. Rev. **170**, 1659 (1968).
[13] C. Callan and D. Gross, Phys. Rev. Letters **21**, 311 (1968); D. Gross, Harvard University Report, 1968 (unpublished).
[14] J. D. Bjorken, Phys. Rev. Letters **16**, 408 (1966).
[15] S. L. Adler, Phys. Rev. **143**, 1144 (1966).
[16] R. Brandt and J. Sucher (Ref. 1).