7. POSSIBLE GENERALIZATIONS AND CONCLUSIONS

One may also consider more complicated diagrams with insertions in the simple Regge vertex [Fig. 5(a)]. It is moderately plausible that the Regge calculus will still hold, but the vertices will acquire additional analytic structure. However, they still ought to be real because their external momenta are spacelike and so their Feynman denominators must be negative definite. It follows that our discussion of signature could be completely general. However, one needs to show that the signature factor of the "odd" Reggeon is always eliminated. This is a problem in multiparticle states and, hence, difficult.

We may expect to retain analyticity in the l variables of the vertex. This would mean that the only j singularities in $q^2 \leqslant 0$ are given by pinches between Regge denominators. These depend only on trajectories and no traces of "elementarity" are visible. We already know that this result is true in detail for the two-Regge intermediate state.[6]

Similar comments may also be valid for n-Regge vertices. We already know from examples[1] that cuts behave in a similar way to poles. It is interesting to note that any Regge graph contributing to 2-2 scattering can be inserted in any of the blobs in our original graph. The leads to various n-Regge vertices. Presumably the detailed extraction of all these graphs is only a matter of extra labor.

There remains the mathematical problem of justifying the limitations (25). At present we have some understanding of this question but no proof (Sec. 4).

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. I. T. Drummond for some very helpful discussions. I also wish to thank the Science Research Council for a studentship.

[6] See, e.g., D. I. Olive and J. C. Polkinghorne, Phys. Rev. **171**, 1475 (1968), and other authors quoted therein.

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Connection between Bound States or Resonances of Two-Particle and Three-Particle Systems*

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The connection between three-particle resonances or bound states and the two-particle resonances or bound states of their component pairs is examined. A simpler and more general derivation is given of the Lovelace equations for isobar or bound-state scattering. As an example, a one-dimensional model of a system of three identical particles, each pair of which interacts only through a single bound or resonant state of energy ν_0, is constructed, and the corresponding off-shell scattering integral equation is obtained. By examining the analytic structure as a function of the off-shell momentum at fixed energy, an expression is obtained for the off-shell scattering amplitude involving explicitly known functions and the solution of a much less singular equation. When ν_0 is the position of a resonance pole of the two-particle system, the three-particle denominator function has a pole at an energy $4\nu_0$ on the unphysical sheet where denominator zeros are associated with resonances, provided that the two-body resonance has a smooth form factor. This suggests the presence of a three-body resonance in this neighborhood if the two-body resonance is narrow. That is not true for bound states. The relation of this to the expected behavior of more realistic problems is discussed.

I. INTRODUCTION

In this paper we study bound and resonant states of a system of three particles, at least one of which can combine with each of the other two to form a two-particle bound or resonant state. In such a system there is an exchange force due to the existence of the two alternative compound two-particle states. There is nothing strange about this force; for example, in the singly ionized hydrogen molecule, in which a single electron can be bound to either of two protons, the molecular binding is produced by exactly such a force. Of course, the detailed structure also depends on the Coulomb repulsion of the two protons and the distortion of the wave function of the bound state due to the presence of the third particle, but it is the exchange force which provides the attraction.

Our main concern here is with the possibility that such an exchange force is itself capable of causing a bound state or resonance of the whole system and with the extent to which the properties of such a bound state or resonance depend only on the parameters of the two-particle resonant states, rather than on the detailed dynamics of the problem. To this end we choose to neglect any interactions between the particles other than the postulated two-particle interactions and assume that there is no distortion of the form factors due to the presence of a third particle.

The motivation for this investigation is the fact that among the short-lived "particles" produced in high-energy collisions are many which decay into just such a three-body system as discussed above, i.e., which can decay via a quasi-two-particle state in at least two alternative modes. Examples include the A_1 meson decaying to three pions with any one of the three possible pairs emerging as a ρ meson, the $N^*(1520)$ decaying into a nucleon and two pions, either one of the two pions together with the nucleon emerging as an $N^*(1238)$, and many others. There has been much discussion[1] of the effects of this kind of overlapping interaction on the decay distributions of the system for fixed total energy, but what we are concerned with here is the effectiveness of these interactions in producing a resonant enhancement as a function of the total energy. Some time ago[2] one of us pointed out the existence of the exchange interaction in such three-particle systems, and argued that the strong dependence of the strength of this exchange force on the total energy might enable one to predict quite simply the resonance energy. Since then there has been a great deal of controversy[3,4] and a considerable amount of misunderstanding about this question.

The main part of this paper deals with a calculation of a model problem which is sufficiently simple to treat exactly. The purpose of this is to clarify the suggestion[2] that there can be a simple relation between the approximate position of a three-particle resonance and the position of the two-particle resonance responsible for the three-particle force, to show that such a connection is possible, to say something about the conditions under which it may reasonably appear, and to understand the apparent contradiction with those papers[3,4] which have argued against such a possibility. In Sec. II we make some general remarks about the description and special features of a three-particle system. In Sec. III we formulate the model, using a nonrelativistic interaction in one dimension, and show how it reduces to an effective two-body problem. The equations are essentially those of Lovelace[5] but the deviation is simpler and more directly related to the physical situation. In Sec. IV we describe the method of solving the equations and give the solution. Section V contains a discussion of the nature of the solution, its relation to three-body resonances, and its connection with other work, and in Sec. VI we discuss the relation to other calculations.

II. MULTIPARTICLE STATES

In order to argue that the results of the specific model calculation are relevant to more realistic problems it is necessary to show that these results depend only on general properties of three-particle scattering rather than on the particular dynamics of the model. We refer the reader to some of the extensive literature[5–7] for a detailed discussion of the three-body scattering problem, and summarize here the properties of the various states and Green's functions with which we shall be concerned, introducing them without reference to any particular dynamics.

In the momentum representation a system of N particles is specified by giving the $4N$ components of the four-momenta $k_{i\mu}$ ($i=1, \cdots, N$, $\mu=0, \cdots, 3$). For an *isolated* system the four quantities

$$K_\mu=\sum_{i=1}^{N} k_{i\mu}$$

are conserved. For *noninteracting* particles each four-momentum $k_{i\mu}$ is separately conserved. A real particle, i.e., one which propagates over macroscopic distances, must have

$$k_{i0}=\nu_i(k_i)=(|k_i|^2+m_i^2)^{1/2}, \tag{2.1}$$

where m_i is the mass. This is the "mass-shell" condition.

For a system of N particles there will in general be some operator $G^{(N)}(\{x\};\{x'\})$ relating the state vector describing the particles at one set of space-time points $\{x\}$ to that describing the same particles at another set $\{x'\}$. In field theory G arises as a vacuum expectation value of N creation and N annihilation operators; in potential theory it occurs as the solution of the differential equation with a δ-function source term, etc. The Fourier transform of G gives the momentum-space Green's function with which we shall be concerned here.

For a single isolated particle (a trivially noninteracting system) the conservation laws mentioned above,

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] I. J. Aitchison and C. Kacser, Phys. Rev. **142**, 1104 (1966); R. D. Amado, *ibid.* **158**, 1414 (1967). These papers contain references to earlier work.

[2] R. F. Peierls, Phys. Rev. Letters **6**, 641 (1961).

[3] C. Goebel, Phys. Rev. Letters **13**, 143 (1964).

[4] C. Schmid, Phys. Rev. **154**, 1363 (1967).

[5] C. Lovelace, Phys. Rev. **135**, 1275 (1964). Similar equations were previously written without relating the operators in the three-particle Green's function. See for example, R. D. Amado, *ibid.* **132**, 485 (1963).

[6] M. Rubin, R. L. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966); **159**, 1348 (1967); **162**, 1555 (1967).

[7] L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* [Publications of the Steklov Mathematical Institute No. 69, 1963 (English transl.: Israel Program for Scientific Translations, Jerusalem, Israel, 1965)].

together with Lorentz invariance, require the matrix elements to be of the form

$$\langle k_{i\mu}'|G_0^{(1)}|k_{i\mu}\rangle=\frac{g(k_i^2)\theta(k_{i0})}{k_i^2-m_i^2-i\epsilon}\delta^{(4)}(k_{i\mu}'-k_{i\mu}). \quad (2.2)$$

For an "elementary" spinless particle $g(k^2)$ is simply a normalization constant. The singularity in G_0 is necessary to prevent the Fourier transform, or configuration-space propagator, from being damped by oscillations at large separations, i.e., to allow macroscopic propagation. The fact that it is a pole corresponds to the most characteristic feature of a single particle, namely, that it has a unique mass, so that macroscopic propagation is allowed for a unique value of k_i^2.

For an N-particle system the Green's function has matrix elements $\langle k_1'\cdots k_N'|G^{(N)}|k_1\cdots k_N\rangle$ (k_i and k_i' are understood to be 4-vectors; we shall omit the explicit subscripts wherever possible without ambiguity). For noninteracting particles we have

$$\langle k_1'\cdots k_N'|G_0^{(N)}|k_1\cdots k_N\rangle=\prod_{i=1}^{N}\langle k_i'|G_0^{(1)}|k_i\rangle. \quad (2.3)$$

For an interacting system the matrix elements of $G^{(N)}$ no longer factorize. However, some simplification comes from the conservation of total four-momentum:

$$\sum k_i=\sum k_i'=K.$$

We choose $N-1$ linearly independent relative momenta $q_1\cdots q_{N-1}$ and, correspondingly, $q_1'\cdots q_{N-1}'$ and write

$$\langle k_1\cdots k_N'|G^{(N)}|k_1\cdots k_N\rangle \equiv\langle q_1'\cdots q_{N-1}'|G^{(N)}(K)|q_1\cdots q_{N-1}\rangle. \quad (2.4)$$

The matrix elements of $G^{(N)}(K)$ will be singular for at least some values of the q_1 and q_1' whenever K_μ is a possible total four-momentum for a macroscopically propagating system, i.e., the sum of N vectors $k_{i\mu}$ satisfying (2.1). Relativistic invariance requires $G^{(N)}$ to be a function of K^2 for fixed q_i, and the singularities described above combine to produce a square-root-type branch cut in $G^{(N)}$ as a function of K^2 with branch point at $K^2=(\sum m_i)^2$. The noninteracting propagator $G_0^{(N)}(K^2)$ has no singularities other than this cut; when the particles interact, other singularities are possible. If $G^{(N)}(K)$ develops a pole of the form $(K^2-M^2)^{-1}$, then we have an N-particle bound state if $M^2<(\sum_i m_i)^2$ and a resonant state if $\mathrm{Re}M^2>(\sum m_i)^2$, $\mathrm{Im}M^2<0$. This corresponds to the empirical characterization of a multiparticle compound state as one which remains localized over macroscopic distances (apart from the usual wave-packet spreading).

For a nonrelativistic system the states are specified by the three-momenta and the conserved total energy. The single-particle propagator is

$$\langle \mathbf{k}'|G(\omega)|\mathbf{k}\rangle=\delta^3(\mathbf{k}'-\mathbf{k})\Big/\left(\omega-\frac{k^2}{2m}-i\epsilon\right), \quad (2.5)$$

that for N noninteracting particles is

$$\langle \mathbf{k}_1'\cdots\mathbf{k}_N'|G_0^{(N)}(W)|\mathbf{k}_1-\cdots\mathbf{k}_N\rangle =\prod_{i=1}^{N}\delta^{(3)}(\mathbf{k}_i'-\mathbf{k}_i)\Big/\left(W-\sum_{i=1}^{N}\frac{k_i^2}{2m_i}-i\epsilon\right), \quad (2.6)$$

while, in general, for N interacting particles

$$\langle \mathbf{k}_1'\cdots\mathbf{k}_N'|G^{(N)}(W)|\mathbf{k}_1\cdots\mathbf{k}_N\rangle =\langle \mathbf{q}_1'\cdots\mathbf{q}'_{N-1}|G^{(N)}(W,\mathbf{K})|\mathbf{q}_1\cdots\mathbf{q}_{N-1}\rangle. \quad (2.7)$$

As in the relativistic case, there is a connection between singularities of G and the possibility of a real system propagating over macroscopic time intervals. For the single particle this is exhibited by the pole at $\omega=k^2/2m$. The important feature of this singularity is that it is a pole at a unique value of ω for fixed k^2. There is always an additive ambiguity in the definition of the energy; in this case the zero point is defined as the on-shell energy of a single particle at rest. The N-particle operator $G^{(N)}$ certainly has a cut starting at $W=0$ in the frame $\mathbf{K}^2=0$ and extending over all positive real W, and this is the only singularity of $G_0^{(N)}$. If there is, in addition, a factor in $G^{(N)}(W,\mathbf{0})$ of the form $(W+W_0)^{-1}$, then this corresponds to a single-particle-like structure with energy scale shifted by W_0, which is positive in the case of a bound state. For a resonance, W_0 is complex with a conjugate pole at W_0^*, both on the sheet $\mathrm{Im}(W_0)^{1/2}<0$. If the nonrelativistic Green's function is written in the usual form $(W-H-i\epsilon)^{-1}$, where H is the Hamiltonian for the system, then the above definition is equivalent to the usual definition[8] of a bound state as a discrete eigenstate of H with eigenvalue $-W_0$.

