Chiral-Symmetric Meson-Nucleon Lagrangians*

G. Kramer†
Argonne National Laboratory, Argonne, Illinois 60439

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A phenomenological Lagrangian for a theory of nucleons and scalar, pseudoscalar, vector, and axial-vector mesons is constructed on the basis of linear representation of chiral $SU(2)\times SU(2)$. The Lagrangian for this extended σ model is used for a study of strong interactions of the particles involved and for the derivation of threshold theorems for pion-nucleon and pion-pion scattering, photoproduction of pions on nucleons, and the production of an s-wave isoscalar resonance by pions on nucleons.

I. INTRODUCTION

In the last few years a number of people have used the $SU(2)\times SU(2)$ [1] commutation relations and the hypothesis of partially conserved axial-vector current[2] (PCAC) to derive interesting formulas for low-energy parameters of various scattering processes and for strong meson-decay constants.[3,4] Recently it was noted by Weinberg[5] that for soft-pion processes the same predictions can also be derived from the lowest-order graphs calculated from an effective Lagrangian partially invariant under chiral-symmetry transformations. The invariance is exact only for zero-mass pions.

Chiral symmetry is a dynamical symmetry. In the formulations discussed so far, it leaves the total Hamiltonian but not the free Hamiltonian invariant. Since the explicit form of the interaction Hamiltonian between the different fields is not known, this requirement does not uniquely define the chiral transformation properties of the fields associated with known particles. The discussion becomes more definite if one requires that the axial-vector current is generated from the free Lagrangian of the nucleon and has the form $\bar{\psi}i\gamma_5\gamma_\mu\boldsymbol{\tau}\psi$. This places the nucleon field in the $(\frac{1}{2},0)+(0,\frac{1}{2})$ representation of $SU(2)\times SU(2)$. The transformations of the mesons are then determined after an appropriate form of interaction Lagrangian is chosen.

The first model of this kind is the so-called σ model introduced by Schwinger[6] and developed further by Gell-Mann and Lévy.[7] In this model, the π and σ fields form a four-vector coupled to the nucleon field ψ in the chiral-invariant combination $\bar{\psi}(\sigma+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi$. In the older work[7] the emphasis was rather on the renormalization of the weak axial-vector current and PCAC than on the usefulness for pure strong-interaction predictions, although Gell-Mann and Lévy mentioned that the nonlinear σ model has the nice property of yielding a very small low-energy s-wave π-N scattering amplitude without charge exchange in agreement with experiment. In the nonlinear σ model of Gell-Mann and Lévy, the σ field is replaced with a nonlinear function of the pion field of the form $[(m/f)^2-\boldsymbol{\pi}^2]^{1/2}$. The nonlinear chiral-invariant models[8] considered earlier on a more general basis by Gürsey[9] have been favored over the linear σ model because of the lack of evidence for a $I=J=0$ π-π resonance to be identified with the σ field.

In the meantime, experimental evidence for a resonance in the $I=J=0$ 2π channel has been accumulated. Feldman *et al.* found a resonance in the $\pi^0\pi^0$ channel in the reaction $\pi^-p\to n\pi^0\pi^0$ with a mass of 730 and a width around 50 MeV.[10] This rather narrow resonance was never confirmed by other experimental groups using similar techniques and beam energies, although one group found a rather broad resonance with a mass around 600 MeV in the same reaction.[11] Further information about the $I=0$ 2π channel comes from $\pi^+\pi^-$ production on protons by negative pions. In this reaction one has contributions from the $I=0$ and $I=1$ states of the $\pi\pi$ system. Since the $I=J=1$ term—in particular,

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† Permanent address: University of Hamburg, Hamburg, Germany.

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[4] Because of the very large number of papers in this area, we refer in Ref. 3 only to the most recent papers with direct relevance to our work. Reference to earlier papers on the same subject can be found in the papers given in Ref. 3 or in B. Renner, *Current Algebras and Their Applications* (Pergamon Press, Inc., New York, 1968); S. Adler and R. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).

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in the ρ region—is much stronger than the $I=J=0$ contribution, information about the s wave comes mostly from interference with the ρ channel and is needed, in particular, to account for the observed asymmetry in the ρ^0 decay. The analysis is based on the peripheral model supplemented with absorption or form factors. Just recently a number of groups have performed such analysis and arrived at $\pi\pi$ s-wave phase shifts passing near or through 90° in the vicinity of the ρ mass.[12] Of course, we would be much more comfortable if there were unambiguous experimental evidence for a resonance in the $I=J=0$ 2π channel. But there appears to be little doubt that the relevant phase shift is large and possibly resonant somewhere between 0.6 and 1 GeV. Further evidence for an $I=J=0$ $\pi\pi$ resonance, even less direct, comes from analysis of elastic πN scattering—in particular, from backward dispersion relations—and from analysis of elastic p-p scattering below the meson-production threshold.[13]

In the following, we shall assume that a broad $I=J=0$ $\pi\pi$ resonance exists and shall call it σ. In such a theory the mesons including σ and π are placed into linear representations of $SU(2)\times SU(2)$. This "linear" theory seems to have some advantage over nonlinear realizations of $SU(2)\times SU(2)$. The pion does not play such an exceptional role as in the nonlinear theories. Therefore, it is easier to incorporate higher-symmetry schemes which combine vector (axial-vector) and scalar (pseudoscalar) mesons. The interactions are polynomials of the fields, as in the conventional theory, and there is some hope that such a theory might be useful also for predictions at higher energies and for processes in which no pions are present in the initial and final state, as in elastic nucleon-nucleon scattering. In fact, one-boson-exchange models based on Lagrangians with structures that we are going to propose have been successfully applied to the analysis of low-energy p-p scattering,[14] and chiral symmetry might be useful to place restrictions on the boson-nucleon coupling constants.

In this paper we construct chiral-symmetric Lagrangians based on linear representations of $SU(2)\times SU(2)$. We start from the old linear σ model of Gell-Mann and Lévy[7] and extend it to incorporate isoscalar and isovector vector and axial-vector mesons. For completeness we include also the $\eta(549)$ and the $\pi_V(1016)$, usually not considered in chiral-symmetry schemes, and introduce their symmetry transformation properties. The other meson fields are σ, π, ρ, a, ω, and d. The latter three are identified with $A_1(1060)$, $\omega(783)$, and D (1285).[15] A chiral-invariant Lagrangian involving interactions of these mesons with each other and with nucleons is constructed in Sec. II. Since the Lagrangian is to be used to derive relations for meson decays and low-energy theorems, we consider only terms with no more than three derivatives of the fields. To introduce the vector and axial-vector mesons in the Lagrangian we do not require gauge invariance of the second kind. Then our currents are not proportional to the ρ and a fields, respectively, but are generated mostly from the conventional kinetic-energy terms of the Lagrangian. We do not intend to investigate the operator structure of the symmetry that we impose on the Lagrangian. Therefore, the question of how our approach is related to current or field algebra remains open.

In Sec. III, after elimination of a coupling between the pseudoscalar and the axial-vector fields, mass relations and normalization conditions for the vector and axial-vector mesons are derived. The mass relations for these mesons as well as for the σ itself essentially stem from the nonzero vacuum expectation value of the σ field. Then symmetry-breaking terms are added to the Lagrangian in the usual way to produce a mass term for the pion field. With this Lagrangian, we calculate in Sec. IV the low-energy parameters for the processes $\pi N\to\pi N$, $\pi\pi\to\pi\pi$, $\gamma N\to\pi N$, and $\pi N\to\sigma N$ from lowest-order preturbation theory. These "low-energy theorems" and the relations for the meson-decay constants are evaluated numerically in Sec. V and compared to experimental data and current-algebra predictions. The only free parameters required as input are essentially the pion-decay constant, the pion-nucleon coupling constant, and the masses of the nucleon, ρ, A_1, and π.

