value for $h^2/4\pi$ less than 1.5 is in poor agreement with experiment, it is highly likely that $M_{\rho^\pm}-M_{\rho^0}\gtrsim -1.2$ MeV for $M_{K^{*0}}-M_{K^{*\pm}}=8$ MeV, $M_{\rho^\pm}-M_{\rho^0}\gtrsim -0.8$ MeV for $M_{K^{*0}}-M_{K^{*\pm}}=6.3$ MeV, and $M_{\rho^\pm}-M_{\rho^0}\gtrsim -0.6$ MeV for $M_{K^{*0}}-M_{K^{*\pm}}=4$ MeV.

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Sum Rules in $\pi\Sigma$ Scattering and the $\pi\Lambda\Sigma$ and $\pi\Sigma\Sigma$ Coupling Constants

ALLEN E. EVERETT*

Department of Physics, Tufts University, Medford, Massachusetts 02155

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Following a procedure proposed recently by Atkinson, we derive sum rules for the A and B amplitudes for the reaction $\pi^-+\Sigma^+\rightarrow\pi^++\Sigma^-$ by equating unsubtracted dispersion relations at $t=0$ and at fixed u or fixed backward angle. Combining these with the already known superconvergence relation for the B amplitude, and assuming the sum rules can be saturated with known resonances, we obtain three equations for two unknown coupling constants $g_{\pi\Lambda\Sigma}{}^2$ and $g_{\pi\Sigma\Sigma}{}^2$. Choosing to fix $u=0$, one obtains values of the coupling constants an order of magnitude larger than expected on the basis of, say, $SU(3)$. We argue that this is probably because of the large extrapolation to unphysical values of $\cos\theta$ required in evaluating the fixed-u dispersion relation for $u=0$. Taking u to be positive in such a way as to minimize the required extrapolations in angle, or choosing fixed $\cos\theta=-1$, one obtains results that are reasonably consistent with one another and with $SU(3)$, to within estimated uncertainties of 50% or more, resulting from experimental error in the resonance widths, large cancellations between the contributions of different resonances, and unknown nonresonant-background and high-energy contributions.

RECENTLY Atkinson has shown that one can obtain sum rules in certain cases, by equating two different unsubtracted dispersion relations, with different variables held fixed, for the same scattering amplitude.[1] To summarize the procedure briefly, he considers an amplitude that satisfies an unsubtracted dispersion relation with either the Mandelstam variable t or u held fixed. (We will work in the channel where s is the total center-of-mass energy squared.) One can then write

$$A(s,t,u)=\int\frac{\mathrm{Im}A(s',t,\tau-s'-t)ds'}{s'-s}=\int\frac{\mathrm{Im}A(s',\tau-s'-u,u)ds'}{s'-s},\quad(1)$$

where

$$\tau=\sum_{i=1}^{4}m_i{}^2=s+t+u\quad(2)$$

and m_i is the mass of one of the two incident or two outgoing particles. Provided no subtractions are needed in either dispersion relation, the second equality in Eq. (1) then yields a sum rule for ImA.

In this paper, we are going to apply this technique to both the A and B amplitudes for the reaction

$$\pi^-+\Sigma^+\rightarrow\Sigma^-+\pi^+.\quad(3)$$

(We use the standard notation, as given, for example, in the paper of Frautschi and Walecka,[2] for the invariant amplitudes in pion-baryon scattering.) This reaction corresponds to isospin 2 in the t ($\pi\pi\rightarrow\Sigma\bar{\Sigma}$) channel. Since at large s and fixed t one has[3]

$$\mathrm{Im}A(s,t)\sim s^{\alpha(t)},\quad \mathrm{Im}B(s,t)\sim s^{\alpha(t)-1},\quad(4)$$

where $\alpha(t)$ is the leading Regge trajectory in the t channel, the usual assumption that no $I=2$ trajectory reaches $j=0$ for $t<0$ implies that B is actually superconvergent, while A satisfies a dispersion relation with no subtractions, provided t is held fixed at a value less than or equal to zero. The superconvergence relation for B has been studied previously.[4,5] If one assumes that the superconvergent sum rule is saturated by the Λ and Σ poles and the known resonances in the $\Sigma\pi$ system, one obtains a relationship between the $\pi\Lambda\Sigma$ and $\pi\Sigma\Sigma$ coupling constants and the experimentally measurable masses and widths of the resonances. Since there is only one equation, the superconvergence relation by itself is not sufficient to determine the values of the coupling constants, unless one invokes $SU(3)$ and a specific value for the d/f ratio. The two additional Atkinson-type sum rules we obtain here, combined with the

* Supported in part by the U. S. Atomic Energy Commission.

[1] R. Atkinson, III, Phys. Rev. **169**, 1293 (1968).

[2] S. Frautschi and J. Walecka, Phys. Rev. **120**, 1486 (1960).

[3] S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

[4] P. Babu, F. Gilman, and M. Suzuki, Phys. Letters **24B**, 65 (1967).