The necessary ingredients for a dynamical calculation for an interacting system are the form of G_0 and a suitable numerical description of the interaction forces. The latter is, in general, given as an operator V in terms of which $G^{(N)}$ can be expressed in the form

$$G=G_0+G_0VG, \quad (2.8)$$

corresponding to the perturbation expansion

$$G=G_0+G_0VG_0+G_0VG_0VG_0+\cdots. \quad (2.9)$$

For the usual nonrelativistic case V is simply the potential $V=H-H_0$ and (2.8) is just the expansion of $1/(W-H)$ in terms of $1/(W-H_0)$. For the relativistic case V is the kernel of the Bethe-Salpeter equation expressed as the total contribution from irreducible graphs. In a general field-theoretical case V represents

[8] See, for example, R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966), p. 236,

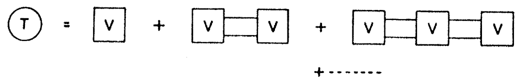

FIG. 1. Graphical representation of the iterative solution of a two-particle scattering equation.

the possible combinations of vertex functions and propagators, and since it can couple states of different particle number, the operators G_0 entering in different places may refer to different numbers of particles. The important part of the structure of G for our purposes, however, comes from those contributions with a fixed number of particles throughout. The expression (2.9) has the form

$$G=G_0+G_0TG_0. \tag{2.10}$$

The matrix elements of the operator T between on-shell states give the amplitudes for the corresponding real scattering processes. T, like G, depends parametrically on the conserved total four-momentum. Since, as we have seen above, G_0 does not possess the bound-state or resonance poles, they must occur in T. The operator T satisfies the equation

$$T=V+VG_0T \tag{2.11}$$

generating the series (2.9). This is an integral equation in $4\cdot(N-1)$ variables $[3\cdot(N-1)$ variables in the nonrelativistic case$]$ corresponding to $q_1\cdots q_{N-1}$, with the total energy W in the over-all c.m. system as a parameter ($W^2=\mathbf{K}^2$).

Up to this point we have made no distinction between different values of $N>1$. There are very important differences between the cases $N=2$ and $N>2$. Consider first the case $N=2$, the usual two-body scattering. The expansion for T can be represented graphically in the form of Fig. 1, where the horizontal lines represent $G_0^{(2)}$, i.e., the propagation of the two particles between successive applications of the interaction. For finite-range forces, V has nonzero matrix elements only between states in which the two particles are close together. For two real particles, which must move with equal and opposite momenta in the c.m. system, any subsequent propagation will increase their separation and they will never again come close enough together for V to act again. Hence in the expansion (2.9) each of the terms G_0 must describe off-shell, or virtual, propagation. In other words, the vanishing of the denominators associated with the various G_0 cannot yield any divergent singularity in T, since, as mentioned above, this would correspond to on-shell propagation;

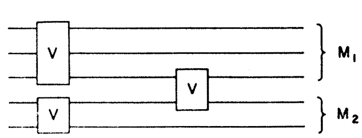

FIG. 2. Example of a possible on-shell rescattering contribution to a multiparticle process.

such a singularity in T can only develop as a failure of the series (2.9) to converge.

For $N>2$ the structure of the equation for T is formally the same as Fig. 1 except that the two lines representing $G_0^{(2)}$ are replaced with N lines.

However, the argument given above does not hold, since V, representing an interaction in which momentum is transferred, can have nonvanishing matrix elements whenever at least two particles are close together. Even if all the particles are affected at once, they can be in distinct, well-separated groups. For example, a contribution of the form shown in Fig. 2 is possible, in which M_1 particles interact and subsequently diverge, as do other M_2 particles ($M_1+M_2=N$). One particle from the first group approaches one from the second until they are close enough to interact. In this case the corresponding G_0 *does* give a singularity in T, since it can be associated with macroscopically propagating on-shell particles. It is the nature and consequences of these singularities that we wish to discuss. It will be sufficient to consider the case $N=3$ and an interaction V involving only pairs of particles. Labeling particles, a typical contribution to T is given by

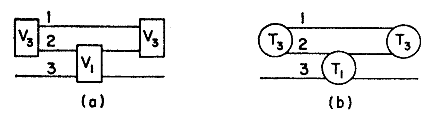

FIG. 3. (a) Contribution to the three-body T matrix from the expansion in the potential. (b) Corresponding contribution from the multiple scattering expansion in terms of the two-particle T matrix.

Fig. 3(a). It can easily be seen that by adding to any such contribution others which differ only in the replacement of any operator V_i with $V_iG_0V_i$, and repeating the procedure, the expansion can be written in terms of the two-particle T matrices. The term corresponding to Fig. 3(a) is given in Fig. 3(b).

Algebraically, we write

$$T^{(3)}=\sum_i T_i^{(2)}+\sum_{i\neq j} T_i^{(2)}G_0T_j^{(2)} + \sum_{i\neq j,\, j\neq k} T_i^{(2)}G_0T_j^{(2)}G_0T_k^{(2)}+\cdots. \tag{2.12}$$

This is the so-called multiple scattering expansion. The operator $T_i^{(2)}$ can be written

$$\langle k_1'k_2'k_3'|T_1^{(2)}(W)|k_1k_2k_3\rangle=\langle q_{12}'|t_1(W-k_3^0)|q_{12}\rangle \times\theta_0(k_3)\delta^{(4)}(k_3'-k_3) \tag{2.13a}$$

relativistically, or

$$\langle \mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'|T_1^{(2)}(W)|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle =\langle \mathbf{q}_{12}'|t_1(W-k_3^2/2m_3)|\mathbf{q}_{12}\rangle\delta(\mathbf{k}_3'-\mathbf{k}_3) \tag{2.13b}$$

nonrelativistically, where t_1 is the two-particle T matrix.

This can be expressed as

$$T^{(3)}=\sum_i T_i^{(2)}-\sum_{ij} T_i^{(2)}X_{ij}T_j^{(2)},$$
$$X_{ij}=G_0\tilde{\delta}_{ij}+\sum_k G_0\tilde{\delta}_{ik}T_kX_{kj}, \tag{2.14}$$

where $\tilde{\delta}_{ij}=\delta_{ij}-1$.
The expansion (2.1) in the form of the integral Eq. (2.14) or equivalent ones has been extensively considered[5,9] and forms the basis of quantitative treatments of three-particle problems. These treatments must take into account the singularities in T discussed above, corresponding to multiple physical rescattering. The crucial point is that there is an upper limit to the number of possible on-shell rescatterings, depending on the kinematics, so that these singularities only occur in the first few terms of the expansion (2.12). Iteration of the equation a sufficient number of times removes the singularities.

We wish to look at a situation where these singularities are overwhelmingly important. This is the case when the two-body Green's function is dominated by a bound-state or resonance pole. An "on-shell" rescattering in this case means that two particles must rescatter at the bound-state or resonance energy. Three successive real rescatterings is the upper limit in this case, for equal-mass particles.

In the next sections we look at this in detail, going to a fairly trivial model to enable an explicit solution to be obtained. However, it will be seen that the origin of the qualitative features of the solution lies in these general properties of three-particle scattering which are present even in much more complex situations and that related considerations can be expected to apply in systems of particles with $N>3$.

III. DERIVATION OF THE MODEL

In this section we derive the equations that we shall use to discuss the unstable-particle scattering. They are precisely those first derived by Lovelace,[5] but the derivation given here is somewhat simpler and more closely related to the interpretation of the equation. In order to simplify the discussion, we shall work in a world of one space dimension. All of the essential features of the problem are still present, and in Appendix A we give the corresponding equations for the more general case. We shall also assume that we work with spinless particles and nonrelativistic kinematics and will make other simplifying assumptions as we proceed.

Consider first the scattering of two particles. Let k_1 and k_2 be their momenta in the c.m. system and let m_1 and m_2 be their masses. Removing the c.m. motion, we work with the relative momentum p:

$$p=\mu(k_1/m_1-k_2/m_2), \tag{3.1}$$

where the reduced mass $\mu=(m_1^{-1}+m_2^{-1})^{-1}$. The scattering is described by the off-shell t matrix,

$$\langle p'|t(\nu)|p\rangle,$$

where ν is the energy in the c.m. system. The primed variables will always refer to the final state. The physical scattering amplitude is obtained when the energy-shell condition is satisfied:

$$\nu=\nu(p)=\nu(p'),$$
where
$$\nu(p)=p^2/2\mu. \tag{3.2}$$

Thus it is clear that for a given value of ν there are four physical scattering amplitudes, corresponding to $p=\pm(2\mu\nu)^{1/2}$, $p'=\pm(2\mu\nu)^{1/2}$, that is, transmission or reflection of an incident wave moving to the "right" or to the "left".

The dynamics of the problem enter when we introduce the equation which determines t. We take this to be the Lippmann-Schwinger equation with a potential v:

$$\langle p'|t(\nu)|p\rangle=\langle p'|v(\nu)|p\rangle -\int_{-\infty}^{\infty}dp''\frac{\langle p'|v(\nu)|p''\rangle\langle p''|t(\nu)|p\rangle}{p''^2/2\mu-\nu-i\epsilon}. \tag{3.3}$$

The potential v is quite arbitrary for the moment and is, in general, nonlocal and energy-dependent. In most cases the problem will be reflection-symmetric, i.e., $\langle p'|v|p\rangle$ will depend only on the magnitudes and relative sign of p and p', so that $\langle p'|v|p\rangle=\langle -p'|v|-p\rangle$, etc. Equation (3.3) preserves this symmetry for t, and so we can define

$$\langle p'|t^{\pm}(\nu)|p\rangle=\tfrac{1}{2}(\langle p'|t(\nu)|p\rangle\pm\langle p'|t(\nu)|-p\rangle),$$
$$\langle p'|v^{\pm}(\nu)|p\rangle=\tfrac{1}{2}(\langle p'|v(\nu)|p\rangle\pm\langle p'|v(\nu)|-p\rangle). \tag{3.4}$$

We can now write Eq. (3):

$$\langle p'|t^{\pm}(\nu)|p\rangle=\langle p'|v^{\pm}(\nu)|p\rangle -\int_{-\infty}^{\infty}dp''\frac{\langle p'|v(\nu)|p''\rangle\langle p''|t^{\pm}(\nu)|p\rangle}{p''^2/2\mu-\nu-i\epsilon}. \tag{3.3'}$$

This is the one-dimensional analog of the partial-wave expansion.

Let us introduce the operator $G_0(\nu)$,

$$\langle p'|G_0(\nu)|p\rangle=\delta(p'-p)(\nu-p^2/2\mu+i\epsilon)^{-1}, \tag{3.5}$$

the free Green's function for the two-body system, and define the kernel K as

$$K(\nu)=v(\nu)G_0(\nu).$$

We can then formally write (3.3) as the operator equation

$$t(\nu)=v(\nu)+K(\nu)t(\nu), \tag{3.6}$$

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[9] S. Weinberg, Phys. Rev. **133**, B232 (1964); L. Rosenberg. *ibid.* **135**, B715 (1964).