II. CONSTRUCTION OF CHIRAL-INVARIANT LAGRANGIAN

In this section we shall construct a chrial-invariant Lagrangian which describes the coupling of nucleons to isovector and isoscalar mesons of spin zero or 1 and with positive or negative parity. These mesons will be labeled as $\pi(0^-,1^+)$, $\delta(0^+,1^+)$, $\rho(1^-,1^-)$, $a(1^+,1^+)$, $\eta(0^-,0^+)$, $\sigma(0^+,0^+)$, $\omega(1^-,0^-)$, and $d(1^+,0^+)$, where the symbol (J^P,I^C) refers to spin, parity, isospin, and C parity of the neutral member of the isotriplet, respectively. Possible identifications of these mesons with observed particles will be discussed in connection with applications of the chiral Lagrangian. From the beginning we assume that the nucleon field ψ is a $(\frac{1}{2},0)+(0,\frac{1}{2})$

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[13] References to these two points can be found in C. Lovelace, *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (North-Holland Publishing Co., Amsterdam, 1968), p. 79, where also recent work of Lovelace and co-workers on πN backward dispersion relations is reported.

[14] See the article of Lovelace (Ref. 13) for a hopefully complete list of references.

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linear representation of $SU(2)\times SU(2)$. Then the isospin $SU(2)$ transformation and the chiral $SU(2)$ transformation for the nucleon field are

$$\begin{aligned}\psi &\to \psi' = e^{(1/2)i\beta\cdot\tau}\psi\,,\\ \psi &\to \psi'' = e^{(1/2)i\alpha\tau\gamma_5}\psi\,.\end{aligned}\tag{1}$$

Before we specify the transformation properties of the mesons, we briefly review the linear σ model. In this model the scalar field σ and the pion-field $\boldsymbol{\pi}$ transform in such a way that the usual Yukawa coupling of σ and $\boldsymbol{\pi}$ together with the nucleon-mass term $m\bar{\psi}\psi$ form a chiral invariant:

$$m\bar{\psi}\psi - f\bar{\psi}(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi = \text{invariant}\,.\tag{2}$$

Then the chiral transformation of σ and $\boldsymbol{\pi}$ must be

$$\begin{aligned}(m-f\sigma) - fi\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi} &\to e^{-\frac{1}{2}i\alpha\cdot\tau\gamma_5}\\ &\quad\times[(m-f\sigma) - fi\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}]e^{-\frac{1}{2}i\alpha\cdot\tau\gamma_5}\end{aligned}\tag{3}$$

or, infinitesimally,

$$\begin{aligned}\delta\sigma &= \boldsymbol{\alpha}\cdot\boldsymbol{\pi}\,,\\ \delta\boldsymbol{\pi} &= \boldsymbol{\alpha}(-m/f+\sigma)\,.\end{aligned}\tag{4}$$

Usually the nucleon-mass term is combined with σ to form a new field

$$\sigma' = -m/f + \sigma\,,$$

which now has a nonvanishing vacuum expectation value $\langle\sigma'\rangle_0 = -m/f$, and σ' and $\boldsymbol{\pi}$ transform like the four-dimensional representation of $SU(2)\times SU(2)$. It is obvious that $\sigma'^2+\boldsymbol{\pi}^2$ is a chiral-invariant, as is

$$\sigma^2 - (2m/f)\sigma + \boldsymbol{\pi}^2\,,\tag{5}$$

which differs from $\sigma'^2+\boldsymbol{\pi}^2$ only by a constant. A simple chiral-invariant Lagrangian $L^{(\sigma)}$ for the ψ-$\boldsymbol{\pi}$-σ system with a direct $\boldsymbol{\pi}$-$\boldsymbol{\pi}$ interaction is easily constructed in terms of the invariants (2), (5), $\bar{\psi}\gamma_\mu\partial_\mu\psi$, and $\partial_\mu{}^2(\sigma^2+\boldsymbol{\pi}^2)$. It has the following form:

$$\begin{aligned}L^{(\sigma)} = &-\tfrac{1}{2}[(\partial\sigma)^2 + (\partial\boldsymbol{\pi})^2] - \lambda[\sigma^2 - (2m/f)\sigma + \boldsymbol{\pi}^2]^2\\ &- \psi(\gamma_\mu\partial_\mu + m)\psi + f\bar{\psi}(\sigma + i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau})\psi\,.\end{aligned}\tag{6}$$

This Lagrangian is the so-called linear σ-model Lagrangian of Gell-Mann and Lévy[7] in the symmetry limit. Before we go on to include the other mesons in this Lagrangian, we want to point out several features of this Lagrangian, some of which have already been noticed by Gell-Mann and Lévy[7] and by Weinberg.[5] First, we remark that the pion mass is zero, whereas the σ mass is $m_\sigma{}^2 = 8m^2\lambda/f^2$. Thus λ must be positive. This essentially comes from the nonvanishing vacuum expectation value of the σ'. The interaction Lagrangian is

$$\begin{aligned}L_{\text{int}}{}^{(\sigma)} = f\bar{\psi}(\sigma + i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau})\psi - \lambda[\sigma^4 + (\boldsymbol{\pi}^2)^2&\\ + 2\sigma^2\boldsymbol{\pi}^2 + (4m/f)\sigma\boldsymbol{\pi}^2]&\,.\end{aligned}\tag{7}$$

If we calculate the pion-nucleon and the pion-pion scattering amplitudes from this interaction Lagrangian in lowest order, we find that they vanish at threshold. In π-N scattering the second-order nucleon-pole contribution which yields $-f^2/4\pi m$ for the scattering length is cancelled by the σ-exchange contribution. Of course, we must consider only the isospin-symmetric amplitude $T^{(+)}$, since the antisymmetric part $T^{(-)}$ vanishes at threshold because of crossing symmetry. For the scattering length $a^{(+)}$ we obtain

$$4\pi a^{(+)} = -\frac{f^2}{m} + \frac{8\lambda m}{f}\frac{f}{m_\sigma{}^2} = 0\,,$$

since $m_\sigma{}^2 = 8m^2\lambda/f^2$. In a similar way one shows that the corresponding π-π scattering lengths are zero. Here the σ-exchange contribution is cancelled by the direct π-π interaction term. Another quantity of direct physical interest is the decay rate of the σ. For zero-mass pions, the total decay rate Γ is given by

$$\Gamma = \frac{g_\sigma{}^2}{4\pi}\frac{3}{m_\sigma}\,,$$

where g_σ is the coupling constant between the σ and the two π's, in our case, $g_\sigma = (4m/f)\lambda$. Thus, with $g_\sigma = 4m\lambda/f = fm_\sigma{}^2/2m$, we have

$$\Gamma = \frac{3}{4}\frac{f^2}{4\pi}\frac{m_\sigma{}^3}{m^2}\,.$$

Of course, we would like to have $\Gamma < m_\sigma$, but with $f^2/4\pi = 14.5$, this is possible only for $m_\sigma < 0.3m$. This limitation on m_σ is not changed appreciably if a symmetry-breaking term is added to the Lagrangian and the actual pion mass is taken into account in the decay rate. On this property alone, Weinberg[5] rejected the σ model as a useful phenomenological model of strong interactions. It is possible, however, that g_σ is modified by further terms which appear quite naturally if all the other meson states are introduced into the model. We are encouraged, therefore, to construct a complete Lagrangian which contains couplings of all mesons to the nucleon but which should account also for the most important meson decays like $\rho\to 2\pi$, $a\to\rho\pi$, $\omega\to\rho\pi$, etc. In the following paragraph we shall still consider the pion as massless. The symmetry breaking will be introduced later in connection with the applications of our Lagrangian.