[5] G. Dass and C. Michael, Phys. Rev. **162**, 1403 (1967).

superconvergence relation, actually overdetermine the coupling constants, so that we can obtain, in principle, not only a value for the coupling constants in terms of the resonance parameters, but also a check on the self-consistency of the theory. As we shall see, there are some problems in carrying out this program. These arise both from the fact that the values obtained for the coupling constants are sensitive to the values at which t and u are fixed in obtaining the sum rules, and also from the experimental uncertainties in evaluating them. We shall make some arguments as to what are reasonable values of t and u to select; using these values, we find the sum rules are mutually consistent, and the values obtained for the coupling constants agree with what one would expect from $SU(3)$ considerations, to within the large estimated experimental uncertainties, of the order of 50%, in evaluating the sum rules.

We now write down our three sum rules. The superconvergence relation for the B amplitude of reaction (3) has the form

$$g_\Lambda^2-g_\Sigma^2=-\int_{s_0}^{\infty}\mathrm{Im}B(s',t)ds'\,,\qquad(5)$$

where $s_0=(1+\Lambda)^2$. We have set the pion mass equal to 1, and will use Λ and Σ to stand for the masses of the corresponding particles. The coupling constants g_Λ and g_Σ are, respectively, equal to the $\pi\Lambda\Sigma$ and $\pi\Sigma\Sigma$ coupling constant $g_{\pi\Lambda}$ and $g_{\pi\Sigma}$ defined by Martin and Wali[6] multiplied by $(4\pi)^{1/2}$. The Atkinson-type sum rules for the A and B amplitudes are

$$\frac{-g_\Lambda^2}{\Lambda^2-u}+\frac{g_\Sigma^2}{\Sigma^2-u}=-\int_{s_0}^{\infty}\frac{\mathrm{Im}B(s',\,t,\,\tau-s'-t)ds'}{s'-s}-\int_{-\infty}^{\tau-t-s_0}\frac{\mathrm{Im}B(s',\,t,\,\tau-s'-t)ds'}{s'-s}+\int_{s_0}^{\infty}\frac{\mathrm{Im}B(s',\,\tau-s'-u,\,u)ds'}{s'-s}\qquad(6)$$

and

$$\frac{(\Sigma-\Lambda)g_\Lambda^2}{\Lambda^2-u}=-\int_{s_0}^{\infty}\frac{\mathrm{Im}A(s',\,t,\,\tau-s'-t)ds'}{s'-s}-\int_{-\infty}^{\tau-t-s_0}\frac{\mathrm{Im}A(s',\,t,\,\tau-s'-t)ds'}{s'-s}+\int_{s_0}^{\infty}\frac{\mathrm{Im}A(s',\,\tau-s'-u,\,u)ds'}{s'-s}\,,\qquad(7)$$

where s, t, and u obey Eq. (2). The direct-channel pole terms are absent from Eqs. (6) and (7), since they cancel out when the difference of the fixed-t and fixed-u dispersion relations is taken.

[6] A. Martin and K. Wali, Phys. Rev. **130**, 2455 (1963).

At fixed u, the asymptotic behavior of both A and B is given by[3,7]

$$\mathrm{Im}A(s,u)\sim s^{\alpha(u)-1/2}\,,\quad \mathrm{Im}B(s,u)\sim s^{\alpha(u)-1/2}\,,\qquad(8)$$

where $\alpha(u)$ is the position of the leading Regge trajectory in the u channel. For our problem, this will be either the Λ trajectory or the $Y_1^*(1385)$ trajectory. Clearly $\alpha_\Lambda(u)<\frac{1}{2}$ for $u<\Lambda^2$. The situation as regards the $Y_1^*(1385)$ trajectory is much less certain. At least two rather different forms for the $N^*(1238)$ trajectory have been suggested[8,9]. It seems reasonable to assume that the $Y_1^*(1385)$ trajectory is probably roughly parallel to that of the $N^*(1238)$. One would appear to be on safe ground in saying that $\alpha(u)<\frac{1}{2}$ for the $Y_1^*(1385)$ trajectory when $u=0$; it is not clear how positive u may be taken and have this condition continue to hold. This question will be important to us later; for the moment we simply remark that the foregoing discussion, combined with Eqs. (8), indicate that our sum rules (6) and (7) should certainly be valid for $u=0$.