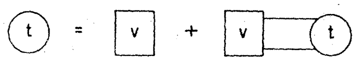

FIG. 4. Graphical representation of the Lippmann-Schwinger equation.

graphically represented in Fig. 4, having the forma solution

$$t(\nu)=[1-K(\nu)]^{-1}v(\nu). \tag{3.7}$$

The structure of G_0 implies that $t(\nu)$ possesses a square-root branch point at $\nu=0$, the cut running along the positive ν axis, and poles at the points where $K(\nu)$ has unit eigenvalue, i.e., where

$$\det[1-K(\nu)]=0. \tag{3.8}$$

The physical scattering amplitude is defined as

$$\lim_{\epsilon\to 0+} t(\nu+i\epsilon)\,,\quad \mathrm{Im}\nu^{1/2}>0.$$

The poles in $t(\nu)$ occur on the physical sheet on the negative real axis, corresponding to bound states of the two-body interaction; on the unphysical sheet they occur on the negative real axis or else at complex conjugate pairs of points corresponding to isolated two-particle resonances when they approach the positive real axis. Thus we may write

$$\langle p'|t(\nu)|p\rangle=\sum_n \frac{r_n(p',p)}{D_n(\nu)}+\langle p'|\tilde{t}(\nu)|p\rangle, \tag{3.9}$$

where $D_n(\nu)$ vanishes linearly at $\nu=\nu_n$ and $\tilde{t}(\nu)$ is analytic except for the branch cut. The residue $r_n(p,p')$ factorizes[5]:

$$r_n(p',p)=g_n(p')g_n^*(p),$$

g being real when ν_n is real, corresponding to a bound state.

We must next consider the scattering of three particles. Let us label the particles A, B, and C and suppose that they have momenta k_A, k_B, and k_C. Following the notation of Lovelace,[5] we then introduce the six momenta

$$p_\alpha=\mu_\alpha(k_\beta/m_\beta-k_\gamma/m_\gamma),$$

$$q_\alpha=M_\alpha\left(\frac{k_\beta+k_\gamma}{m_\beta+m_\gamma}-\frac{k_\alpha}{m_\alpha}\right), \tag{3.10}$$

where

$$\mu_\alpha^{-1}=m_\beta^{-1}+m_\gamma^{-1},$$

$$M_\alpha^{-1}=m_\alpha^{-1}+(m_\beta+m_\gamma)^{-1},$$

$\alpha\beta\gamma$ being some cyclic permutation of ABC. With this definition, if we work in the over-all c.m. system, $-q_\alpha$ is just the momentum of particle α, and p_α is the momentum of particle β in the c.m. system of β and γ. μ_α is the reduced mass of β and γ, and M_α is the reduced mass of α and the c.m. of β and γ. The energies corresponding to p_α and q_α are

$$\nu_\alpha=p_\alpha^2/2\mu_\alpha,\quad \omega_\alpha=q_\alpha^2/2M_\alpha.$$

Any pair of the six vectors p_α and q_α can be chosen as independent, the transformation between them being specified by

$$\begin{aligned} p_\alpha&=-(\mu_\alpha/m_\gamma)q_\alpha-q_\beta,\\ p_\beta&=q_\alpha+(\mu_\beta/m_\gamma)q_\beta. \end{aligned} \tag{3.11}$$

The phase-space volume element is $dq_\alpha dp_\alpha$. The three-body scattering is described by an off-shell amplitude

$$\langle p'q'|T(W)|pq\rangle,$$

where W is the total energy of the three-particle system. We denote by $|pq\rangle$ a general three-particle state without specifying which pair of momenta is being used to label it. The energy-shell condition is

$$\omega_\alpha+\nu_\alpha=W=\omega_\alpha'+\nu_\alpha'. \tag{3.12}$$

One of the complications of three-particle scattering is that if one pair possesses a bound state, it is possible to have a transition from an initial state of three well-separated particles to a final state of the bound system well separated from the third particle, or a rearrangement collision, in which one bound pair breaks up and another is formed. All these processes are contained in the behavior of $\langle p'q'|T(W)|pq\rangle$ for different values of the variables.

Suppose that particles B and C have a bound state of binding energy $-\nu_{0A}$, i.e., their scattering amplitude takes the form

$$\langle p_A'|t_A(\nu_A)|p_A\rangle=\frac{g(p_A')g(p_A)}{D_0(\nu_A)}+\cdots. \tag{3.13}$$

Then the three-body scattering may be written[7]

$$\begin{aligned} \langle p_A'q_A'|T(W)|p_Aq_A\rangle=&\frac{g(p_A')}{D_0(p_A'^2/2\mu_A)}F_A(q_A',p_A,q_A;W)\\ &+\tilde{F}_A(p_A',q_A',q_A;W)\frac{g(p_A)}{D_0(p_A^2/2\mu_A)}+\frac{g(p_A')}{D_0(p_A'^2/2\mu_A)}\\ &\times\tau_{AA}(q_A',q_A;W)\frac{g(p_A)}{D_0(p_A{}^2/2\mu_A)}\\ &+R_{AA}(p_A',q_A',p_A'q_A';W), \end{aligned} \tag{3.14}$$

where R, F, and $\tilde{F}$ are all regular near $p_A^2/2\mu_A=\nu_{0A}$, $p_A'^2/2\mu_A=\nu_{0A}$. Figure 5 shows Eqs. (13) and (14) graphically: $g(p)$ is a vertex form factor for formation of the state BC, and $1/D_0$ is the propagator of the state BC.

The figure immediately suggests, and it can be proved,[7] that the quantity $\tau_{AA}(q_A',q_A;w)$ should be identified with the off-shell scattering amplitude for the two-particle scattering of particle A and the bound particle BC, which is a perfectly well-defined object

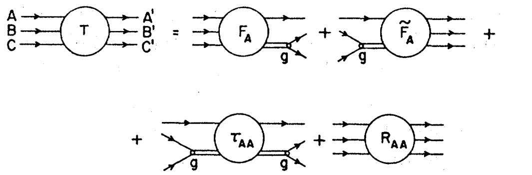

FIG. 5. Graphical representation of the expansion (3.14).

independent of this discussion of the three-particle scattering. The other functions F and $\tilde{F}$ have corresponding interpretations as the amplitudes for formation or breakup of a bound state by interaction with a third particle. Similar quantities τ_{AB}, τ_{BC}, $\cdots$, can obviously be introduced and will describe rearrangement collisions.

Note that the same discussion can be carried through when the vanishing of D_0 corresponds to a resonance or unstable-particle pole rather than to a bound state. The only essential difference is that now there exists no independent definition of τ as a scattering amplitude, since the absence of asymptotic states does not allow such amplitudes to be constructed. We shall therefore use Eq. (3.14) and similar ones to define unstable-particle scattering amplitudes.

We now wish to obtain the equations satisfied by τ_{AA}, τ_{BC}, etc. To do this, and again merely for simplicity, we make two further assumptions. First, we assume that for each pair of particles there is a single resonance or bound state, instead of a sum as in Eq. (3.9). Second, we assume that the potential $\langle p'q'|V(W)|pq\rangle$ responsible for the three-particle scattering described by $T(W)$ consists only of the separate two-particle potentials. The latter assumption allows us to work with the one-dimensional version of Eqs. (2.13) and (2.14):

$$T=\sum_{\alpha} T_\alpha-\sum_{\alpha,\beta} T_\alpha X_{\alpha\beta}T_\beta, \tag{3.15}$$

$$X_{\alpha\beta}=\bar{\bar{\delta}}_{\alpha\beta}G_0+\sum_{\gamma} \bar{\bar{\delta}}_{\alpha\gamma}G_0T_\gamma X_{\gamma\beta}, \tag{3.16}$$

where

$$\langle p_\alpha'q_\alpha'|T_\alpha(W)|p_\alpha q_\alpha\rangle =\delta(q_\alpha-q_\alpha')\langle p_\alpha'|t_\alpha(W-\omega_\alpha)|p_\alpha\rangle \tag{3.17}$$

and

$$\langle p_\alpha'q_\alpha'|G_0(W)|p_\alpha q_\alpha\rangle =\delta(p_\alpha'-p_\alpha)\delta(q_\alpha'-q_\alpha)(W-\omega_\alpha-\nu_\alpha+i\epsilon)^{-1}. \tag{3.18}$$

We now recall Eq. (3.13) and define the state

$$|A\rangle=\int dp_A g(p_A)|p_A\rangle$$

and, correspondingly,

$$|q_A,A\rangle=\int dp_A g(p_A)|p_A,q_A\rangle; \tag{3.19}$$

we can then write

$$t_A(\nu)=\frac{|A\rangle\langle A|}{D_0(\nu_A)}+\tilde{t}_A, \tag{3.20}$$

where $\tilde{t}$ is orthogonal to $|A\rangle$. Similarly

$$T_A(W)=\int dq_A \frac{|q_A,A\rangle\langle q_A A|}{D_0(W-\omega_A)}+\tilde{T}_A. \tag{3.21}$$

Now using (3.15), inserting (3.21), taking its matrix element between states $|p_A,q_A\rangle$ and $|p_A'q_A'\rangle$, and comparing with (3.14), we easily see that

$$\tau_{AA}(q_A',q_A;W)=-\langle q_A',A|X_{AA}|q_A,A\rangle.$$

Thus the matrix element of X_{AA} between states $|q_A,A\rangle$ and $|q_A',A\rangle$ gives the off-shell scattering amplitude of compound particle BC from particle A with initial and final c.m. momenta q_A and q_A'. In general, the scattering amplitude is

$$\tau_{\alpha\beta}(q_\alpha',q_\beta;W)=-\langle q_\alpha',\alpha|X_{\alpha\beta}(W)|q_\beta,\beta\rangle, \tag{3.22}$$

where X is defined by (3.16).

From (3.21) we can obtain the equations satisfied by these amplitudes. Up to this point we have merely assumed that the two-particle systems possessed the bound-state or resonant poles indicated explicitly for B and C by the first term in (3.13). We now make the rather drastic assumption that only these terms are important, i.e., that each two-particle scattering consists simply of a single bound-state or resonant term, and that therefore the three-particle interaction consists purely of rearrangements between different compound states. Equations (3.16) then reduce to a set of coupled, one-dimensional integral equations:

$$\tau_{\alpha\beta}(q_\alpha,q_\beta';W)=Z_{\alpha\beta}(q_\alpha,q_\beta';W) -\sum_{\gamma}\int_{-\infty}^{\infty} dq_\gamma''\frac{Z_{\alpha\gamma}(q_\alpha,q_\gamma'';W)\tau_{\gamma\beta}(q_\gamma'',q_\beta';W)}{D_0(W-\omega_\gamma'')}, \tag{3.23}$$

where $Z_{\alpha\beta}$ is the matrix element of $-G_0\bar{\bar{\delta}}_{\alpha\beta}$:

$$-G_0(W)=\int dp_\alpha dq_\alpha \frac{|p_\alpha q_\alpha\rangle\langle p_\alpha q_\alpha|}{\omega_\alpha+\nu_\alpha-W-i\epsilon}. \tag{3.24}$$

Using the kinematic relations (3.11), we can write, for $\alpha\neq\beta$,

$$\langle p_\alpha q_\alpha|p_\beta'q_\beta'\rangle=\delta\left(p_\alpha+\frac{\mu_\alpha}{m_\gamma}p_\beta'+\left(1-\frac{\mu_\alpha\mu_\beta}{m_\gamma^2}\right)q_\beta'\right) \times\delta\left(q_\alpha-q_\beta'+\frac{\mu_\beta}{m_\gamma}q_\beta'\right). \tag{3.25}$$

Also, since $\omega_\alpha=q_\alpha^2/2M_\alpha$ and $\nu_\alpha=p_\alpha^2/2\mu_\alpha$, we can use (3.11) to express

$$\omega_\alpha+\nu_\alpha=\omega_\beta+\nu_\beta=\frac{q_\alpha^2}{2\mu_\beta}+\frac{q_\beta^2}{2\mu_\alpha}+\frac{q_\alpha q_\beta}{m_\gamma}, \tag{3.26}$$

and thus, using the definitions (3.19) and (3.22), we finally obtain

$$Z_{\alpha\beta}(q_\alpha, q_\beta; W) = \bar{\delta}_{\alpha\beta} g_\alpha\left(-\frac{\mu_\alpha}{m_\gamma}q_\alpha - q_\beta\right) g_\beta^*\left(q_\alpha + \frac{\mu_\beta}{m_\gamma}q_\beta\right) \Big/ \left(\frac{q_\alpha^2}{2\mu_\beta} + \frac{q_\beta^2}{2\mu_\alpha} + \frac{q_\alpha q_\beta}{m_\gamma} - W - i\epsilon\right), \quad (3.27)$$

where, as above, $\alpha\beta\gamma$ form a cyclic permutation of ABC, and g_A, g_B, and g_C are the momentum-space form factors corresponding to the single compound state comprising the scattering of particles B and C, C and A, and A and B, respectively. These are precisely the Lovelace equations,[5] apart from a difference of normalization.