Unfortunately, there is no *a priori* principle which determines the transformation properties of δ, ρ, a, η, ω, and d similarly, as there is in the case of σ and π, but once we have chosen a particular coupling of these mesons to the nucleon field as part of the interaction Lagrangian L_{int}, the transformation properties of the mesons are determined from the condition that L_{int} is a chiral-invariant.[16] Just as in the σ model, we choose

[16] Instead of this, we could list the representations of $SU(2)\times SU(2)$, in which these fields are placed.

the usual Yukawa coupling without derivatives:

$$L' = \bar{\psi}\{[f_\rho(i\gamma_\mu \boldsymbol{\varrho}_\mu + i\gamma_5\gamma_\mu \mathbf{a}_\mu) + f_\eta \boldsymbol{\delta}]\cdot\boldsymbol{\tau} + f_d i\gamma_5\gamma_\mu d_\mu + f_\omega i\gamma_\mu\omega_\mu + f_\eta i\gamma_5\eta\}\psi = \text{invariant}. \quad (8)$$

Then the δ, ρ, etc., must transform infinitesimally as

$$\begin{aligned} \delta\boldsymbol{\delta} &= \boldsymbol{\alpha}\eta, & \delta_\eta &= -\boldsymbol{\alpha}\cdot\boldsymbol{\delta}, \\ \delta\boldsymbol{\varrho}_\mu &= [\boldsymbol{\alpha}\times\mathbf{a}_\mu], & \delta\mathbf{a}_\mu &= [\boldsymbol{\alpha}\times\boldsymbol{\varrho}_\mu], \\ \delta\omega_\mu &= 0, & \delta d_\mu &= 0. \end{aligned} \quad (9)$$

We see that the choice for the coupling above connects the ρ and the a meson, which should lead to a relation between the coupling constants of $\rho\pi\pi$ and $a\rho\pi$. Furthermore, we notice that the couplings of ω and d are invariant by themselves. Thus, we cannot gain any relations for the coupling of these mesons if their transformation properties are such as stated in (9). Another possibility to couple the mesons π, ρ, δ, a, η, σ, ω, and d to the nucleon field ψ in a charge-conjugation-invariant way is a derivative coupling of the form

$$L'' = \tfrac{1}{4}\bar{\psi}\{\phi,\gamma_\mu\}\partial_\mu\psi - \tfrac{1}{4}(\partial_\mu\bar{\psi})\{\phi,\gamma_\mu\}\psi, \quad (10)$$

with

$$\phi = \sigma + i\gamma_5\eta + i\gamma_\mu\omega_\mu + i\gamma_5\gamma_\mu d_\mu + \boldsymbol{\tau}\cdot(\boldsymbol{\delta} + i\gamma_5\boldsymbol{\pi} + i\gamma_\mu\boldsymbol{\varrho}_\mu + i\gamma_5\gamma_\mu\mathbf{a}_\mu).$$

Here we omitted the corresponding coupling constants, so that the meson fields are not normalized as usual. Unfortunately, the interaction L'' is not chiral-invariant in itself, as the coupling (8) is, even if the meson fields fulfill specific chiral transformation rules. To obtain a chiral-symmetric coupling together with L'' above, we must introduce mesons with opposite C parities. Such mesons can be coupled only with derivatives of the nucleon field in a similar way to (10), but with commutators instead of anticommutators with the γ_μ. It is clear that these derivative couplings do not relate the π and the $(0^+,1^-)$ particle and would be chiral-invariant only if particular transformation rules between mesons of opposite parity and C parity are fulfilled. These couplings would lead, for example, to relations between the couplings of the ω and a $(1^+,1^-)$ particle which could be identified with the $B(1220)$ meson. In this paper we shall not pursue these possibilities for other representations of some of the mesons and for different chiral-invariant couplings than (8) any further, and shall limit ourselves to the exploration of the nonderivative Yukawa coupling (8) with the transformation rules (9).

The construction of the chiral-invariant Lagrangian with couplings for the most important meson decays is straightforward after we have constructed chiral covariants which are products of two meson fields. Of particular importance are products transforming like σ and $\boldsymbol{\pi}$. The covariants must contain space-time derivatives of the scalar and pseudoscalar meson fields if couplings to the vector fields are to be present. Such products of meson fields which transform like $\partial_\mu\boldsymbol{\pi}$, $\partial_\mu\sigma$, $\partial_\mu\eta$, and $\partial_\mu\boldsymbol{\delta}$ are the following:

$$\begin{aligned} \boldsymbol{\pi}_\mu &= \partial_\mu\boldsymbol{\pi} + g_1(\boldsymbol{\varrho}_\mu\times\boldsymbol{\pi} - \mathbf{a}_\mu\sigma') - g_2 d_\mu\boldsymbol{\delta}, \\ \sigma_\mu &= \partial_\mu\sigma' + g_1(\mathbf{a}_\mu\cdot\boldsymbol{\pi} + g_2 d_\mu\eta), \\ \eta_\mu &= \partial_\mu\eta + f_2\mathbf{a}_\mu\cdot\boldsymbol{\delta} + f_3 d_\mu\sigma', \\ \boldsymbol{\delta}_\mu &= \partial_\mu\boldsymbol{\delta} - f_2(\boldsymbol{\varrho}_\mu\times\boldsymbol{\delta} + \mathbf{a}_\mu\eta) - f_3 d_\mu\boldsymbol{\pi}, \end{aligned} \quad (11)$$

with arbitrary real constants g_1, g_2, f_2, and f_3. We notice that all these "fields" have positive C parity and that $\boldsymbol{\pi}_\mu$ has the quantum numbers of the $\mathbf{a}_\mu$ and η_μ has the quantum numbers of the d_μ, but, of course, the chiral transformation properties are different. We have

$$\begin{aligned} \delta\boldsymbol{\pi}_\mu &= -\boldsymbol{\alpha}\sigma_\mu, & \delta\eta_\mu &= -\boldsymbol{\alpha}\cdot\boldsymbol{\delta}_\mu, \\ \delta\sigma_\mu &= \boldsymbol{\alpha}\cdot\boldsymbol{\pi}_\mu, & \delta\boldsymbol{\delta}_\mu &= \boldsymbol{\alpha}\eta_\mu. \end{aligned} \quad (12)$$

We notice that we obtain the desired $\rho\pi\pi$ coupling if $\partial_\mu\boldsymbol{\pi}$ and $\partial_\mu\sigma$ are replaced with $\boldsymbol{\pi}_\mu$ and σ_μ in the Lagrangian $L^{(\sigma)}$ [Eq. (6)]. This leads immediately to a relation between the $\rho\pi\pi$ and $a\rho\pi$ coupling constants. Products of meson fields with the same chiral transformation properties as $\boldsymbol{\pi}_\mu$, σ_μ, η_μ, and $\boldsymbol{\delta}_\mu$ but with negative C parities are the following:

$$\begin{aligned} \boldsymbol{\pi}_\mu' &= g_3(\mathbf{a}_\mu\times\boldsymbol{\delta} + \boldsymbol{\varrho}_\mu\eta) + g_4\omega_\mu\boldsymbol{\pi}, \\ \sigma_\mu' &= g_3\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\delta} + g_4\omega_\mu\sigma', \\ \eta_\mu' &= f_1\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\pi} + f_4\omega_\mu\eta, \\ \boldsymbol{\delta}_\mu' &= -f_1(\mathbf{a}_\mu\times\boldsymbol{\pi} - \boldsymbol{\varrho}_\mu\sigma') + f_4\omega_\mu\boldsymbol{\delta}. \end{aligned} \quad (13)$$