We make the usual approximation that the integrals in (5), (6), and (7) are saturated by pole terms and low-lying resonances. One simplification here, over the πN case studied by Atkinson, is that the t channel, having isospin 2, has no low-lying particles or resonances. Hence, in accord with the spirit of our approximations, we neglect the integral over the left-hand cut in the fixed-u dispersion relation, since the discontinuity across that cut is proportional to the t-channel absorptive part. We take the contribution of a resonance of isospin I, total angular momentum j, and orbital angular momentum l to the amplitudes for reaction (3) to be given by

$$f_{lj}=c_I(\Gamma_e q^{2l}/2)/(M-W-\tfrac{1}{2}i\Gamma q^{2l+1})\,,\qquad(9)$$

where f_{lj} corresponds to the elastic partial-wave amplitude

$$f_{lj}=e^{i\delta_{lj}}\sin\delta_{lj}/q\qquad(10)$$

and the relations for A and B in terms of the f_{lj} are given, for example, in Ref. 2. In Eq. (9), $W=s^{1/2}$, $c_I=\frac{1}{2}$, and $-\frac{1}{3}$ for $I=0$ and 1 respectively, q equals the magnitude of the center-of-mass three-momentum, and Γ_e and Γ are the elastic and total widths. In most cases we have used the narrow-width approximation. We have, however, done an approximate integration of the complete Breit-Wigner formula of Eq. (9) for the $Y_1^*(1385)$ whose position is quite near threshold, and also, in most of our calculations, near the zero of the dispersion denominator $s'-s$; the use of Eq. (9) reduced the $Y_1^*(1385)$ contributions, on the average, to about

[7] D. Freedman and J.-M. Wang, Phys. Rev. **153**, 1596 (1967).

[8] D. Beder and J. Finkelstein, Phys. Rev. **160**, 1363 (1967); D. Griffiths and W. Palmer, *ibid.* **161**, 1606 (1967); R. Ramachandran, *ibid.* **166**, 1528 (1968).

[9] A. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. Price, M. Roos, P. Soding, W. Willis, and C. G. Wohl, Rev. Mod. Phys. **40**, 77 (1968).

70% or 80% of the value obtained using the narrow-width approximation.

We note here one attractive feature of the sum rules (6) and (7). In many of our calculations, we shall be taking s rather close to threshold. In such a case, the S-wave scattering lengths, being weighted by small values of the dispersion denominators, might be expected to make appreciable contributions to the first and third integrals on the right side of Eqs. (6) and (7). The magnitude of these contributions would be unknown. Fortunately, however, because of their independence of angle, the S-wave contributions to the first and third integrals cancel. There remains, of course, a contribution to the second integral from the S wave in the crossed channel, but since, for $s>s_0$, this is not weighted by a large dispersion denominator it is presumably reasonable to ignore it.

We turn now to the problem of choosing the values of t and u. What we wish to do, of course, is to choose t and u in such a way as to maximize the validity of our assumption that the integrals in the sum rules are saturated by the relatively low-lying resonances. As far as t is concerned, we shall follow Refs. 1, 4, and 5, and most other workers in the field, and consider forward scattering, setting $t=0$ in the superconvergence relation (5) and in the fixed-t integrals in Eqs. (6) and (7). Besides being the most natural choice, this has the great advantage of not requiring any extrapolations into unphysical regions with $|\cos\theta|>1$. Unfortunately, for the fixed-u dispersion relations in (6) and (7), the situation is more complicated. The natural choice to make for u is again $u=0$, and this is what has been done in general in considering fixed-u dispersion relations.[1,5,8] However, such a choice here leads to highly unrealistic results. We shall describe these briefly, and then attempt to understand why $u=0$ is not likely to be the optimum choice of u in this problem.

Equations (5), (6), and (7) may be analyzed in the following way. Since the Σ pole does not appear in A, Eq. (7) by itself gives a value for g_Λ^2. We will write (7) in the form

$$a g_\Lambda^2/4\pi = b\,, \tag{7'}$$

where the quantities a and b depend, of course, on the value of u chosen. We designate the value of the $\pi\Lambda\Sigma$ coupling constant resulting from (7′) as $g_{\Lambda1}$. The superconvergence relation (5) does not depend on u. Once we have decided to set $t=0$, we may evaluate the right-hand side in the resonance saturation approximation. We take the resonance parameters from the review of Rosenfeld *et al.*[9] with the exception of the $\pi\Sigma$ decay width of the $Y_1^*(1660)$. It is pointed out in Ref. 9 that this is very uncertain experimentally. We have chosen to use the value of 21 MeV given by Armenteros *et al.*[10]

[10] R. Armenteros, M. Ferro-Luzzi, D. Leith, R. Levi-Setti, A. Minten, R. D. Tripp, H. Filthuth, V. Hepp, E. Kluge, H. Schneider, R. Barloutaud, P. Granet, J. Meyer, and J. Porte, Phys. Letters **24B**, 198 (1967).

TABLE I. The columns headed $A(0)$, $A(u_1)$, and $A(u_2)$ give the contributions of the resonances whose masses are listed in the first column to the parameter b defined by Eq. (7′) for $u=0$, 43.2, and 48.3 m_π^2 respectively. The columns headed $B(0)$, $B(u_1)$, and $B(u_2)$ give the resonant contributions to h, defined in Eq. (6′), for the same three values of u. The last five rows of the table give the values of the indicated parameters, defined in Eqs. (5′)–(7′), for $u=0$, $u=u_1$, and $u=u_2$. Contributions left blank in the table, or from known resonances not listed in the first column, are negligible.