In the case of three identical particles of unit mass, this can be reduced to the single equation

$$\tau(x,x'; W) = Z(x,x'; W) - \int_{-\infty}^{\infty} dx'' \frac{Z(x,x''; W)\tau(x'',x'; W)}{D_0(W - \frac{3}{4}x''^2)}, \quad (3.28)$$

$$Z(x,x'; W) = -2\frac{g(-\frac{1}{2}x - x')g^*(x + \frac{1}{2}x')}{x^2 + x'^2 + xx' - W - i\epsilon}.$$

It will be convenient to separate the symmetric and antisymmetric parts as in Eq. (3.4); we write

$$\tau^\pm(x,x'; W) = \tfrac{1}{2}[\tau(x,x'; W) \pm \tau(-x, x'; W)],$$
$$Z^\pm(x,x'; W) = \tfrac{1}{2}[Z(x,x'; W) \pm Z(-x, x'; W)], \quad (3.29)$$

so that

$$\tau^\pm(x,x'; W) = Z^\pm(x,x'; W) - \int_{-\infty}^{\infty} \frac{Z(x,x''; W)\tau^\pm(x'',x'; W)}{D_0(W - \frac{3}{4}x''^2)} dx''. \quad (3.30)$$

IV. EXPLICIT SOLUTION OF THE EQUATIONS

Equation (3.30) resembles the one-dimensional Lippmann-Schwinger equation (3.3), with Z taking the place of the potential and $D_0^{-1}(w - \frac{3}{4}x''^2)$ taking the place of the propagator $(x''^2/2m - E)^{-1}$. The usual procedure for solving such an equation is via the Fredholm expansion. However, for certain values of W the kernel develops singularities corresponding to the possibility of multiple on-shell rescatterings, as discussed in Sec. II, which prevent the convergence of the Fredholm solution. These can be removed by iterating the equation a sufficient number of times, restoring it again to Fredholm form, but even then the presence of nearby energy-dependent singularities makes the convergence slow.

In a recent paper one of us[10] has presented an alternative technique for dealing with equations of this form when the kernels have a reasonably simple (though important) singularity structure, and applied it to various problems of potential scattering. This method is particularly suitable for the present problem; the existence of the rescattering singularities is not only easy to take into account but also actually simplifies the interpretation of the solution.

To illustrate the method we outline here the procedure in detail in the case of a simpler equation and then quote the corresponding results for Eq. (3.30). Consider the equation

$$a_0(z,u) = b(z,u) + \int_{-\infty}^{\infty} \frac{dw}{w - \lambda} k(z,w) a_0(w,u). \quad (4.1)$$

Here u and λ are (complex) parameters, the w integration runs along the real axis, and λ is assumed to lie off the real axis. The functions $k(z,w)$ and $b(z,w)$ are known explicitly and, together with $a_0(z,u)$, depend, in general, on the parameter λ as well as on their explicit arguments. The equation determines $a_0(z,u)$ as the solution of an integral equation for z real. For z complex and in the neighborhood of the real axis $a_0(z,u)$ is therefore determined by the expression (4.1) in terms of its values on the real axis, provided that the integral exists.

We wish to consider the case when k has a pole as a function of z:

$$k(z,w) = \frac{R(z,w)}{z - \phi(w)}. \quad (4.2)$$

The residue and position of the pole depend explicitly on w and parametrically on λ. We consider first the case where ϕ is a linear function, and assume where necessary that

$$\phi(z) = \alpha z + i\beta, \quad \alpha, \beta \text{ real} \quad (4.3)$$

but shall leave ϕ as an arbitrary function in the formulas. This maps the real axis into the line Φ a distance β above it, maps Φ into the line Φ_1, etc. (see Fig. 6). We assume that the singularities of R and b are khown and isolated and ignore their effect.

Equation (4.1) defines $a_0(z,u)$ except on the line Φ. It has a discontinuity across Φ given by

$$a_0(z_0 + i\epsilon, u) - a_0(z_0 - i\epsilon, u) = 2\pi i \int_{-\infty}^{\infty} \frac{dw}{w - \lambda} R(z_0,w)\delta(z_0 - \phi(w)) a_0(w,u)$$
$$= 2\pi i \int_{-\infty}^{\infty} \frac{dw}{w - \lambda} R(z_0,w) \frac{\delta(w - x_0)}{\phi'(x_0)} a_0(w,u)$$
$$= 2\pi i \frac{R(z_0,x_0)}{x_0 - \lambda} \frac{a_0(x_0,u)}{\phi'(x_0)}, \quad (4.4)$$

where z_0 is some point on Φ and $\phi(x_0) = z_0$, i.e., $x_0 = (z_0 - i\beta)/\alpha$. Let us now define, for arbitrary z, the

[10] D. Brayshaw, Phys. Rev. **167**, 1505 (1968).

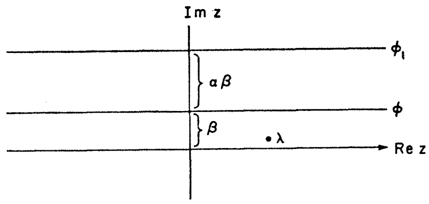

FIG. 6. Singular lines associated with continuing Eq. (4.1).

quantities z_1, $\Gamma(z,z')$, and $\theta_1(z)$:

$$\begin{aligned} \phi(z_1)&=z\,,\\ \Gamma_1(z,z')&=2\pi i R(z,z')/\phi'(z')\,,\\ \theta_1(z)&=0\,,\quad \text{for } z \text{ below } \Phi\\ &=1\,,\quad \text{otherwise}. \end{aligned} \tag{4.5}$$

Then the function

$$a_1(z,u)=b(z,u)+\int_{-\infty}^{\infty}\frac{dw}{w-\lambda}k(z,w)a_1(w,u) +\theta_1(z)\Gamma_1(z,z_1)a_1(z_1,u)/(z_1-\lambda) \tag{4.6}$$

is identical with $a_0(z,u)$ below Φ, is continuous across Φ, and is defined except when z_1 lies on Φ, i.e., when z lies on Φ_1. The discontinuity across Φ_1 can be removed in the same way. After the nth step the equation picks up an extra term,

$$D_n(z,u)=\theta_n(z)\Gamma_n(z,z_n)a(z_n,u)/(z_n-\lambda)\,, \tag{4.7}$$

where

$$\begin{aligned} \phi(z_n)&=z_{n-1}\,,\\ \Gamma_n(z,z_n)&=2\pi i\frac{\Gamma_{n-1}(z,z_{n-1})}{z_{n-1}-\lambda}\frac{R(z_{n-1},z_n)}{\phi'(z_n)}\,,\\ \theta_n(z)&=0\,,\quad \text{for } z \text{ below } \Phi_n\\ &=1\,,\quad \text{otherwise}. \end{aligned} \tag{4.8}$$

Thus, finally,

$$a(z,n)=b(z,n)+\int_{-\infty}^{\infty}\frac{dw}{w-\lambda}k(z,w)a(w,n) +\sum_{n=1}^{\infty}D_n(z,u) \tag{4.9}$$

defines a function in the entire plane free from any discontinuities due to the pole of k. We can now write an integral representation for $a(t,u)$:

$$a(t,u)=\frac{1}{2\pi i}\oint\frac{a(z,u)}{t-z}dz\,, \tag{4.10}$$

where the integration contour surrounds all singularities of $a(z,u)$ and some suitably convergent region at large $|z|$. The singularities of $a(z,u)$ consist of the poles in D_n of the form $(z_n-\lambda)^{-1}$ together with all those resulting from the behavior of b and R. [The other poles in D_n of the form $z_r-\lambda$ do not contribute, since, when $z_r=\lambda$, $\theta_n(z_r)=0$ for $r<n$.] We combine the latter contributions with any remaining from the large-$|z|$ integral into some function $B(t,u)$ and evaluate the former ones explicitly. The contribution of the nth term to (4.10) is

$$\alpha_n(t,\lambda)a(\lambda,u)\,,$$

where

$$\alpha_n(t,\lambda)=\frac{(2\pi i)^n R(\phi(\lambda),\lambda)}{t-\phi^n(\lambda)}\prod_{r=1}^{n-1}\frac{R(\phi^{r+1}(\lambda),\phi^r(\lambda))}{\phi^r(\lambda)-\lambda}\,, \tag{4.11}$$

and

$$\phi^1(\lambda)=\phi(\lambda)\,,\quad \phi^r(\lambda)=\phi(\phi^{r-1}(\lambda))\,.$$

Thus we can write

$$\begin{aligned} a(t,u)&=B(t,n)+C(t,\lambda)a(\lambda,u)\,,\\ C(t,\lambda)&=\sum_n \alpha_n(t,\lambda)\,. \end{aligned} \tag{4.12}$$

Setting $t=\lambda$ and solving for a, we find

$$a(\lambda,u)=\frac{B(\lambda,u)}{1-C(\lambda,\lambda)}\,, \tag{4.13}$$

and hence

$$a(t,u)=B(t,u)+\frac{C(t,\lambda)B(\lambda,u)}{1-C(\lambda,\lambda)}\,. \tag{4.14}$$

This is the central result that we need.

The procedure remains substantially the same for more complicated equations of this general type. For example, either or both of the poles $[z-\phi(w)]^{-1}$ and $(w-\lambda)^{-1}$ may be replaced with a product of similar poles, $\prod_i[z-\phi_i(w)]^{-1}$ and $\prod_j(w-\lambda_j)^{-1}$. The first simply increases the number of lines across which the continuation must be made, these now being obtained from the real axis by successively applying the different mappings $\phi_i(W)$ an arbitrary number of times in all possible orders. The second leads to the replacing of (4.12)-(4.14) with coupled equations. If two of the poles $\phi_i(\lambda)$ coincide for some λ_0 in such a way as to pinch the W integration contour, then the corresponding contribution to $C(\lambda,\lambda)$ will develop a pole as a function of λ at the position λ_0.

A further complication arises if the form of the mapping $\phi(w)$ becomes other than linear. For example, if $\phi=\alpha z^2+i\beta$, then

$$z_n=[(z_{n-1}-i\beta)/\alpha]^{1/2}\,,$$

so that each term D_n introduces a new square-root branch cut as a function of z. In this case the continuation procedure must be carried out over all sheets of the resulting Riemann surface.