Therefore, as before, $\boldsymbol{\pi}_\mu'\cdot\boldsymbol{\pi}_\mu' + \sigma_\mu'\sigma_\mu'$ and $\eta_\mu'\eta_\mu' + \boldsymbol{\delta}_\mu'\cdot\boldsymbol{\delta}_\mu'$ are chiral-invariants, which also should be included in the Lagrangian. The second invariant involves again a connection between the $\rho\pi\pi$ and $a\rho\pi$ vertices. However, the couplings for $a\rho\pi$ constructed either with (11) or (13) do not contain any tensor coupling of the a to the $\rho\pi$ system. In general, by Lorentz invariance alone, the $a\rho\pi$ vertex depends on two coupling constants, whereas the $\rho\pi\pi$ vertex depends only on one coupling constant. This tensor coupling is easily generated in a chiral-invariant way by considering covariant tensors $\boldsymbol{\pi}_{\mu\nu}$, $\sigma_{\mu\nu}$, $\eta_{\mu\nu}$, and $\boldsymbol{\delta}_{\mu\nu}$, defined by

$$\begin{aligned} \boldsymbol{\pi}_{\mu\nu} &= \gamma_1(\boldsymbol{\varrho}_{\mu\nu}\times\boldsymbol{\pi} - \mathbf{a}_{\mu\nu}\sigma') - \gamma_2 d_{\mu\nu}\boldsymbol{\delta} + \gamma_1'(\boldsymbol{\varrho}_\mu\times\partial_\nu\boldsymbol{\pi} - \boldsymbol{\varrho}_\nu\times\partial_\mu\boldsymbol{\pi} \\ &\quad - \mathbf{a}_\mu\partial_\nu\sigma' + \mathbf{a}_\nu\partial_\mu\sigma') - \gamma_2'(d_\mu\partial_\nu\boldsymbol{\delta} - d_\nu\partial_\mu\boldsymbol{\delta}), \\ \sigma_{\mu\nu} &= \gamma_1\mathbf{a}_{\mu\nu}\cdot\boldsymbol{\pi} + \gamma_2 d_{\mu\nu}\eta + \gamma_1'(\mathbf{a}_\mu\cdot\partial_\nu\boldsymbol{\pi} - \mathbf{a}_\nu\cdot\partial_\mu\boldsymbol{\pi}) \\ &\quad + \gamma_2'(d_\mu\partial_\nu\eta - d_\nu\partial_\mu\eta), \\ \eta_{\mu\nu} &= \epsilon_2\mathbf{a}_{\mu\nu}\cdot\boldsymbol{\delta} + \epsilon_3 d_{\mu\nu}\sigma' + \epsilon_2'(\mathbf{a}_\mu\cdot\partial_\nu\boldsymbol{\delta} - \mathbf{a}_\nu\cdot\partial_\mu\boldsymbol{\delta}) \\ &\quad + \epsilon_3(d_\mu\partial_\nu\sigma' - d_\nu\partial_\mu\sigma'), \\ \boldsymbol{\delta}_{\mu\nu} &= -\epsilon_2(\boldsymbol{\varrho}_{\mu\nu}\times\boldsymbol{\delta} + \mathbf{a}_{\mu\nu}\eta) - \epsilon_3 d_{\mu\nu}\boldsymbol{\pi} \\ &\quad - \epsilon_2'(\boldsymbol{\varrho}_\mu\times\partial_\nu\boldsymbol{\delta} - \boldsymbol{\varrho}_\nu\times\partial_\mu\boldsymbol{\delta} + \mathbf{a}_\mu\partial_\nu\eta - \mathbf{a}_\nu\partial_\mu\eta) \\ &\quad - \epsilon_3'(d_\mu\partial_\nu\boldsymbol{\pi} - d_\nu\partial_\mu\boldsymbol{\pi}). \end{aligned} \quad (14)$$

$\boldsymbol{\pi}_{\mu\nu}$, $\sigma_{\mu\nu}$, $\eta_{\mu\nu}$, and $\boldsymbol{\delta}_{\mu\nu}$ are antisymmetric tensors with positive C parity and transform under chiral symmetry like $\boldsymbol{\pi}$, σ', η, and $\boldsymbol{\delta}$. Antisymmetric tensors with negative

C parity are

$$\begin{aligned}
\boldsymbol{\pi}_{\mu\nu}' &= \gamma_3(\boldsymbol{a}_{\mu\nu}\times\boldsymbol{\delta}+\boldsymbol{\rho}_{\mu\nu}\eta)+\gamma_4\omega_{\mu\nu}\boldsymbol{\pi} \\
&\quad +\gamma_3'(\boldsymbol{a}_\mu\times\partial_\nu\boldsymbol{\delta}-\boldsymbol{a}_\nu\times\partial_\mu\boldsymbol{\delta}+\boldsymbol{\rho}_\mu\partial_\nu\eta-\boldsymbol{\rho}_\nu\partial_\mu\eta) \\
&\qquad +\gamma_4(\omega_\mu\partial_\nu\boldsymbol{\pi}-\omega_\nu\partial_\mu\boldsymbol{\pi}), \\
\sigma_{\mu\nu}' &= \gamma_3\boldsymbol{\rho}_{\mu\nu}\cdot\boldsymbol{\delta}+\gamma_4\omega_{\mu\nu}\sigma'+\gamma_3'(\boldsymbol{\rho}_\mu\partial_\nu\boldsymbol{\delta}-\boldsymbol{\rho}_\nu\partial_\mu\boldsymbol{\delta}) \\
&\qquad +\gamma_4(\omega_\mu\partial_\nu\sigma'-\omega_\nu\partial_\mu\sigma'), \\
\eta_{\mu\nu}' &= \epsilon_1\boldsymbol{\rho}_{\mu\nu}\cdot\boldsymbol{\pi}+\epsilon_4\omega_{\mu\nu}\eta+\epsilon_1'(\boldsymbol{\rho}_\mu\cdot\partial_\nu\boldsymbol{\pi}-\boldsymbol{\rho}_\nu\cdot\partial_\mu\boldsymbol{\pi}) \\
&\qquad +\epsilon_4'(\omega_\mu\partial_\nu\eta-\omega_\nu\partial_\mu\eta), \\
\boldsymbol{\delta}_{\mu\nu}' &= -\epsilon_1(\boldsymbol{a}_{\mu\nu}\times\boldsymbol{\pi}-\boldsymbol{\rho}_{\mu\nu}\sigma')+\epsilon_4\omega_{\mu\nu}\boldsymbol{\delta} \\
&\quad -\epsilon_1'(\boldsymbol{a}_\mu\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_\nu\times\partial_\mu\boldsymbol{\pi}-\boldsymbol{\rho}_\mu\partial_\nu\sigma'+\boldsymbol{\rho}_\nu\partial_\mu\sigma') \\
&\qquad +\epsilon_4'(\omega_\mu\partial_\nu\boldsymbol{\delta}-\omega_\nu\partial_\mu\boldsymbol{\delta}).
\end{aligned} \tag{15}$$