M (MeV)	$A(0)$	$B(0)$	$A(u_1)$	$B(u_1)$	$A(u_2)$	$B(u_2)$
1385	2.97	0.37	4.21	0.55	0.90	0.59
1405	−0.25	···	−0.22	···	−0.25	···
1520	7.25	−2.91	7.58	−2.80	7.70	−2.90
1660	−6.00	1.67	−6.77	1.46	−6.05	1.53
1690	4.50	−1.25	5.08	−1.09	4.50	−1.15
1815	1.57	−0.38	−0.23	0.23	−0.72	0.36
1830	−4.72	0.41	−2.81	0.45	−2.50	0.37
a	0.113		0.356		0.533	
b	5.37		6.84		3.60	
d		−0.261		−0.210		−0.370
e		−0.550		1.34		1.78
h		−2.09		−1.20		−1.20

for $\Gamma_{\pi\Sigma}$ for this resonance, largely in order to have a number with which to calculate. The large uncertainty in this quantity is, of course, still present, and is, in fact, a serious source of uncertainty in our results, since the contribution of the $Y_1^*(1660)$ is large in all of our sum rules. We write Eq. (5) in the form

$$g_\Sigma^2/4\pi = g_\Lambda^2/4\pi + c\,. \tag{5'}$$

Upon calculating the contributions of the various resonances to the superconvergence relation, we obtain for c the expression

$$\begin{aligned} c = 2.32\ (1385) + 1.64\ (1520) - 2.21\ (1660) \\ +1.81\ (1690) + 0.54\ (1815) - 1.68\ (1830) \\ +0.54\ (2035) + 0.28\ (2100) = 3.24\,, \end{aligned} \tag{11}$$

where the numbers in parenthesis give the mass of the resonance to which a particular term in the equation corresponds. Equation (11) includes the contributions of all resonances for which the relevant parameters are reasonably well known and which make a non-negligible contribution to c. The 1690-MeV resonance referred to is the one with $I=0$, the properties of the $I=1$, 1690-MeV resonance having not been established. Substituting Eq. (11) into Eq. (6), we write the latter in the form

$$d g_\Lambda^2/4\pi + e = h\,, \tag{6'}$$

where d, e, and h depend on the choice of u. We refer to the value of g obtained from (6′) as $g_{\Lambda2}$. Clearly, if our three sum rules are consistent, we should have $g_{\Lambda1}=g_{\Lambda2}$. Moreover, if the assumptions which have gone into deriving the sum rules are correct, and if the coupling constants obey approximate $SU(3)$, then the two constants f_Λ and f_Σ defined by

$$g_\Lambda^2 = \tfrac{4}{3}(1-f_\Lambda)^2 g^2 \tag{12a}$$

TABLE II. The second and third columns give the values of $g_\Lambda^2/4\pi$, the $\pi\Lambda\Sigma$ coupling constant, as obtained from the sum rule (7) and from the sum rules (5) and (6) taken together when u is fixed at the values given in the first column. The fourth column gives the value of $g_\Sigma^2/4\pi$, the $\pi\Sigma\Sigma$ coupling constant, obtained from Eqs. (5) and (6) at the same values of u, while the remaining columns give the values of the $SU(3)$ parameter f obtained from $g_{\Lambda1}$, $g_{\Lambda2}$, and g_Σ using Eqs. (12a) or (12b) for $u=43.2$ and $48.3\ m_\pi^2$. Values of f are not given for $u=0$, since the results in that case are in such extreme violation of $SU(3)$.

u (m_π^2)	$g_{\Lambda1}^2/4\pi$	$g_{\Lambda2}^2/4\pi$	$g_\Sigma^2/4\pi$	$f_{\Lambda1}$	$f_{\Lambda2}$	f_Σ
0	47.4 ± 14.1	101 ± 20	104 ± 21			
43.2	19.2 ± 7.1	11.4 ± 5.0	14.6 ± 5.8	0.01 ± 0.18	0.24 ± 0.15	0.50 ± 0.09
48.3	6.76 ± 4.0	8.05 ± 3.1	11.3 ± 3.9	0.41 ± 0.17	0.36 ± 0.11	0.44 ± 0.07

and

$$g_\Sigma^2=4f_\Sigma^2g^2\,, \tag{12b}$$

with $g^2/4\pi=14.7$, should be roughly equal to one another, and to the $SU(3)$ parameter f defined by Martin and Wali.[6] Experimentally, the most direct determination of f is perhaps that of Kim,[11] who finds $f=0.41\pm0.07$. As f increases through this range, $g_\Sigma^2/4\pi$ ranges from 7.6 to 13.5, while $g_\Lambda^2/4\pi$ decreases from 10.7 to 5.3. The values corresponding to $f=0.4$ are $g_\Lambda^2/4\pi=7.05$, and $g_\Sigma^2/4\pi=9.4$.

If we choose $u=0$, the second and third columns of Table I give the contributions of the various resonances to the constants b and h, as well as the values of the constants a, b, d, e, and h themselves. Substituting these into Eqs. (6′) and (7′), one finds the results listed in the first row of Table II. As will be seen, one finds values of the coupling constants an order of magnitude larger than what one expects on the basis of at least some very rough validity of $SU(3)$.