Let us now return to Eq. (3.30). It has precisely the form of (4.1) together with all the complications just described. The denominator $D_0(W-\frac{3}{4}x''^2)$ plays the role of the denominator $w-\lambda$ in (4.1), but has two roots. For the case of a bound-state denominator, with

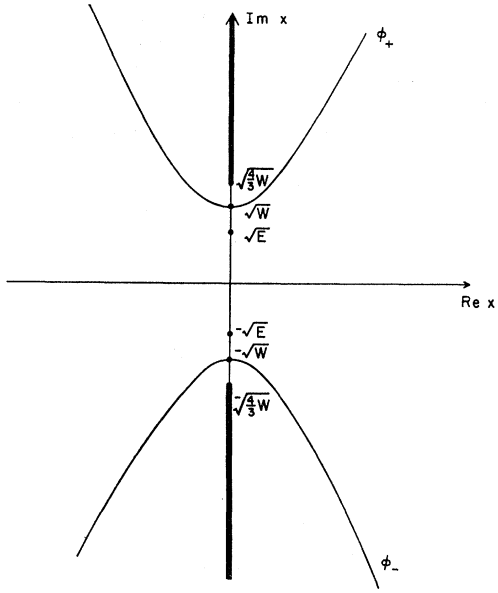

FIG. 7. Singularity structure associated with Eq. (3.30) for $W<\nu_0$, $\nu_0<0$.

a zero at $W-\frac{3}{4}x''^2=\nu_0<0$, or $x''^2=E=\frac{4}{3}(W-\nu_0)$, the roots are equal and opposite and the existence of both is already taken into account by the symmetrization procedure. The term $Z(x,x';W)$ plays the role of $k(z,w)$ but has a double-pole structure because of the quadratic denominator:

$$x^2+x'^2+xx'-W-i\epsilon=[x-\phi_+(x')][x-\phi_-(x')],$$

where

$$\phi_\pm(x)=-\tfrac{1}{2}x\pm(W-\tfrac{3}{4}x^2)^{1/2}. \tag{4.15}$$

In addition, it may have additional poles coming from the form factors $g(p)$. Let us assume these to be absent and treat g as a constant for the moment, though we shall return to this point in Sec. V. The mappings $\phi_\pm$ are illustrated in Fig. 7, for the case when W is real, negative, and less than ν_0, i.e., $E<0$. The lines $\Phi_\pm$ are given by $z=\phi_\pm(x)$ for x real. The branch points of the mapping occur at $\pm(\frac{4}{3}W)^{1/2}$, and the functions $\phi_\pm$ have the properties

$$\begin{aligned}\phi_\pm(\phi_\pm(z))&=\phi_\mp(z) \quad\text{or}\quad z,\\ \phi_\pm(\phi_\mp(z))&=z \quad\text{or}\quad \phi_\pm(z),\end{aligned} \tag{4.16}$$

the two alternatives corresponding to the two different sheets linked by the branch cuts beginning at $\pm(\frac{4}{3}W)^{1/2}$.

Continuation across the lines $\Phi_\pm$ gives the term analogous to $\alpha_1(t,\lambda)$ in the above discussion. (The contributions α_n for $n>1$ all correspond to continuing across the same lines, or across the real axis, but on one of the other sheets.) The only possible pinch of the x'' integration comes when $\phi_+(E^{1/2})$ coincides with $\phi_-(-E^{1/2})$. None of the contributions from the higher iterations can ever pinch the contour, since they lie on a different sheet. The sum of all these nonsingular contributions can be expressed as a contour integral along the branch cut.

Carrying out this procedure in detail, we obtain the following results. For convenience we recall some earlier definitions. We are dealing with three identical particles of unit mass and total kinetic energy W. The off-shell amplitude for two-body scattering at energy ν has been assumed to be

$$t(p,p';\nu)=gg^*/D_0(\nu).$$

$D_0(\nu)$ vanishes linearly near $\nu=\nu_0$:

$$D_0(\nu)\approx(\nu-\nu_0)D_0'(\nu_0).$$

The compound state corresponding to the zero of D_0 can scatter from the third particle with off-shell amplitude $\tau(q,q';\frac{3}{4}E+\nu_0)$, where $E=\frac{4}{3}(W-\nu_0)$ is the energy above threshold for the quasi-two-particle system. The symmetric and antisymmetric parts of τ are $\tau_\pm$.

We now define the following functions, W being understood to be a parameter where necessary:

$$\beta_\pm(x,y)=\tfrac{1}{2}gg^*\left(\frac{1}{y+x}\pm\frac{1}{y-x}\right),$$

$$B_\pm(x,y)=\pm\frac{\beta_\pm(x,\phi_-(x))}{(W-\frac{3}{4}y^2)^{1/2}}, \tag{4.17}$$

$$C_\pm(x,y)=B_\pm(x,y)+2\int_{\Gamma(W)}\frac{dx'}{(W-\frac{3}{4}x'^2)^{1/2}}\times\beta_\pm(x,x')D_\pm(x',y).$$

$\Gamma(W)$ runs along the cut from $(\frac{4}{3}W)^{1/2}$ to $+i\infty$, and $D_\pm$ will be defined below.

Let

$$\Lambda_\pm(x,y)=C_\pm(x,y)-\frac{8\pi iC(x,E^{1/2})C(E^{1/2},y)}{3E^{1/2}D_0'(\nu_0)+8\pi iC(E^{1/2},E^{1/2})}. \tag{4.18}$$

Then

$$\tau_\pm(q,q';W)=\Lambda_\pm(q,q')\pm\Lambda_\pm(q,-q'). \tag{4.19}$$

The functions $D_\pm(x,y)$ are defined as follows. Let

$$\xi_\pm(x,x')=\frac{\beta_\pm(x,x')}{D_0(W-\frac{3}{4}x^2)},$$

$$K_\pm(x,x')=\pm[\xi_\pm(\phi_+(x),x')+\xi_\pm(\varphi_-(x),x')], \tag{4.20}$$

$$V_\pm(x,y)=\pm\left(\frac{B_\pm(\phi_+(x),y)}{D_0(W-\frac{3}{4}\phi_+^2(x))}+\frac{B_\pm(\phi_-(x),y)}{D_0(W-\frac{3}{4}\phi_-^2(x))}\right),$$

where $\phi_\pm$ are defined by (4.15).

Then $D_\pm$ satisfies the equation

$$D_\pm(x,y)=V_\pm(x,y)+2\int_{\Gamma(W)}\frac{dx'}{(W-\frac{3}{4}x'^2)^{1/2}}\times K_\pm(x,x')D_\pm(x',y). \tag{4.21}$$

The solution to (4.21) may be obtained by standard numerical techniques.

V. BEHAVIOR OF THE SOLUTION: THREE-BODY BOUND STATES AND RESONANCES

In Sec. IV we have not yet "solved" the equations for $\tau_\pm(q,q';W)$ but we have written the solution in a form which exhibits explicitly the effects of the important singularities of the kernel and which involves the solution of a much less singular equation which can be solved numerically. We now discuss the analytic structure of the expressions (4.18)–(4.20).

We first note that these expressions were derived for fixed W in the region

$$\begin{aligned} &\mathrm{Re}W>0\,,\\ &\mathrm{Im}W^{1/2}\gtrsim 0\,, \end{aligned} \tag{5.1}$$

i.e., the physical scattering region. It was also assumed that ν_0 was a bound-state energy and hence real and negative. In order to discuss the properties of our result for other values of W and for the case when ν_0 is a resonance energy, we must analytically continue it to the desired region of these variables. The form that we had obtained for $\tau_\pm(q,q',W)$ does not represent the correct continuation in all regions.

If we allow W to vary, holding ν_0 fixed, then as we cross the curve

$$\mathrm{Im}(\tfrac{3}{2}E^{1/2}-\nu_0^{1/2})=0$$

our expression for $\Lambda_\pm(x,y)$ develops a discontinuity, which comes about because of the appearance of an end-point singularity of the integral over the contour $\Gamma(W)$ in (4.17). This boundary curve is a parabola, illustrated in Fig. 8(a). [As above, we use $E=\frac{4}{3}(W-\nu_0)$.] If we replace (4.18) with

$$\Lambda_\pm(x,y)=C_\pm(x,y)-\tilde{\theta}(W)\times\frac{8\pi iC_\pm(x,E^{1/2})C_\pm(E^{1/2},y)}{3E^{1/2}D_0'(\nu_0)+8\pi iC_\pm(E^{1/2},E^{1/2})}\,, \tag{5.2}$$

where

$$\begin{aligned} \tilde{\theta}(W)&=1\,, \quad \text{if } \mathrm{Im}(\tfrac{3}{2}E^{1/2}-\nu_0^{1/2})<0\\ &=0\,, \quad \text{if } \mathrm{Im}(\tfrac{3}{2}E^{1/2}-\nu_0^{1/2})>0 \end{aligned}$$

then $\Lambda_\pm$ is continuous across the boundary and (4.19) is valid for all W on the physical sheet $\mathrm{Im}W^{1/2}>0$.

To discuss the case when ν_0 is a resonance energy, $\mathrm{Re}\nu_0>0$, $\mathrm{Im}\nu_0^{1/2}<0$, we must consider what happens for fixed W when we allow ν_0 to vary. As we go from the bound-state to the resonance value of ν_0, allowing ν_0 to pass above the origin and then down through the square-root branch cut on to the second sheet, we reach the situation shown in Fig. 8(b). In this case a discontinuity would develop in our expression as the boundary curve moves through the fixed physical value of W with which we started. The curve moves through the $\sqrt{W}$ cut and the $\sqrt{E}$ cut. Again the expression (5.2) gives the correct continuation. In this case, however, on the physical sheet we have

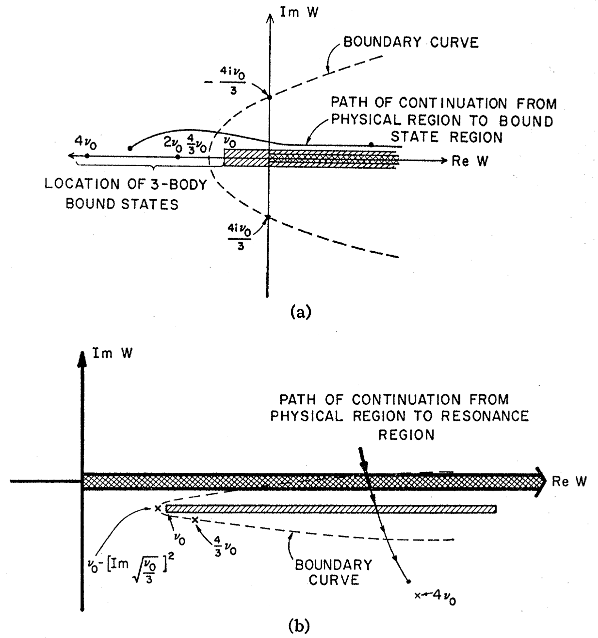

FIG. 8. (a) Singularity structure of the solution (5.2) for the bound-state case $\nu_0<0$. (b) Singularity structure of the solution (5.2) for the resonant case $\mathrm{Re}\nu_0>0$, $\mathrm{Im}(\nu_0^{1/2})<0$. The heavy lines indicate the physical sheet, the intermediate lines correspond to the sheet $\mathrm{Im}(E^{1/2})>0$, $\mathrm{Im}(W^{1/2})>0$, and the finest lines correspond to the sheet $\mathrm{Im}(E^{1/2})<0$, $\mathrm{Im}(W^{1/2})<0$.

$$\Lambda_\pm(x,y)=C_\pm(x,y)$$

and the second term does not occur until we continue as indicated in Fig. 8(b) through both branch cuts to the sheet on which

$$\mathrm{Im}W^{1/2}<0\,,\quad \mathrm{Im}E^{1/2}<0\,,$$

and cross the boundary curve.

Having thus obtained an expression for $\tau_\pm(q,q';W)$ which is valid for all the possible values of W and ν_0 which are of interest, we can now discuss the analytic structure of this expression.