In Eqs. (14) and (15) the tensors $\boldsymbol{\rho}_{\mu\nu}$, etc., are defined as usual: $\boldsymbol{\rho}_{\mu\nu}=\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu$. The tensors defined in Eq. (15) are important for generating the $\omega\rho\pi$ coupling and for the tensor coupling of ρ and ω to the nucleon field ψ. Of course, $\boldsymbol{\pi}_{\mu\nu}\cdot\boldsymbol{\pi}_{\mu\nu}+\sigma_{\mu\nu}\sigma_{\mu\nu}$, $\boldsymbol{\pi}_{\mu\nu}'\cdot\boldsymbol{\pi}_{\mu\nu}'+\sigma_{\mu\nu}'\sigma_{\mu\nu}'$, $\eta_{\mu\nu}\eta_{\mu\nu}+\boldsymbol{\delta}_{\mu\nu}\cdot\boldsymbol{\delta}_{\mu\nu}$, and $\eta_{\mu\nu}'\eta_{\mu\nu}'+\boldsymbol{\delta}_{\mu\nu}'\boldsymbol{\delta}_{\mu\nu}'$ are chiral-invariants, but also

$$\epsilon_{\mu\nu\rho\sigma}(\boldsymbol{\pi}_{\mu\nu}\cdot\boldsymbol{\delta}_{\rho\sigma}-\sigma_{\mu\nu}\eta_{\rho\sigma})$$

and

$$\epsilon_{\mu\nu\rho\sigma}(\boldsymbol{\pi}_{\mu\nu}'\cdot\boldsymbol{\delta}_{\rho\sigma}'-\sigma_{\mu\nu}'\eta_{\rho\sigma}') \tag{16}$$

are chiral-invariant and are invariant against C conjugation. We notice that the second invariant in (16) has a $\rho\omega\pi$ coupling of the following form:

$$L_{\rho\omega\pi}=-(2m/f)\epsilon_{\mu\nu\rho\sigma}(\gamma_4'\epsilon_1\omega_\mu\partial_\nu\boldsymbol{\pi}\cdot\boldsymbol{\rho}_{\rho\sigma}-\gamma_4\epsilon_1'\omega_{\mu\nu}\boldsymbol{\rho}_\rho\cdot\partial_\sigma\boldsymbol{\pi}). \tag{17}$$

Unfortunately, the constants $\gamma_4'\epsilon_1$ and $\gamma_4\epsilon_1'$ are not related to other interesting meson vertices, but they do appear in scattering processes, as, for instance, $\pi+\omega\to\pi+a$. We see from Eq. (17) that a $\rho\omega\pi$ coupling does not exist if gauge invariance of the second kind for the vector fields is demanded. In the same way, symmetric traceless tensors with chiral transformation properties like σ', $\boldsymbol{\pi}$, $\boldsymbol{\delta}$, and η can be introduced, but we shall not write them down here. They should be considered in connection with chiral interactions for spin-2 mesons like f, f', and A_2 which we shall not attempt in this paper. Of some interest are the traces of symmetric tensors which transform like scalar particles, for example,

$$\begin{aligned}
\boldsymbol{\pi}_s &= \alpha_1(\boldsymbol{\rho}_\mu\times\partial_\mu\boldsymbol{\pi}-\boldsymbol{a}_\mu\partial_\mu\sigma')-\alpha_2 d_\mu\partial_\mu\boldsymbol{\delta}, \\
\sigma_s &= \alpha_1\boldsymbol{a}_\mu\cdot\partial_\mu\boldsymbol{\pi}+\alpha_2 d_\mu\partial_\mu\eta, \\
\eta_s &= \beta_2\boldsymbol{a}_\mu\cdot\partial_\mu\boldsymbol{\delta}+\beta_3 d_\mu\partial_\mu\sigma', \\
\boldsymbol{\delta}_s &= -\beta_2(\boldsymbol{\rho}_\mu\times\partial_\mu\boldsymbol{\delta}+\boldsymbol{a}_\mu\partial_\mu\eta)-\beta_3 d_\mu\partial_\mu\boldsymbol{\pi},
\end{aligned} \tag{18}$$

which have positive C parity. Of course, $\boldsymbol{\pi}_s^2+\sigma_s^2$ and $\eta_s^2+\boldsymbol{\delta}_s^2$ are chiral-invariant. Other combinations with the same transformation properties involve $\partial_\mu\boldsymbol{\rho}_\mu$, $\partial_\mu\boldsymbol{a}_\mu$, $\partial_\mu\omega_\mu$, or $\partial_\mu d_\mu$, which vanish in lowest order of perturbation theory. Traces of symmetric tensors with opposite C parity can also be written immediately. We shall not make use of these traceless symmetric tensors for later applications, but they are useful for meson-meson scattering like π-ρ or σ-a scattering and production of ρ, a, etc., by nucleons.

So far we have introduced only chiral-covariants with one derivative, so that the corresponding term in the Lagrangian does not have more than two derivatives, but later we shall make a translation in the a field which will bring one additional derivative into the Lagrangian. Therefore, we must also allow independent chiral-invariants with three derivatives. Such invariants are easily constructed with new vectors defined like $\boldsymbol{\pi}_\mu$, σ_μ, η_μ, and $\boldsymbol{\delta}_\mu$ but with an additional derivative. The following vectors transform like $\boldsymbol{\pi}_\mu$, σ_μ, etc.:

$$\begin{aligned}
&\hat{g}_1(\boldsymbol{\rho}_{\mu\nu}\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_{\mu\nu}\partial_\nu\sigma')-\hat{g}_2 d_{\mu\nu}\partial_\nu\boldsymbol{\delta}, \\
&\hat{g}_1\boldsymbol{a}_{\mu\nu}\cdot\partial_\nu\boldsymbol{\pi}+\hat{g}_2 d_{\mu\nu}\partial_\nu\eta, \\
&\hat{f}_2\boldsymbol{a}_{\mu\nu}\cdot\partial_\nu\boldsymbol{\delta}+\hat{f}_3 d_{\mu\nu}\partial_\nu\sigma', \\
&-\hat{f}_2(\boldsymbol{\rho}_{\mu\nu}\times\partial_\nu\boldsymbol{\delta}+\boldsymbol{a}_{\mu\nu}\partial_\nu\eta)-\hat{f}_3 d_{\mu\nu}\partial_\nu\boldsymbol{\pi}.
\end{aligned} \tag{19}$$

Vectors with opposite C parity are defined analogously on the basis of $\boldsymbol{\pi}_\mu'$, etc., given in (13). It is obvious that all these vectors transform like $\boldsymbol{\pi}$, σ', η, and $\boldsymbol{\delta}$ as π_μ, σ_μ, etc., did. We add these two "derivative vectors" to the $\boldsymbol{\pi}_\mu$ introduced in (11) and define new $\boldsymbol{\pi}_\mu$, σ_μ, η_μ, and $\boldsymbol{\delta}_\mu$ by

$$\begin{aligned}
\boldsymbol{\pi}_\mu &= \partial_\mu\boldsymbol{\pi}+g_1(\boldsymbol{\rho}_\mu\times\boldsymbol{\pi}-\boldsymbol{a}_\mu\sigma')-g_2 d_\mu\boldsymbol{\delta} \\
&\qquad +\hat{g}_1(\boldsymbol{\rho}_{\mu\nu}\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_{\mu\nu}\partial_\nu\sigma')-\hat{g}_2 d_{\mu\nu}\partial_\nu\boldsymbol{\delta}
\end{aligned} \tag{20}$$

and analogous definitions for σ_μ, η_μ, and $\boldsymbol{\delta}_\mu$. The same construction is done for $\boldsymbol{\pi}_\mu'$, σ_μ', etc. A further generalization is achieved by substituting in (19) for $\partial_\nu\boldsymbol{\pi}$, $\partial_\nu\sigma'$, $\partial_\nu\eta$, and $\partial_\nu\boldsymbol{\delta}$ in the terms proportional to $\hat{g}_1$ and $\hat{g}_2$ the complete $\boldsymbol{\pi}_\nu$, σ_ν, η_ν, and $\boldsymbol{\delta}_\nu$. In this way we can generate higher covariant polynomials.