Table II also gives an estimated uncertainty in the results for the coupling constants. This estimate was made in the following way. We have taken the errors in each of the resonance contributions to be $\pm15\%$, which is consistent with the general magnitude of the errors in the total and partial widths as given in Ref. 9, except for the $Y_1^*(1660)$, whose contribution we have rather arbitrarily taken to have a 20% uncertainty. There is, of course, no real way to determine the error due to the neglect of nonresonant and high-energy contributions to the integrals. For purposes of estimating the total error, we took the nonresonant background contribution to be 15% of the largest resonance contribution, and the remaining high-energy contribution to be equal to that of the 2100-MeV resonance, the highest resonance we have included explicitly. Following this prescription, we find the parameter c, from Eq. (11), to have an estimated uncertainty of $\pm0.85=\pm26\%$. This uncertainty in c is, of course, reflected in a corresponding uncertainty in e in Eq. (6′). The errors in e and h are, to a considerable degree, correlated, since they receive contributions from the same resonances. Inspection of Table I and of Eq. (11) shows that, for the most part the resonances contribute with the same sign to h and to $-e$. Therefore, in estimating the error in $g_{\Lambda2}$, using Eq. (6′), we have added the errors in $-e$ and h, rather than taking the square root of the sum of the squares. Independently of the details of the procedure for estimating the errors, we conclude that the results for the coupling constants obtained when $u=0$ are unreasonably high.

In order to attempt to understand what has gone wrong, we write down the formula for the center-of-mass angle between the incident and outgoing pions, which is given by

$$\cos\theta(s,u)=1+(\tau-s-u)/2\nu\,, \tag{13}$$

where $\nu=q^2$. The resonance contributions to the third integrals on the right side of Eqs. (6) and (7), the integrals carried out at fixed u, involve, for $u=0$, Legendre polynomials of argument $\cos\theta(s_r,0)$, where s_r is the squared mass of the resonance in question. For the low-lying resonances, this becomes very large because of the unequal masses of the π and Σ as may be seen from the second column in Table III, where we give the values of $\cos\theta(s_r,0)$ for the three resonances which make the largest contributions to our sum rules. It is this fact which results in the very large contributions of the 1385- and 1520-MeV resonances to b and h at $u=0$. [Although $\cos\theta(s_r,0)$ is larger for the $Y_1^*(1385)$, the angle-dependent term in that case is suppressed by kinematic factors, whereas the angle-dependent term in the $Y_0^*(1520)$ case is kinematically enhanced, which explains why its contribution at $u=0$ is even larger than that of the $Y_1^*(1385)$.] We feel that it is not realistic to take seriously the idea of keeping only the resonant terms in the partial-wave expansion at values of s and u where these become anomalously large due to their angular dependence, and that this is probably the reason for the (presumably) poor results obtained from (6) and (7) when u is set equal to zero.

If this diagnosis is correct, the question then arises as to whether it is possible to extract useful information from our sum rules. One way of proceeding is to go to positive values of u in order to decrease the values of $\cos\theta(s_r,u)$, and, hopefully, bring them within the physical region. It turns out not to be possible to make $|\cos\theta(s_r,u)|<1$ for all of the resonances for any choice of u. If one chooses u large enough so that $\cos\theta(s_r,u)=1$ for the $Y_1^*(1385)$, which requires $u=48.3$, then $\cos\theta(s_r,u)$ is between -1.5 and -2 for some of the higher resonances. We have considered two choices for u which seem to us to be reasonable. The first is the one

[11] J. K. Kim, Phys. Rev. Letters 19, 1079 (1967).