A bound state or resonance of the three-particle system will appear, as discussed in Sec. II, when $\tau_\pm(q,q';W)$ develops a pole as a function of W independent of q and q'. Such a pole will certainly occur when the denominator

$$\Delta_\pm(E)=3E^{1/2}D_0'(\nu_0)+8\pi iC_\pm(E^{1/2},E^{1/2}) \tag{5.3}$$

vanishes, unless cancelled by the terms

$$C(q,E^{1/2})C(E^{1/2},q')$$

in the numerator. [As above, we have set $E=\frac{4}{3}(W-\nu_0)$.] There are three sources of singularities in $\Delta_\pm(E)$. First, there is the square-root branch point at $E=0$. This corresponds to the threshold for the quasi-two-particle system of the two-particle resonance or bound state interacting with the third particle. $C(E^{1/2},E^{1/2})$ contains the term $B(E^{1/2},E^{1/2})$ and an integral. The latter possesses a square-root branch point at $W=0$, corresponding to the three-particle threshold. If $\nu_0<0$, corresponding to a bound state of the two-body system, then the point $E=0$ lies on the negative real axis for W. Both square-root branch cuts lie to the right along the real axis. The physical region for scattering of the third particle by the bound state is reached by approaching the real axis from above in the sheet defined by

$$\mathrm{Im}E^{1/2}>0\,,\quad \mathrm{Im}W^{1/2}>0\,.$$

A three-particle bound state, or bound state of the quasi-two-particle system, occurs as a pole on the negative real axis $E=E_0<0$, reached from the physical region by passing above both branch points, as illustrated in Fig. 8(a). If $\mathrm{Re}\nu_0>0$, $\mathrm{Im}\nu_0^{1/2}<0$, corresponding to a resonance of the two-particle system, the point $E=0$ moves above the branch point $W=0$ and onto the lower sheet $\mathrm{Im}W^{1/2}<0$, below the axis. A resonance of the whole system now occurs as a pole at E_0 lying below both cuts on the sheet $\mathrm{Im}E^{1/2}<0$, $\mathrm{Im}W^{1/2}<0$, as shown in Fig. 8(b).

The term $B_\pm(E^{1/2},E^{1/2})$ has poles when $\frac{1}{2}E^{1/2}+\nu_0^{1/2}=\pm E^{1/2}$. The region of E in which we are interested is the region where resonances or bound states may develop. For both cases, discussed above, this can be seen to be the sheet where $\mathrm{Im}E^{1/2}$ has the same sign as $\mathrm{Im}\nu_0^{1/2}$. Thus only the pole corresponding to the upper sign (i.e., to the point $E=4\nu_0$) lies on the relevant sheet. The pole at $E=(4/9)\nu_0$ is far away from the physical region. Examination of the integral occurring in $C(E^{1/2},E^{1/2})$ leads us to the conclusion that its only singularity, other than the threshold cuts, is a branch point at $E=(4/9)\nu_0$ which again is on the sheet $\mathrm{Im}E^{1/2}/\mathrm{Im}\nu_0^{1/2}<0$, and is therefore far from the physical region. Thus in the neighborhood of $E=4\nu_0$ the denominator $\Delta_\pm(E^{1/2})$ contains the pole from $B_\pm(E^{1/2},E^{1/2})$ together with a relatively slowly varying background. Provided that the residue of the pole, which is proportional to gg^*, is not too large, $\Delta_\pm(E^{1/2})$ will develop a zero in the vicinity of the pole. Thus $\tau_\pm(q,q';W)$ will develop a pole near the point $E=4\nu_0$, or $W=4\nu_0$, if $\tilde{\theta}(4\nu_0)\neq0$. The latter condition is crucial, since, as discussed above, the denominator $\Delta_\pm(E^{1/2})$ does not appear in $\tau_\pm(q,q';W)$ unless $\tilde{\theta}(4\nu_0)\neq0$. Considering the definition of $\tilde{\theta}(W)$ given in (5.2), we see that the $\tilde{\theta}(4\nu_0)$ vanishes when ν_0 is a bound-state energy but does not vanish when ν_0 is a resonant energy.

We have thus arrived at the principal result of the present paper:

If the scattering of two identical spinless particles moving in one dimension can be described by a single separable term corresponding to a resonance of energy ν_0, with a slowly varying form factor, then the corresponding three-particle scattering amplitude will develop a denominator pole at an energy $4\nu_0$ on the second unphysical sheet. For a narrow two-body resonance this suggests the occurrence of a three-body resonance near $4\nu_0$. The corresponding statement in the case of a two-particle bound state is not true.

This result has, of course, been obtained in a highly specialized model. Before we discuss its relation to other work it is important to examine the special assumptions of the model and enquire to what extent the result depends on them and to what extent it illustrates a general connection between the positions of two- and three-particle resonances. These assumptions can be divided into essentially kinematical and essentially dynamical ones. In the former class are the assumptions of one-dimensional motion, identity of the particles, absence of spin, and nonrelativistic kinematics. These assumptions are introduced to simplify the details of the calculations, and similar behavior occurs even when any of these assumptions is relaxed. In Appendix A we shall give the explicit form of the equations and their solutions when various of these restrictions are relaxed. The dynamical assumptions, which we wish to consider here, are the assumption of a single separable term for the two-body amplitude and the constancy of the form factors.

Clearly, in any real problem there will always be background terms in addition to a separable-pole contribution. Part of this background may consist of other separable poles, in which case Eq. (3.30) is replaced with a set of coupled equations, all the functions discussed in this section become matrices, and the denominator $\Delta_\pm$ becomes a determinant. There will now be a number of points corresponding to poles of the matrix elements $B_{ij}{}^\pm(E^{1/2},E^{1/2})$, and if they are well separated, the above argument will apply to each. If the background is smooth, it can be regarded as adding an additional slowly varying contribution to Eq. (3.30). Since the argument of this section already assumes that there exists a pole in the discontinuity of (3.30) across the kinematic cut together with a slowly varying background, nothing is changed in principle by the addition of such a term.

The constant-form-factor assumption is certainly not justified in practice; in general, the form factors must be normalized and obviously cannot be constant. In fact, typically the form factor $g(p)$ behaves like $(p^2+\Lambda^2)^{-1}$ and develops poles when $p=\pm i\Lambda$. In Eq. (3.30) the arguments of the form-factor terms are $q+\frac{1}{2}q'$ and $q'+\frac{1}{2}q$. The corresponding lines in the q plane across which Eq. (3.30) develops a discontinuity will be the straight lines parallel to the real axis at a distance $\pm\Lambda$ from it. Thus, provided that $|\Lambda|$ is large compared with $2|\mathrm{Im}(\nu_0)^{1/2}|$, this will not give rise to any rapidly varying background to the discontinuity across the lines $\Phi_\pm$ in the relevant region, and thus the conclusion about the occurrence of a pole near $E=4\nu_0$ will be unchanged. The leading contribution of the form-factor singularities can, in fact, be explicitly taken into account,[11] confirming the above conclusion.

We therefore argue that the only features of the model which are crucial to obtain the result quoted earlier in this section are the singularity structure arising from the free three-body Green's function and the two-particle scattering poles, which will be present in almost any three-body dynamical scheme dominated by a two-body resonance or bound state, and the assumption of slowing varying form factors. This structure is illustrated graphically in Fig. 9. The denominator will contain terms which result from the on-shell discontinuities of the various terms on the right-hand side of Fig. 9 across the lines $\Phi_\pm$, i.e., the lines where the three-particle Green's function is singular. The first term is itself singular but does not give rise to a singular discontinuity. The second term, or "box graph" in S-matrix language, is responsible for the singular term in the denominator, in addition to possessing itself singularities whose positions depend on the off-shell momenta q and q'.

The requirement on the form factor ensures that each successive two-particle interaction must be sufficiently localized to be distinguished from the next, so that the structure illustrated in Fig. 9 can have some meaning. Mathematically, if $g(p)$ has too close a singularity in momentum space, then the lines $\Phi_\pm$ cease to be the most important sources of singularities in the structure of τ. The two dimensions which are important are the size of the compound particle and the wavelength associated with the three-body kinetic energy. If the latter is large compared with the former, then the three-particle intermediate states can really be assumed to propagate freely, giving rise to the structure of Fig. 9. Again this can be expressed in momentum space as the requirement that the form-factor singularities should be far away compared with the Green's-function singularities.

VI. RELATION TO OTHER CALCULATIONS

In Sec. V we discussed in detail the denominator occurring in Eq. (4.18). The numerator term contains the dependence on the off-shell momenta and includes the singularities of the first terms of the iterative expansion of Eq. (3.30). Although it is the singularities of the term $Z^\pm(x,x';W)$ which are responsible, via the kernel, for the analytic structure of the amplitude, the denominator singularity is, in fact, a property of the sum of the entire perturbation series. On the energy shell (i.e., for $x^2=x'^2=E$) the numerator has poles at both $W=4\nu_0$ [from $B(E^{1/2},E^{1/2})$] and $W=\frac{4}{3}\nu_0$ [from $B(E^{1/2},-E^{1/2})$ on the physical sheet, where $\mathrm{Im}W^{1/2}$, $\mathrm{Im}E^{1/2}$, and $\mathrm{Im}\nu_0{}^{1/2}$ all have the same sign]. Singularities are also present in the other terms of the numerator. However, the positions of these singularities are dependent on the values of the variables x and x'. In any actual experimental situation a three-particle state always occurs as a final state, and a three-particle scattering amplitude always enters a calculation as a half-on-shell expression integrated over the off-shell momentum, e.g.,

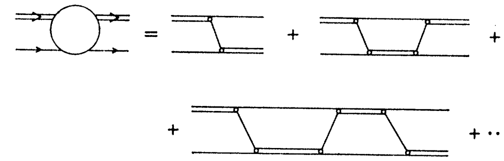

FIG. 9. Graphical representation of Eq. (3.30).

$$\int dq\, f(q)\tau(q,E^{1/2};W)\,, \tag{6.1}$$

where $f(q)$ represents the amplitude for producing the three-particle state. In general, this integration will remove the numerator singularities but, of course, will not affect the denominator zeros. The fact that the numerator singularities, which themselves disappear after integration, are responsible for a denominator singularity, which remains, has been the cause of much confusion. This confusion stems at least in part from a previous paper by one of us[2] essentially pointing out the existence of the singularities in the inhomogeneous term Z for the $\pi\pi N$ system and observing without any explanation that a candidate for a three-body resonance occurred nearby. Since then, many authors have carried out calculations to study the question that we have considered here, that is, whether there could be a connection between the singularities of the on-shell Born term and an observable three-body effect at a near position.

These calculations have been of two kinds. On the one hand, many authors have done calculations which would correspond to looking at the first terms in the expansion for $\tau_\pm$ and effectively inserting them into an integral of the type (5.1). Most of these calculations have been done in the framework of S-matrix theory, working entirely with on-shell amplitudes, but the result is the same. A good summary of these calculations is given in a paper by Schmid,[4] and the conclusion is correctly drawn that the direct effects of the numerator singularities will never be observable.

Another class of calculations has been to examine soluble models and to examine their solutions for three-body resonances or bound states. The present work is in this spirit. However, most of the work has been done on the framework of the Lee model,[12] which has the complication of inextricably coupling two- and three-particle channels and of limited flexibility in the parameters of the two-particle state. More important, it has usually been carried out for static kinematics.

[11] D. Brayshaw, Ph.D. thesis, Rockefeller University, 1968 (unpublished).

[12] P. K. Srivastava, Phys. Rev. **131**, 461 (1963); F. S. Chen-Cheung and C. M. Sommerfield, *ibid.* **152**, 1401 (1966).

For static kinematics the singularity structure degenerates. A good example is discussed by Sawyer,[13] where he shows that formally a resonance exists at the expected energy but that the numerator vanishes at the same time in the static limit.

A calculation by Hwa[14] looked, in the framework of S-matrix theory, at the singularity structure in general, and obtained a relation between the denominator function and the on-shell behavior of the three-particle scattering amplitude. However, the only contribution to the singularities of the amplitude that he considered explicitly was that of the first term on the right-hand side of Fig. 9. The principal effect in the off-shell analysis comes from the singularity involving one intermediate state, i.e., from the "box graph," Fig. 9. It is likely that applying Hwa's procedure to the singularities of this term would show a denominator pole on the correct sheet and consequently a nearby resonance.