Now we consider which of these products of meson fields can be coupled to nucleons to preserve chiral invariance. It is clear that $\boldsymbol{\pi}_s$, σ_s, η_s, and $\boldsymbol{\delta}_s$ are coupled like $\boldsymbol{\pi}$, σ', η, and $\boldsymbol{\delta}$ as stated in (2) and (8), but we shall not include them in our Lagrangian. Furthermore, it is easily seen that $\boldsymbol{\pi}_{\mu\nu}'$, $\sigma_{\mu\nu}'$, $\eta_{\mu\nu}'$, and $\boldsymbol{\delta}_{\mu\nu}'$ defined in (15) can be coupled invariantly with nonderivative tensor coupling to form L_n:

$$L_n=\kappa\bar{\psi}\{-\tfrac{1}{2}[\gamma_\mu,\gamma_\nu](\sigma_{\mu\nu}'+\boldsymbol{\tau}\cdot\boldsymbol{\delta}_{\mu\nu}' -i\gamma_5(\eta_{\mu\nu}'+\boldsymbol{\tau}\cdot\boldsymbol{\pi}_{\mu\nu}'))\}\psi, \tag{21}$$

whereas $\boldsymbol{\pi}_{\mu\nu}$, $\sigma_{\mu\nu}$, $\eta_{\mu\nu}$, and $\boldsymbol{\delta}_{\mu\nu}$ cannot be coupled in this way because of wrong C parity. A C-invariant coupling, on the other hand, which contains derivatives of the nucleon field is not chiral-invariant. We notice that L_n yields the tensor coupling for ρ and ω. The situation is different for the "vector fields" defined in Eqs. (11) and (13). Here both kinds—those with positive and those with negative C parity—can couple in a chiral-invariant way to ψ, but only with derivatives acting on ψ, giving L_n':

$$L_n'=\bar{\psi}\tfrac{1}{2}[\gamma_\mu,\gamma_\nu]\overleftrightarrow{\partial}_\nu(\sigma_\mu+\boldsymbol{\tau}\cdot\boldsymbol{\delta}_\mu+i\gamma_5(\eta_\mu+\boldsymbol{\tau}\cdot\boldsymbol{\pi}_\mu))\psi$$
$$-i\bar{\psi}\overleftrightarrow{\partial}_\mu(\sigma_\mu'+\boldsymbol{\tau}\cdot\boldsymbol{\delta}_\mu'+i\gamma_5(\eta_\mu'+\boldsymbol{\tau}\cdot\boldsymbol{\pi}_\mu'))\psi, \quad (22)$$

where $\bar{\psi}\partial_\nu\psi\equiv\bar{\psi}\partial_\nu\psi-(\partial_\nu\bar{\psi})\psi$ is supposed to act only on the nucleon fields and in $\boldsymbol{\pi}_\mu$, σ_μ, η_μ, and $\boldsymbol{\delta}_\mu$ the terms linear in the basic fields must be removed.

All meson couplings considered so far transformed like $\boldsymbol{\pi}$, σ', η, and $\boldsymbol{\delta}$. In the same way meson couplings of the tensor type which transform like $\boldsymbol{\rho}_\mu$ and $\boldsymbol{a}_\mu$ are usually defined

$$\begin{aligned}\mathbf{f}_{\mu\nu}&=\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu+\gamma(\boldsymbol{\rho}_\mu\times\boldsymbol{\rho}_\nu+\boldsymbol{a}_\mu\times\boldsymbol{a}_\nu),\\ \mathbf{g}_{\mu\nu}&=\partial_\mu\boldsymbol{a}_\nu-\partial_\nu\boldsymbol{a}_\mu+\gamma(\boldsymbol{\rho}_\mu\times\boldsymbol{a}_\nu+\boldsymbol{a}_\mu\times\boldsymbol{\rho}_\nu).\end{aligned} \quad (23)$$

Since $\boldsymbol{\rho}_\mu^2+\boldsymbol{a}_\mu^2$ is chiral-invariant, we can easily write down a chiral-invariant $L^{(\rho)}$ which contains the free part of the ρ and a meson:

$$L^{(\rho)}=-\tfrac{1}{4}(\mathbf{f}_{\mu\nu}^2+\mathbf{g}_{\mu\nu}^2)-\tfrac{1}{2}m_V^2(\boldsymbol{\rho}_\mu^2+\boldsymbol{a}_\mu^2), \quad (24)$$

with the same mass m_V for the fields $\boldsymbol{\rho}_\mu$ and $\boldsymbol{a}_\mu$. Of course, the Lagrangian $L^{(\omega)}$ for the ω and d meson is not restricted by chiral invariance, and is

$$L^{(\omega)}=-\tfrac{1}{4}(\omega_{\mu\nu}^2+d_{\mu\nu}^2)-\tfrac{1}{2}m_\omega^2\omega_\mu^2-\tfrac{1}{2}m_d^2d_\mu^2. \quad (25)$$

But $L^{(\eta)}$, which contains the free part for the η and $\boldsymbol{\delta}$, is restricted to the following form:

$$L^{(\eta)}=-\tfrac{1}{2}[(\partial_\mu\eta)^2+(\partial_\mu\boldsymbol{\delta})^2]-\tfrac{1}{2}m_S^2(\eta^2+\boldsymbol{\delta}^2). \quad (26)$$

This completes the construction of the chiral-invariant Lagrangian L. It is given by the sum of the chiral-invariant parts defined in Eqs. (6), (8), (24), (26), (21), and (22):

$$L=L^{(\sigma)}+L'+L^{(\rho)}+L^{(\omega)}+L^{(\eta)}+L_m+L_n+L_n', \quad (27)$$

but with $\partial_\mu\boldsymbol{\pi}$ replaced with $\boldsymbol{\pi}_\mu$, etc., in $L^{(\sigma)}$ and with $\partial_\mu\eta$ replaced with η_μ, etc., in $L^{(\eta)}$, and where L_m contains all the other four meson couplings transforming like $\boldsymbol{\pi}^2+\sigma'^2$ and $\eta^2+\boldsymbol{\delta}^2$, which are

$$\begin{aligned}L_m=&-\tfrac{1}{2}(\boldsymbol{\pi}_\mu^{12}+\sigma_\mu'^2+\boldsymbol{\delta}_\mu'^2+\eta_\mu'^2)+(\boldsymbol{\pi}_{\mu\nu}^2+\sigma_{\mu\nu}^2+\eta_{\mu\nu}^2+\boldsymbol{\delta}_{\mu\nu}^2)\\&+(\boldsymbol{\pi}_{\mu\nu}'^2+\sigma_{\mu\nu}'^2+\eta_{\mu\nu}'^2+\boldsymbol{\delta}_{\mu\nu}'^2)+\epsilon_{\mu\nu\rho\sigma}(\boldsymbol{\pi}_{\mu\nu}\cdot\boldsymbol{\delta}_{\rho\sigma}\\&-\sigma_{\mu\nu}\eta_{\rho\sigma}+\boldsymbol{\pi}_{\mu\nu}'\cdot\boldsymbol{\delta}_{\rho\sigma}'-\sigma_{\mu\nu}'\eta_{\rho\sigma}').\end{aligned} \quad (28)$$

Of course, many more chiral-invariant Lagrangian terms can be constructed if we allow higher derivatives and higher products of the basic meson fields. Since we wish only to establish consequences for three-meson vertices and threshold behavior of some two-particle scattering processes, the Lagrangian L is sufficient. In Sec. III we shall eliminate two-meson interactions in the Lagrangian and shall derive relations for masses and decays of the more interesting mesons like ρ, π, and σ.