which we have just mentioned, of choosing $u=48.3$; this has the merit that $|\cos\theta(s_r,u)|<2$ for all of the resonances. The values of $\cos\theta(s_r,u)$ which result for the three most important resonances are shown in Table III,, along with those for $u=0$ and for our remaining choice $u=43.2$. The latter was determined by requiring $\cos\theta=-1$, just at the negative edge of the physical region, for the important and strongly angular dependent contribution of the $Y_0^*(1520)$. We will denote these two choices of u, 43.2 and 48.3, by u_1 and u_2, respectively. The choice of u_1 leaves $\cos\theta=4.9$ for the 1385-MeV resonance. As we have already noted, the term in the $Y_1^*(1385)$ contribution proportional to $P_1(\cos\theta)$ is kinematically suppressed relative to the constant term. This suppression is by a factor of the order of $(M-\Sigma)/(M+\Sigma)$ in A, and $(M-\Sigma)^2/(M+\Sigma)^2$ in B, where M is the mass of the resonance. Hence, one might hope that the required extrapolation in angle for the choice $u=u_1$ is not too serious, especially for the case of the B amplitude. Unfortunately, in going to these values of u, questions arise as to whether the fixed-u dispersion relations are still unsubtracted. This depends on the unknown behavior of the $Y_1^*(1385)$ trajectory. If Barger and Cline[12] are correct as to the form of the $N^*(1238)$ trajectory, and if the Y_1^* and N^* trajectories are parallel, then the fixed-u dispersion relation remains unsubtracted only up to u of the order of 40. (Their results, obtained from a fit to backward πN scattering data at high energies, are quite similar to those obtained from a Chew-Frautschi[13] plot of the πN resonances.) On the other hand, if the Y_1^* trajectory is parallel to the N^* trajectory of Desai, Gregorich, and Ramachandran,[14] which does have the advantage of avoiding unobserved parity-doublet states, Eqs. (6) and (7) hold out to u of the order of 60. In the former case, there is the possibility, discussed by Dass and Michael,[5] that resonance saturation might be a good approximation to the dispersion integral of the amplitude with the divergent Regge contribution subtracted off, so that the results of using (6) and (7) together with the resonance saturation approximation could conceivably be valid even if the amplitude itself, without the nonconvergent Regge contributions subtracted off, does not satisfy an unsubtracted dispersion relation. Dolen, Horn, and Schmid[15] have shown that a necessary condition for this to be true is that there be resonance contributions of opposite sign and comparable strength, which is true here as may be seen from Table I. However, Schmid[16] has also found that, at least in some cases, the Regge terms may contain a reasonable approximation to

TABLE III. The columns headed $\cos\theta(u)$ give the values of $\cos\theta(s,u)$, defined by Eq. (13), for the indicated values of u (in units of m_π^2), and s equal to the square of the resonance masses in the first column.

M (MeV)	$\cos\theta(0)$	$\cos\theta(43.2)$	$\cos\theta(48.3)$
1385	32.3	4.90	1.00
1519	5.73	-1.00	-1.90
1660	1.40	-1.47	-1.94

intermediate energy resonances. If this should be true in general, then it seems unlikely that the resonance contributions would be a good approximation to the integral over the convergent part of the amplitude; in that case our sum rules would be expected to be valid only if the $Y_1^*(1385)$ trajectory is such that the full amplitudes, without any Regge terms subtracted off, obey unsubtracted dispersion relations. And of course, even if the fixed u dispersion relation is, strictly, unsubtracted at $u=u_1$ or u_2, its convergence will be worse than at $u=0$, so that the error due to the neglect of the large s contributions will probably be worse. We choose to investigate the results of taking u in the range of u_1 or u_2, despite all of these uncertainties, because the results seem to indicate clearly that $u=0$ is not a satisfactory choice. We reject a choice of u in the range $0<u<u_1$ on two grounds. First of all, there seems to be no physical argument for choosing one u in this range over any other, so that, if the "correct" value of u lies in this range we do not know how to pick it out, whereas a choice of u near u_1 or u_2 can be justified on the ground of minimizing the required extrapolation to unphysical angles. And as a practical matter, we have investigated briefly the results of choosing u between 0 and u_1 and find that they offer, in general, no improvement over $u=0$. In fact, choosing $u\approx30$ leads to a prediction of negative values for the squares of the coupling constants.

The resonant contributions to b and h, and the values of the parameters in Eqs. (6′) and (7′), for $u=u_1$ and u_2 are given in Table I, and the corresponding results for the coupling constants and for f_Λ and f_Σ are given in the second and third rows of Table II. While the uncertainties are very large, all of the results obtained with $u=u_1$ or u_2, with the possible exception of the results obtained from the sum rule (7) at $u=u_1$, are compatible with one another and with the results expected from $SU(3)$, namely, $g_\Lambda^2/4\pi\approx7$, $g_\Sigma^2/4\pi\approx9.5$, $f\approx0.4$. As far as the result for $g_{\Lambda1}^2$ at $u=u_1$ is concerned, we note from Table I that it is large mainly because it receives an unusually large contribution from the $Y_1^*(1385)$. As we have already remarked, $\cos\theta(s_r,u_1)$ is appreciably larger than 1 for this resonance, and this will have a more severe effect in the sum rule for A than in Eq. (6) for B. Therefore it is not a matter of too great concern that Eq. (7) is not well satisfied at $u=u_1$; indeed, it seems to further support our hypothesis about the importance of avoiding the application of the saturation approximation in cases where any of the resonances make ab-

[12] V. Barger and D. Cline, Phys. Rev. Letters **21**, 392 (1968); **19**, 1504 (1967).
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normally large contributions because of a large value of $\cos\theta$. In any event, if our error estimates are reasonable, the discrepancy between the results of Eq. (7) at $u=u_1$ and the other results is of marginal significance.