In the three-dimensional case, the singularity analogous to the pole of $B(E^{1/2},E^{1/2})$ is of a logarithmic nature. At first sight it might seem too "weak" a singularity to dominate. However, one can check this by direct numerical calculation. A calculation has been done on the three-α system, using a separable form for the two-α scattering amplitude which satisfies unitarity, and which has the poles corresponding to the 0^+ ground and the 2^+ excited state of Be^8. We then have two resonant energies ν^0 and ν^2, and, according to the above discussion, we expect singularities in the three-α amplitude at the points $4\nu^0$ and $4\nu^2$. If three-α resonances are indeed associated with these points, they should appear as excited states of C^{12} with excitation energies 7.652 and 18.9 MeV, respectively. Furthermore, they should correspond to the spin-parity assignments even-plus and decay via the $Be^8+\alpha$ channel. Experimentally,[15] C^{12} is known to possess such a 0^+ state at 7.656 MeV, a 0^+ state at 17.8 MeV, and a 2^+ state at 18.34 MeV. It is clear that the agreement is quite good. More detailed calculations[11] of this system confirm that the corrections to these values are small and in the right direction.

Another numerical calculation which has been done is that of Basdevant and Kreps,[16] who looked at an equation analogous to the three-dimensional form of Eq. (3.30) but with relativistic kinematics, and applied it to the three-pion problem, putting in the ρ-meson pole as the sole pion-pion scattering. They obtained various resonances: In a subsequent study[17] it was shown that their solutions indeed had singularities near the analog of the point $\frac{4}{3}\nu_0$ on the unphysical sheet, and they did not search high enough on the physical sheet to investigate the vicinity of $4\nu_0$. If any case, since the ρ width is already large, this point is rather far from the real axis and would yield a very broad resonance. The resonances that they did see where observed to correspond to the π-p threshold. If one examines their kernel, it is easily shown that it contains a threshold pole which induces a resonance by precisely the same mechanism as the pole at $4\nu_0$ in the present example. It is not clear to us whether that threshold pole should really be present. It does not seem to be there in the kernel derived from the relativistic Lovelace equations.

[13] R. F. Sawyer, Phys. Rev. **139**, B151 (1965).
[14] R. C. Hwa, Phys. Rev. **130**, 2580 (1963).
[15] W. C. Olsen *et al.*, Nucl. Phys. **61**, 625 (1965); V. V. Balashov and I. Rotter, *ibid.* **61**, 138 (1965).
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[17] J. L. Basdevant and R. L. Omnes, Phys. Rev. Letters **17**, 775 (1966).

VII. CONCLUSIONS

We have examined in detail a highly simplified, though consistent, model of the scattering process indicated schematically in Fig. 9: the propagation of two particles, held together by the successive resonant scattering of a third particle from each in turn. In this model it was found that a three-particle resonance might be expected to develop near the higher of the two energies determined by the singularities of the first term in Fig. 9 interpreted as an on-shell process. This conclusion stemmed basically from the kinematic structure of the three-particle process and, in particular, is associated with the possible singularities of the intermediate-state propagator associated with the on-shell rescattering. Such singularities are present in more realistic situations and the corresponding neighborhoods for seeking three-particle resonances can be determined. The principal dynamical assumption made is that the singularities of the momentum-space form factors for the decay of the two-particle resonance are far away; this ensures that the successive vertices in Fig. 9 can easily be resolved. The occurrence of such resonances is still a quantitative problem. An example was given from the three-α particle problem in nuclear physics. In the relativistic case, which is still further removed from the simple model but contains the essential ingredients, it is clear that low-lying excitations of three-particle systems involving a baryon isobar would be good candidates to examine.

ACKNOWLEDGMENTS

The work contained in this paper owes a great deal to many colleagues who have listened patiently and commented constructively on various earlier versions of these ideas over the past several years. We are indebted to Dr. G. Tiktopoulos and Dr. N. Khuri for particularly helpful discussions relating to the analytic behavior of the solution.

APPENDIX A: EQUATIONS AND SOLUTION FOR THE THREE-DIMENSIONAL PROBLEM

In order to gain some insight into more realistic models, we consider a model identical in all respects with the one discussed in the main body of the paper, excepting the restriction to one-dimensional motion. We are thus dealing with the three-dimensional version of (3.28), which has the form

$$\tau_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=Z_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)+\sum_{\sigma}\int d\mathbf{q}''\frac{Z_{\lambda\sigma}(\mathbf{q}',\mathbf{q}'';W)}{D(W-\frac{3}{4}q''^2)}\tau_{\sigma\mu}(\mathbf{q}'',\mathbf{q};W), \quad (A1)$$

where

$$Z_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=-\frac{2g_\lambda^*(\mathbf{p}')g_\mu(\mathbf{p})}{q'^2+\mathbf{q}'\cdot\mathbf{q}+q^2-W-i\epsilon}.$$

The above equation assumes a two-body bound state of angular momentum l, and the subscripts $\lambda\mu$ refer to the initial and final magnetic quantum numbers of the composite two-particle system, i.e., $-l\leq\mu\leq l$. The vectors $\mathbf{p}$ and $\mathbf{p}'$ are related to $\mathbf{q}$ and $\mathbf{q}'$ by

$$\mathbf{p}=\tfrac{1}{2}\mathbf{q}+\mathbf{q}',\qquad \mathbf{p}'=-\mathbf{q}-\tfrac{1}{2}\mathbf{q}'. \quad (A2)$$

The form factor $g_\mu(\mathbf{p})$ can be written in the form

$$g_\mu(\mathbf{p})=g(p)Y_{l\mu}(\hat{p}\cdot\hat{n})e^{i\mu\phi}, \quad (A3)$$

in terms of some spherical coordinate system with the z axis along the $\hat{n}$ direction. If we choose this direction to that of $\hat{q}$, the amplitude $\tau_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)$ becomes a helicity amplitude. To remove the angular dependence we project out partial waves of total angular momentum J:

$$\tau_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=\frac{1}{2\pi}\sum_{J=0}^{\infty}\tfrac{1}{2}(2J+1)\tau_{\lambda\mu}{}^J(q',q;W)d_{\lambda\mu}{}^J(\hat{q}'\cdot\hat{q}),$$
$$\tau_{\lambda\mu}{}^J(q',q;W)=2\pi\int_{-1}^{1}dz\, d_{\lambda\mu}{}^J(z)\tau_{\lambda\mu}(\mathbf{q}',\mathbf{q};W), \quad (A4)$$

where $z=\hat{q}'\cdot\hat{q}$, and we write similar equations for $Z_{\lambda\mu}$ and $Z_{\lambda\mu}{}^J$. In fact,

$$Z_{\lambda\mu}{}^J(q',q;W)=-4\pi\times\int_{-1}^{1}\frac{dz\,d_{\lambda\mu}{}^J(z)Y_{l\lambda}(\hat{p}'\cdot\hat{q}')Y_{l\mu}(\hat{p}\cdot\hat{q})g^*(p')g(p)}{q'^2+qq'z+q^2-W-i\epsilon}, \quad (A5)$$

with

$$p'=(q^2+qq'z+\tfrac{1}{4}q'^2)^{1/2},\quad p=(\tfrac{1}{4}q^2+qq'z+q'^2)^{1/2},\quad \hat{p}'\cdot\hat{q}'=-(zq+\tfrac{1}{2}q')/p',\quad \hat{p}\cdot\hat{q}=(\tfrac{1}{2}q+zq')/p. \quad (A6)$$

The integral equation for the amplitude $\tau_{\lambda\mu}{}^J$ is then

$$\tau_{\lambda\mu}{}^J(q',q;W)=Z_{\lambda\mu}{}^J(q',q;W)+\sum_{\sigma}\int_0^{\infty}\frac{q''^2dq''\,Z_{\lambda\sigma}{}^J(q',q'';W)\tau_{\sigma\mu}{}^J(q'',q;W)}{D(W-\frac{3}{4}q''^2)}. \quad (A7)$$

We thus obtain a set of coupled one-dimensional equations to which we can apply the same method of solution discussed in the one-dimensional model. In the latter calculation we assumed the $g(p)$ to be constant for convenience. Here we shall assume that $g(p)$ has the form

$$g(p)=p^lh(p^2), \quad (A8)$$

where $h(p^2)$ is an entire function of p^2. In terms of the function $h(p^2)$ we rewrite Eq. (A5) as

$$Z_{\lambda\mu}{}^J(q',q;W)=\int_{-1}^{1}\frac{dz\,f_{\lambda\mu}{}^J(z;q',q)h^*(p'^2)h(p^2)}{q^2+q'^2+qq'z-W-i\epsilon}, \quad (A9)$$

where

$$f_{\lambda\mu}{}^J(z;q',q)\equiv-4\pi d_{\lambda\mu}{}^J(z)Y_{l\lambda}(\hat{p}\cdot\hat{q}')Y_{l\mu}(\hat{p}\cdot\hat{q})(p'p)^l.$$

It is elementary to verify that $f_{\lambda\mu}{}^J(z;q',q)$ is a polynomial in all its variables, and that

$$f_{\lambda\mu}{}^J(-z;-q',q)=(-1)^{J-\lambda}f_{-\lambda\mu}{}^J(z;q',q)=(-1)^{J-\mu}f_{\lambda-\mu}{}^J(z;q',q). \quad (A10)$$

In solving Eq. (A7), it proves more convenient to work in terms of amplitudes of definite parity. We therefore define

$$f_{\lambda\mu}{}^{J\pm}(z;q',q)=f_{\lambda\mu}{}^J(z;q',q)\pm(-1)^\lambda f_{-\lambda\mu}{}^J(z,q,q),$$
$$\tau_{\lambda\mu}{}^{J\pm}(q',q;W)=\tau_{\lambda\mu}{}^J(q',q;W)\pm(-1)^\lambda\tau_{-\lambda\mu}{}^J(q',q;W), \quad (A11)$$
$$Z_{\lambda\mu}{}^{J\pm}(q',q;W)=Z_{\lambda\mu}{}^J(q',q;W)\pm(-1)^\lambda Z_{-\lambda\mu}{}^J(q',q;W),$$

from which it follows that

$$\tau_{\lambda\mu}{}^{J\pm}(q',q;W)=Z_{\lambda\mu}{}^{J\pm}(q',q;W)+\tfrac{1}{2}\sum_{\sigma}\int_0^{\infty}\frac{dq''dq''^2\,Z_{\lambda\sigma}{}^{J\pm}(q',q'';W)\tau_{\sigma\mu}{}^{J\pm}(q'',q;W)}{D(W-\frac{3}{4}q''^2)}. \quad (A12)$$

From Eqs. (A9) and (A10) it also follows that

$$Z_{\lambda\mu}{}^{J\pm}(-q',q;W)=\pm(-1)^JZ_{\lambda\mu}{}^{J\pm}(q',q;W), \quad (A13)$$

and Eq. (A12) implies that $\tau^{J\pm}$ has the same symmetry. This allows us to rewrite the integral in (A12) as extending from $-\infty$ to $+\infty$. We obtain a set of equations very similar to the one-dimensional problem, the only difference being that the "potential term" $Z_{\lambda\mu}{}^{J\pm}$ has logarithmic cuts in the complex q' plane instead of poles. Applying the method developed in Sec. IV, we obtain the following solution:

Let M be the largest integer such that $M\leq\frac{1}{2}|J-l|$, and define the functions

$$\beta^{J\pm}(q,q')=\frac{1}{q'-q}\pm\frac{(-1)^J}{q'+q},$$

$$\alpha_{\lambda\mu}{}^{J\pm}(Q,q)=\frac{qh^*(W-\frac{3}{4}Q^2)h(W-\frac{3}{4}q^2)}{4D(W-\frac{3}{4}q^2)}\times f_{\lambda\mu}{}^{J\pm}\left(\frac{W-Q^2-q^2}{qQ};Q,q\right),$$

$$V_{\lambda\mu}{}^{J\pm}(q,W)=-\frac{2\pi ih(\nu_0)}{3D'(\nu_0)}q^{2M}\int_{\frac{1}{2}E^{1/2}+\nu_0{}^{1/2}}^{i\infty}\frac{dx}{x^{2M+1}}\times\beta^{J\pm}(q,x)h^*(W-\tfrac{3}{4}x^2)f_{\lambda\mu}{}^{J\pm}\times\left(\frac{W-x^2-E}{xE^{1/2}};x,E^{1/2}\right),\quad\text{(A14)}$$

$$A_{\lambda\mu}{}^{J\pm}(q,x)=q^{2M}\left(1-\frac{\frac{3}{2}x}{(W-\frac{3}{4}x^2)^{1/2}}\right)\times\int_x^{i\infty}\frac{dQ\,\beta^{J\pm}(q,Q)\alpha_{\lambda\mu}{}^{J\pm}(Q,(W-\frac{3}{4}x^2)^{1/2}+\frac{1}{2}x)}{Q^{2M+1}},$$

$$B_{\lambda\mu}{}^{J\pm}(q,x)=q^{2M}\left(1+\frac{\frac{3}{2}x}{(W-\frac{3}{4}x^2)^{1/2}}\right)\times\int_x^{i\infty}\frac{dQ\,\beta^{J\pm}(q,Q)\alpha_{\lambda\mu}{}^{J\pm}(Q,(W-\frac{3}{4}x^2)^{1/2}-\frac{1}{2}x)}{Q^{2M+1}}.$$

In terms of the above we define functions $U_{\lambda\mu}{}^{J\pm}$ and $W_{\lambda\mu}{}^{J\pm}$ to be the solutions of the coupled integral equations

$$U_{\lambda\mu}{}^{J\pm}(q',q;W)=Z_{\lambda\mu}{}^{J\pm}((W-\tfrac{3}{4}q'^2)^{1/2}+\tfrac{1}{2}q',q;W)+\sum_\sigma\int_{\Gamma(W)}dx\times[A_{\lambda\sigma}{}^{J\pm}((W-\tfrac{3}{4}q'^2)^{1/2}+\tfrac{1}{2}q',x)U_{\sigma\mu}{}^{J\pm}(x,q;W)+B_{\lambda\sigma}{}^{J\pm}((W-\tfrac{3}{4}q'^2)^{1/2}+\tfrac{1}{2}q',x)\times W_{\sigma\mu}{}^{J\pm}(x,q;W)],\quad\text{(A15)}$$

$$W_{\lambda\mu}{}^{J\pm}(q',q;W)=Z_{\lambda\mu}{}^{J\pm}((W-\tfrac{3}{4}q'^2)^{1/2}+\tfrac{1}{2}q',q;W)+\sum_\sigma\int_{\Gamma(W)}dx\times[A_{\lambda\sigma}{}^{J\pm}((W-\tfrac{3}{4}q'^2)^{1/2}-\tfrac{1}{2}q',x)U_{\sigma\mu}{}^{J\pm}(x,q;W)+B_{\lambda\sigma}{}^{J\pm}((W-\tfrac{3}{4}q'^2)^{1/2}-\tfrac{1}{2}q',x)\times W_{\sigma\mu}{}^{J\pm}(x,q;W)].\quad\text{(A16)}$$

We also define additional functions $X_{\lambda\mu}{}^{J\pm}(q',W)$ and $Y_{\lambda\mu}{}^{J\pm}(q',W)$ which also satisfy (A15), but with the inhomogeneous terms replaced with $V_{\lambda\mu}{}^{J\pm}((W-\frac{3}{4}q'^2)^{1/2}+\frac{1}{2}q',W)$ and $V_{\lambda\mu}{}^{J\pm}((W-\frac{3}{4}q'^2)^{1/2}-\frac{1}{2}q',W)$, respectively. The functions U, W, X, and Y are well defined by these equations, which can be done numerically. They are less complicated than they appear, and can be decoupled by a simple change of variables. Furthermore, we shall not need their solutions explicitly for our purposes here; it is sufficient to know that the functions U, W, X, and Y so defined do not have the singularity at $W=4\nu_0$, which is easily established. Regarding these functions as known, we can then define

$$\psi_{\lambda\mu}{}^{J\pm}(q',q;W)=Z_{\lambda\mu}{}^{J\pm}(q',q;W)+\sum_\sigma\int_{\Gamma(W)}dx\,[A_{\lambda\sigma}{}^{J\pm}(q',x)U_{\sigma\mu}{}^{J\pm}\times(x,q;W)+B_{\lambda\sigma}{}^{J\pm}(q',x)W_{\sigma\mu}{}^{J\pm}(x,q;W)]$$

and (A17)

$$\phi_{\lambda\mu}{}^{J\pm}(q',W)=V_{\lambda\mu}{}^{J\pm}(q',W)+\sum_\sigma\int_{\Gamma(W)}dx\,[A_{\lambda\sigma}{}^{J\pm}(q',x)\times X_{\sigma\mu}{}^{J\pm}(x,W)+B_{\lambda\sigma}{}^{J\pm}(q',x)X_{\sigma\mu}{}^{J\pm}(x,W)].$$

The solution to (A12) is given by

$$\tau_{\lambda\mu}{}^{J\pm}(q',q;W)=\psi_{\lambda\mu}{}^{J\pm}(q',q;W)+\sum_\sigma\tilde{\theta}(W)\phi_{\lambda\sigma}{}^{J\pm}(q',W)\tau_{\sigma\mu}{}^{J\pm}(E^{1/2},q;W),\quad\text{(A18)}$$

in terms of the half-on-shell amplitudes $\tau_{\sigma\mu}{}^{J\pm}(E^{1/2},q;W)$. The latter are to be determined by substituting the value $q'=E^{1/2}$ into (A17) and solving the resulting system of simultaneous linear equations.

A pole in the amplitude $\tau_{\lambda\mu}{}^{J\pm}(q',q;W)$ will then occur as the result of a zero in the determinant of this system of linear equations; denote this by

$$\mathfrak{D}^{J\pm}(W)=\det|1-\phi^{J\pm}(E^{1/2},W)|.\quad\text{(A19)}$$

The quantity $\mathfrak{D}^{J\pm}(W)$ plays the role in the three-dimensional problem that the $\Delta_\pm(E)$ defined in (5.3) played in the one-dimensional problem. In the latter case we found that $\Delta_\pm(E)$ contained a term which had a pole at $E=4\nu_0$. Examination of the functions involved in the definition (A17) of $\phi_{\lambda\mu}{}^{J\pm}$ shows that $V_{00}{}^{J+}(E^{1/2},W)$ has a logarithmic singularity at $W=4\nu_0$; no other terms are singular. This corresponds to the $\lambda=\mu=0$ element of the determinant given in (A19) being singular for the (+) case. The quantity $\mathfrak{D}^{J-}(W)$ may indeed have zeros, but they are not connected with the $W=4\nu_0$ singularity.

To determine whether or not the singular term in $\mathfrak{D}^{J+}(W)$ can give rise to a zero, we must in general know the magnitude and sign of the coefficient of $V_{00}{}^{J+}(E^{1/2},W)$ and of the background terms. To see this we note that $\mathfrak{D}^{J+}(W)$ can be written in the form

$$\mathfrak{D}^{J+}(W)=R^J(W)-a^J(W)\ln\xi,\quad\text{(A20)}$$

where $\xi\equiv(\frac{3}{2}E^{1/2}+\nu_0{}^{1/2})/(\nu_0{}^{1/2}-\frac{1}{2}E^{1/2})$ is dimensionless and $\rightarrow\infty$ as $W\rightarrow4\nu_0$.

If, in a given problem, we find that

$$\mathrm{Re}[R^J(W)/a^J(W)]\gg 1\,, \tag{A21}$$

this is sufficient to assert that a zero occurs for $W\approx 4\nu_0$. If, however, $\mathrm{Re}a^J/\mathrm{Re}R^J<0$, a zero will certainly not occur in this neighborhood. The third possibility is that $\mathrm{Re}a^J$ and $\mathrm{Re}R^J$ have the same sign but are comparable in magnitude. In this case a zero may or may not occur. It is therefore clear that the singularity may induce a nearby zero in some systems but not in others.

If we consider the case where the quantities

$$\phi_{\lambda\mu}{}^{J+}(E^{1/2},W)$$

are small [except for $\phi_{00}{}^{J+}(E^{1/2},W)$] in the neighborhood of $W=4\nu_0$, we can write

$$\mathfrak{D}^{J+}(W)=1-V_{00}{}^{J+}(E^{1/2},W)+\text{small terms}\,. \tag{A22}$$

The quantity $V_{00}{}^{J+}(E^{1/2},W)$ will itself be small except in the immediate vicinity of $W=4\nu_0$. A zero of $\mathfrak{D}^{J+}(W)$ will then occur in that neighborhood, provided that $\mathrm{Re}V_{00}{}^{J+}(E^{1/2},W)>0$. Examination of the explicit form for $V_{00}{}^{J+}$ then shows that this will only be true if J is *even*.

We therefore conclude that the singularity may give rise to a zero in $\mathfrak{D}^{J+}(W)$, and hence a pole in the off-shell amplitude, and that this is particularly likely when J is even. However, the nonappearance of this zero in certain cases is also compatible with our analysis. In any case, the singular term is always important in the vicinity of $4\nu_0$ and dominates at that point.

APPENDIX B: SINGULARITIES FOR THREE UNLIKE PARTICLES

In the case of unequal masses and different bound-state (resonant) energies between two-body pairs, it is necessary to work with coupled equations, even in the one-dimensional problem. However, there is no difficulty in applying the method of solution discussed in Sec. IV, which results in an expression for the off-shell amplitudes in terms of the half-on-shell amplitudes. The latter are obtained by solving a system of simultaneous linear equations analogous to those discussed in Appendix A for the three-dimensional problem. A pole in the off-shell amplitude is then due to a zero in the determinent for this system of equations. Examination of the elements of this determinant shows that certain of them are singular at values of W which are the generalizations of the points $\frac{4}{3}\nu_0$ and $4\nu_0$ which arise in the identical-particle case. In terms of the notation developed in Sec. III we define

$$\lambda_{\alpha\beta}=(\mu_\alpha\mu_\beta)^{1/2}/m_\gamma\,. \tag{B1}$$

The values of W for which singularities occur are then

$$W=W_\pm=\frac{\nu_{0\alpha}+\nu_{0\gamma}\pm 2\lambda_{\alpha\gamma}\nu_{0\alpha}{}^{1/2}\nu_{0\gamma}{}^{1/2}}{1-\lambda_{\alpha\gamma}{}^2} \tag{B2}$$

for $\alpha\neq\gamma$.

From the derivation of (B2) it can be shown that the W_+ singularity is always on the physical sheet; in the identical-particle case this reduces to $W_+=4\nu_0$. The singularity at W_- may or may not occur on the physical sheet, depending on the relative magnitudes of the masses and bound-state or resonant energies. For the case where $\nu_{0\alpha}$ and $\nu_{0\gamma}$ are resonant energies, the condition for W_- to be physical is that

$$\mathrm{Im}(\lambda_{\alpha\gamma}\nu_{0\gamma}{}^{1/2}-\nu_{0\alpha}{}^{1/2})<0\,. \tag{B3}$$

However, regardless of the status of W_- in a given system, we know that at least three singularities occur at physical values of W, i.e., the three values of W_+ as α and γ vary. From (B2) it follows that

$$|\nu_{0\alpha}|+|\nu_{0\gamma}|<|W_+|<\infty\,. \tag{B4}$$

If we imagine $\nu_{0\alpha}$ and $\nu_{0\gamma}$ to be fixed and consider the change in W_+ as we vary m_β, we find that $|W_+|$ approaches the lower limit in (B4) as $m_\beta\to\infty$, and the upper limit as $m_\beta\to 0$.

In conclusion, we consider the special case of one heavy particle of mass M interacting with two identical light particles of unit mass. If the two-body resonant energy of the one heavy-one light subsystem is ν_0, and we assume that there is no interaction between the light particles, the formulas become

$$W_+=\frac{2(M+1)}{M}\nu_0\,,\quad W_-=\frac{2(M+1)}{M+2}\nu_0\,. \tag{B5}$$