III. RELATIONS FOR MESON MASSES AND DECAYS

In this section we shall be concerned only with meson properties. Here we shall discuss also the connection of the fields $\boldsymbol{\pi}$, σ, η, $\boldsymbol{\delta}$, $\boldsymbol{\rho}$, $\boldsymbol{a}$, ω, and d with the observed meson resonances. First, we consider only the interaction of $\boldsymbol{\pi}$, $\boldsymbol{\rho}$, $\boldsymbol{a}$, and σ. These fields will be associated with the familar $\pi(140)$, $\rho(765)$, and $A_1(1060)$, whereas the σ can be the s-wave $\pi\pi$ resonance, encountered in π-π phase-shift analyses mentioned in the Introduction. Other possibilities for σ are $\eta_V(1070)$[15] or a linear combination of η_V with the $\pi\pi$ resonance of lower mass. But the 2π decay mode of the η_V seems to be rare,[15] so that it is unlikely that it is a candidate for our σ. The field $\boldsymbol{a}_\mu$ is actually a linear combination of $\partial_\mu\boldsymbol{\pi}$ and the A_1, since $L^{(\sigma)}$ with $\partial_\mu\boldsymbol{\pi}\to\boldsymbol{\pi}_\mu$ contains a direct $\boldsymbol{a}_\mu\to\partial_\mu\boldsymbol{\pi}$ transition interaction which can be removed by a linear transformation:

$$\boldsymbol{a}_\mu=\boldsymbol{a}_\mu'+z\partial_\mu\boldsymbol{\pi}. \quad (29)$$

Then, up to a factor, a_μ' will be associated with the A_1. The direct $\boldsymbol{a}_\mu\partial_\mu\boldsymbol{\pi}$ term comes about because σ' contains a constant term forced upon us by chiral symmetry. $L^{(\sigma)}$ also has terms which contribute to the mass of the ρ and a meson, which leads to relations of the masses of these mesons with the coupling constants introduced in $\boldsymbol{\pi}_\mu$ and σ_μ. In order to derive these relations, we start from that part of the Lagrangian L which contributes to the free part of $\boldsymbol{\rho}_\mu$, $\boldsymbol{a}_\mu$, σ, and $\boldsymbol{\pi}$, and which originates from $L^{(\sigma)}+L^{(\rho)}+L_m$. It will be called $\hat{L}$ and consists only of the meson-meson interactions for $\boldsymbol{\rho}_\mu$, $\boldsymbol{a}_\mu$, $\boldsymbol{\pi}$, and σ. The η-δ part will be considered separately.

$$\begin{aligned}\hat{L}=&-\tfrac{1}{2}\{[\partial_\mu\boldsymbol{\pi}+g_1(\boldsymbol{\rho}_\mu\times\boldsymbol{\pi}-\boldsymbol{a}_\mu\sigma')+\hat{g}_1(\boldsymbol{\rho}_{\mu\nu}\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_{\mu\nu}\times\partial_\nu\sigma')]^2+(\partial_\mu\sigma'+g_1\boldsymbol{a}_\mu\cdot\boldsymbol{\pi}+\hat{g}_1\boldsymbol{a}_{\mu\nu}\cdot\partial_\nu\boldsymbol{\pi})^2+(f_1\boldsymbol{\rho}_\mu\cdot\boldsymbol{\pi}+\hat{f}_1\boldsymbol{\rho}_{\mu\nu}\cdot\partial_\nu\boldsymbol{\pi})^2\\&+[f_1(\boldsymbol{a}_\mu\times\boldsymbol{\pi}-\boldsymbol{\rho}_\mu\sigma')+\hat{f}_1(\boldsymbol{a}_{\mu\nu}\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{\rho}_{\mu\nu}\partial_\nu\sigma')]^2+m_V^2(\boldsymbol{\rho}_\mu^2+\boldsymbol{a}_\mu^2)+m_\sigma^2\sigma^2\}-\tfrac{1}{4}\{[\boldsymbol{\rho}_{\mu\nu}+\gamma(\boldsymbol{\rho}_\mu\times\boldsymbol{\rho}_\nu+\boldsymbol{a}_\mu\times\boldsymbol{a}_\nu)]^2\\&+[\boldsymbol{a}_{\mu\nu}+\gamma(\boldsymbol{\rho}_\mu\times\boldsymbol{a}_\nu+\boldsymbol{a}_\mu\times\boldsymbol{\rho}_\nu)]^2\}+[\gamma_1(\boldsymbol{\rho}_{\mu\nu}\times\boldsymbol{\pi}-\boldsymbol{a}_{\mu\nu}\sigma')+\gamma_1'(\boldsymbol{\rho}_\mu\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{\rho}_\nu\times\partial_\mu\boldsymbol{\pi}-\boldsymbol{a}_\mu\partial_\nu\sigma'-\boldsymbol{a}_\nu\partial_\mu\sigma)]^2\\&+[\gamma_1\boldsymbol{a}_{\mu\nu}\times\boldsymbol{\pi}+\gamma_1'(\boldsymbol{a}_\mu\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_\nu\times\partial_\mu\boldsymbol{\pi})]^2+[\epsilon_1\boldsymbol{\rho}_{\mu\nu}\cdot\boldsymbol{\pi}+\epsilon_1'(\boldsymbol{\rho}_\mu\cdot\partial_\nu\boldsymbol{\pi}-\boldsymbol{\rho}_\nu\cdot\partial_\mu\boldsymbol{\pi})]^2+[\epsilon_1(\boldsymbol{a}_{\mu\nu}\times\boldsymbol{\pi}-\boldsymbol{\rho}_{\mu\nu}\sigma')\\&+\epsilon_1'(\boldsymbol{a}_\mu\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_\nu\partial_\mu\boldsymbol{\pi}-\boldsymbol{\rho}_\mu\partial_\nu\sigma+\boldsymbol{\rho}_\nu\partial_\mu\sigma)]^2+(4m\lambda/f)\sigma\boldsymbol{\pi}^2.\end{aligned} \quad (30)$$

With the substitution $\boldsymbol{a}_\mu=\boldsymbol{a}_\mu'+z\partial_\mu\boldsymbol{\pi}$ this Lagrangian is transformed into

$$\hat{L}=-\tfrac{1}{2}[z_\pi(\partial_\mu\boldsymbol{\pi})^2+z_\rho\tfrac{1}{2}(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu)^2+z_a\tfrac{1}{2}(\partial_\mu\boldsymbol{a}_\nu'-\partial_\mu\boldsymbol{a}_\nu')^2+(\partial_\mu\sigma)^2+m_a^2\boldsymbol{a}_\mu^2+m_\rho^2\boldsymbol{\rho}_\mu^2+m_\sigma^2\sigma^2]+L_3, \quad (31)$$

where L_3 contains all three- and four-meson interactions to be considered later. The normalization constants and masses are given in terms of the "symmetry" masses m_V^2 and coupling constants:

$$\begin{aligned} z_\pi &= (1+g_1 zm/f)^2+m_V^2 z^2\,,\\ z_\rho &= 1-4\epsilon_1^2 m^2/f^2\,,\\ z_a &= 1-4\gamma_1^2 m^2/f^2\,, \end{aligned} \tag{32}$$