In addition to choosing u to be rather large and positive, there is another procedure by which one may attempt to obtain sum rules of the general type of Eqs. (6) and (7) while avoiding the difficulties resulting from extrapolation to large unphysical values of $\cos\theta$. This is to make use of the fact that the amplitudes A and B also satisfy dispersion relations at fixed $\cos\theta=-1$, as has been discussed by Atkinson.[17] Therefore, in place of the fixed-u dispersion relations, one may combine the dispersion relations at fixed $\cos\theta=-1$ with those at fixed $t=0$ to obtain sum rules analogous to (6) and (7). Clearly, since $\cos\theta$ is fixed at -1, no problem of unphysical angles arises in this method for any value of s, so that the use of fixed backward angle, rather than fixed u, dispersion relations might well be advantageous. Since, for large s, $\cos\theta=-1$ corresponds to $u=0$, the convergence of the fixed-angle dispersion relations should be the same as those for fixed $u=0$ so that, from our previous discussion, Regge pole theory indicates no subtractions are required.

In Ref. 17, the discussion is carried out "dispersing" in the variable ν rather than s. In the general case this is preferable, since the configuration of the cuts of the backward amplitude is simpler when it is considered as a function of ν instead of s. However, a simplification arises in our case, since we have isospin $=2$ in the t channel and so are neglecting the t-channel absorptive part. As a result the analytic properties of $A(s,\cos\theta=-1)$ in the s plane are very simple. The singularities consist of the s- and u-channel pole terms plus cuts along the real axis for $0\leqq s\leqq s_L$ and $s_0\leqq s<\infty$, where $s_L=(\Sigma^2-1)^2/s_0^2$ is the value of s at which $u=\tau-s-t=s_0$ when $\cos\theta=-1$, i.e., when $t=-4\nu(s)$. The discontinuity across the left-hand cut is, of course, due to the u-channel absorptive part. In the general case there is also a circular cut in the complex s plane, as well as a cut along the negative real axis with discontinuities determined by the t-channel absorptive part; since we are neglecting the latter discontinuity, these cuts are not present in our calculation.

We now write down the analogs of Eqs. (6) and (7) for the case that the fixed-u dispersion relation is replaced by a dispersion relation with $\cos\theta=-1$ or $t=-4\nu$. We find

$$
\begin{aligned}
&-\frac{g_\Lambda^2}{\Lambda^2-u}+\frac{C_\Lambda g_\Lambda^2}{s_\Lambda-s}+\frac{g_\Sigma^2}{\Sigma^2-u}-\frac{C_\Sigma g_\Sigma^2}{s_\Sigma-s}\\
&=-\int_{s_0}^{\infty}\frac{\operatorname{Im}B(s',0,\tau-s')ds'}{s'-s}-\int_{-\infty}^{\tau-s_0}\frac{\operatorname{Im}B(s',0,\tau-s')ds'}{s'-s}\\
&\qquad+\int_{s_0}^{\infty}\frac{\operatorname{Im}B(s',4\nu(s'),\tau-s'+4\nu(s'))}{s'-s}ds'+\int_0^{s_L}\frac{\operatorname{Im}B(s',4\nu(s'),\tau-s'+4\nu(s'))}{s'-s}ds', \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(\Sigma-\Lambda)g_\Lambda^2}{\Lambda^2-u}-\frac{C_\Lambda(\Sigma-\Lambda)g_\Lambda^2}{s_\Lambda-s}\\
&=-\int_{s_0}^{\infty}\frac{\operatorname{Im}A(s',0,\tau-s')}{s'-s}ds'-\int_{-\infty}^{\tau-s_0}\frac{\operatorname{Im}A(s',0,\tau-s')}{s'-s}ds'\\
&\qquad+\int_{s_0}^{\infty}\frac{\operatorname{Im}A(s',4\nu(s'),\tau-s'+4\nu(s'))ds'}{s'-s}+\int_0^{s_L}\frac{\operatorname{Im}A(s',4\nu(s'),\tau-s'+4\nu(s'))ds'}{s'-s}. \qquad (15)
\end{aligned}
$$

The constant s_Λ appearing in Eqs. (14) and (15) is defined, analogously to s_L, as the root of the equation $\tau-s-4\nu(s)=\Lambda^2$, and similarly for s_Σ; the constants C_Λ and C_Σ arise because the Λ and Σ in the cross channel give rise to poles at Λ^2 and Σ^2 in the variable $u'=\tau-s'-4\nu(s')$, rather than in s'. The values of these constants are given by

$$
\begin{aligned}
&s_\Lambda=(\Sigma^2-1)^2/\Lambda^2,\quad s_\Sigma=(\Sigma^2-1)^2/\Sigma^2,\\
&C_\Lambda=(-1+4d\nu/ds|_{s=\Lambda^2})^{-1}=-(\Sigma^2-1)^2/\Lambda^4, \qquad (16)\\
&C_\Sigma=(-1+4d\nu/ds|_{s=\Sigma^2})^{-1}=-(\Sigma^2-1)^2/\Sigma^4.
\end{aligned}
$$

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The last integrals in (14) and (15) may be evaluated by introducing the new variable $u'=\tau-s'-4\nu(s')$ and making use of the antisymmetry (symmetry) of $\operatorname{Im}A$ ($\operatorname{Im}B$) under interchange of s and u.

Equations (14) and (15) are, of course, very similar in structure to Eqs. (6) and (7); they share with the latter the advantage that any large S-wave contributions at threshold will cancel out. In analogy with (6′) and (7′), we write Eqs. (14) and (15) in the form

$$d'g_\Lambda^2/4\pi+e'=h', \qquad (14')$$

$$a'g_\Sigma^2/4\pi=b', \qquad (15')$$

where use has been made of (5) in (14′). Table IV gives the contributions of the various resonances to b' and h', as well as the values of the constants a', b', d', e', and h'. Equation (15′) leads to a value of 18.5 ± 14 for $g_\Lambda^2/4\pi$; in other words, because of the large cancellations among the various resonance contributions, essentially no conclusion can be drawn from the sum rule (15) as to either the value of g_Λ^2 or the consistency of (5), (14), and (15). From (14′), i.e., from the combination of Eq. (14) with the superconvergence relation (5), one obtains $g_\Lambda^2/4\pi=26.0\pm12.6$. Combined with (11) this gives $g_\Sigma^2/4\pi=29.2\pm13.5$. These two coupling constant values are not in remote agreement with $SU(3)$ for any value of f.

The large discrepancy between these last results and the more or less expected value is disappointing, in view of the fact that one would expect the use of the fixed angle dispersion relations, avoiding, as it does, both extrapolation and convergence problems, to yield the most reliable result of any of our sum rules. In view of the fact that the discrepancy is only about 1.5 times the estimated error, one need not, however, take it too seriously, either as a reflection on the assumptions which go into the sum rules or the validity of approximate $SU(3)$ for the coupling constants. The most likely conclusion would seem to be that we have been statistically unfortunate with respect to the tendency of the various experimental errors and neglected background terms to reinforce rather than cancel one another. It may be worthwhile noting that Eq. (14) by itself is consistent with coupling constants close to the $SU(3)$ values. Inserting the numerical values, (14) can be put in the form

$$g_\Sigma^2/4\pi-g_\Lambda^2/4\pi=0.2(g_\Lambda^2/4\pi-7.0)\pm1.3. \quad (14'')$$

Taking $g_\Lambda^2/4\pi=7.5$, for example, (14″) leads to $g_\Sigma^2/4\pi=7.6\pm1.3$, hence giving results consistent with $SU(3)$ with reasonably small symmetry breaking. Equations (5′) and (11), however, lead to the relation

$$g_\Sigma^2/4\pi-g_\Lambda^2/4\pi=3.24\pm0.85\,, \quad (5'')$$

and it is the requirement of the simultaneous validity of (5″) and (14″) which forces a very large value of the coupling constants. From Table IV and Eq. (11) one can see that most of the resonances contribute with opposite sign to $(g_\Sigma^2-g_\Lambda^2)/4\pi$ in (5″) and (14″); hence it is quite possible for the errors in the resonance terms, if they have the proper sign, to reconcile the simultaneous validity of (5) and (14) with a value of $g_\Lambda^2/4\pi$ of about 7.

It is, of course, possible that the discrepancy between the results of Eq. (14) and the expected values of the coupling constants is real, and that either the saturation approximation fails badly or the coupling constants are actually much larger than $SU(3)$ would suggest. Earlier results[18] on the KN coupling constants had indicated deviations from $SU(3)$ by factors of 2 or so, though Kim's results[11] seem to indicate that the deviations in that case are, in fact, small. In any event, the uncertainties are too large, at the present time, to allow us to make any definite statements on the basis of (14).

TABLE IV. Contributions of the indicated resonances to the parameters b' and h'. Contributions left blank, or from known resonances not listed in the first column, are negligible. The last five rows of the table give the values of the parameters a', b', d', e', and h'; the parameters in question are defined in Eqs. (15′) and (14′).

M (MeV)	b'	h'
1385	1.68	0.31
1405	−0.16	...
1520	4.50	−1.01
1660	−5.10	0.64
1690	3.80	−0.48
1815	0.28	0.04
1830	−2.78	0.21
a'	0.12	
b'	2.22	
d'		−0.033
e'		−0.57
h'		−0.29

In conclusion, then, all that one can say is that, following Atkinson's procedure,[1] one can obtain several sum rules for the $\pi\Lambda\Sigma$ and $\pi\Sigma\Sigma$ coupling constants. With the exception of those involving fixed $u=0$ and requiring large extrapolations to unphysical angles, all are consistent with values for these coupling constants corresponding to $SU(3)$ with little or no symmetry breaking. However, the uncertainties which result from the summation of several resonance terms of both signs and each with appreciable experimental error, coupled with the presence of unknown nonresonant background and high-energy contributions, are too large to allow one to draw any firm conclusions either as to the values of the coupling constants or the mutual consistency of the various sum rules and the approximations used in evaluating them. It may also be worth emphasizing again the sensitivity of our fixed-u results to the value of u chosen, which suggests that considerable caution may be required in using fixed-u dispersion relations at $u=0$.

It is a pleasure to thank Professor Reilly Atkinson for several very helpful discussions on various aspects of this work.

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