$$m_a^2=m_V^2+g_1^2 m^2/f^2\,,\quad m_\rho^2=m_V^2+f_1^2 m^2/f^2\,. \tag{33}$$

With the constant z equal to

$$z=-\frac{g_1 m}{f(m_V^2+g_1^2 m^2/f^2)}=-\frac{g_1 m}{f m_a^2}\,, \tag{34}$$

the direct $\partial_\mu \boldsymbol{\pi}\boldsymbol{a}_\mu'$ interaction term is eliminated. By substituting z into (32), z_π can be cast into a simpler form:

$$z_\pi=m_V^2/m_a^2\,. \tag{35}$$

The necessity of making a shift in the $\boldsymbol{a}_\mu$ field to eliminate the $\partial_\mu \boldsymbol{\pi}\cdot\boldsymbol{a}_\mu'$ coupling seems to be characteristic of chiral-invariant Lagrangians and has been encountered also by Wess and Zumino[17] and by Gasiorowicz and Geffen[18] with their Lagrangians. Furthermore, we observe that the masses of the $\hat{\boldsymbol{a}}_\mu$ and $\boldsymbol{\varrho}_\mu$ split even with chiral invariance, a fact which is connected with the nonzero vacuum expectation value of the σ' field.

Next, we derive the decay interactions for $\rho\pi\pi$, $a\rho\pi$, and $a\sigma\pi$ which follow from $\hat{L}$ as given in (30). $L_{\rho\pi\pi}$ gets contributions in $L^{(\sigma)}$, in $L^{(\rho)}$, and in L_m. We have

$$L_{\rho\pi\pi}=-G_1\partial_\mu\boldsymbol{\pi}\cdot\boldsymbol{\varrho}_\mu\times\boldsymbol{\pi}-G_2\partial_\mu\boldsymbol{\pi}\times\partial_\nu\boldsymbol{\pi}\cdot(\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu)\,, \tag{36}$$

where

$$G_1=g_1\left(1+z\frac{g_1 m}{f}\right)-f_1^2\frac{zm}{f} \tag{37}$$

and

$$G_2=\tfrac{1}{2}z^2\gamma-4z\epsilon_1\epsilon_1'\frac{m}{f}+\hat{g}_1\left(1-zg_1\frac{m}{f}\right)^2.$$

In G_1 and G_2 we eliminate z and f_1 with Eqs. (32) and (33) and obtain

$$G_1=g_1\frac{m_\rho^2}{m_a^2}\,,\quad G_2=-\hat{g}_1\frac{m_V^4}{m_a^4}+\frac{1}{2m_a^4}(m_a^2-m_V^2)\gamma-\frac{\epsilon G_1}{m_\rho^2}\,,$$

$$\epsilon=\frac{4\epsilon_1\epsilon_1' m^2}{f^2}\,. \tag{38}$$

On the mass shell, $L_{\rho\pi\pi}$ is equivalent to

$$L_{\rho\pi\pi}=\frac{-m_\rho^2 g_1}{m_a^2}\left(1-\frac{m_a^2-m_V^2}{2m_a^2}\frac{\gamma}{g_1}+\frac{m_V^4}{m_a^2}\frac{\hat{g}_1}{g_1}+\epsilon\right)\times\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\,, \tag{39}$$

whereas off the mass shell, extrapolated to $\partial^2\rho_\mu=0$, we have

$$L_{\rho\pi\pi}=-G_1\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\,.$$

$L_{a\rho\pi}$ has a somewhat more complicated structure. First, we read off from Eq. (30)

$$\begin{aligned} L_{a\rho\pi}=&(g_1^2-f_1^2)\frac{m}{f}\boldsymbol{a}_\mu'\cdot\boldsymbol{\varrho}_\mu\times\boldsymbol{\pi}-z\gamma(\boldsymbol{\varrho}_{\mu\nu}\cdot\boldsymbol{a}_\mu'\times\partial_\nu\boldsymbol{\pi}\\ &+\boldsymbol{a}_{\mu\nu}'\cdot\boldsymbol{\varrho}_\mu\times\partial_\nu\boldsymbol{\pi})+2\frac{m}{f}(\gamma_1^2-\epsilon_1^2)\boldsymbol{a}_{\mu\nu}'\cdot\boldsymbol{\varrho}_{\mu\nu}\times\boldsymbol{\pi}\\ &+\frac{4m}{f}\gamma_1\gamma_1'\boldsymbol{a}_{\mu\nu}'\pi\cdot\boldsymbol{\varrho}_\mu\times\partial_\nu\boldsymbol{\pi}+\frac{4m}{f}\epsilon_1\epsilon_1'\boldsymbol{\varrho}_{\mu\nu}\cdot\boldsymbol{a}_\mu'\times\partial_\nu\boldsymbol{\pi}\\ &+2g_1\hat{g}_1\frac{m}{f}\left(1+zg_1\frac{m}{f}\right)\boldsymbol{\varrho}_{\mu\nu}\cdot\boldsymbol{a}_\mu'\times\partial_\nu\boldsymbol{\pi}\\ &+2f_1\hat{f}_1\frac{m}{f}\left(1+zg_1\frac{m}{f}\right)\boldsymbol{a}_{\mu\nu}'\cdot\boldsymbol{\varrho}_\mu\times\partial_\nu\boldsymbol{\pi}\,. \end{aligned} \tag{40}$$

On the mass shell, $L_{a\rho\pi}$ can be reduced to

$$L_{a\rho\pi}=-h_1\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\pi}\times\boldsymbol{a}_\mu'-h_2\boldsymbol{\varrho}_{\mu\nu}\cdot\boldsymbol{\pi}\times\boldsymbol{a}_{\mu\nu}'\,, \tag{41}$$

where

$$\begin{aligned} h_1=&(g_1^2-f_1^2)\frac{m}{f}+z\gamma(m_a^2-m_\rho^2)\\ &-\frac{4m}{f}(\gamma_1\gamma_1' m_a^2-\epsilon_1\epsilon_1' m_\rho^2)\\ &+\frac{m}{f}\left(1+zg_1\frac{m}{f}\right)(2g_1\hat{g}_1 m_\rho^2-f_1\hat{f}_1 m_a^2)\,, \end{aligned} \tag{42}$$

$$\begin{aligned} h_2=&\frac{2m}{f}(\epsilon_1^2-\gamma_1^2)+\frac{2m}{f}(\epsilon_1\epsilon_1'-\gamma_1\gamma_1')\\ &+\frac{m}{f}\left(1+zg_1\frac{m}{f}\right)(g_1\hat{g}_1-\tfrac{1}{2}f_1\hat{f}_1)\,. \end{aligned}$$

With the definitions $\epsilon'=4\gamma_1\gamma_1' m^2/f^2$ and $\delta=2(\epsilon_1^2-\gamma_1^2)\times m^2/f^2$ and Eqs. (33) and (34) for z, m_ρ^2, and m_a^2, respectively, h_1 and h_2 are simplified to

$$\begin{aligned} h_1=&(m_a^2-m_\rho^2)\frac{f}{m}-\gamma\frac{g_1 m}{f}\frac{m_a^2-m_\rho^2}{m_a^2}-\frac{f}{m}(\epsilon m_\rho^2-\epsilon' m_a^2)\\ &-\frac{m}{f}\frac{m_V^2}{m_a^2}(2m_\rho^2 g_1\hat{g}_1-m_a^2 f_1\hat{f}_1)\,, \end{aligned} \tag{43}$$

$$h_2=\frac{f}{m}[\delta-\tfrac{1}{2}(\epsilon-\epsilon')]+\frac{m}{f}\frac{m_V^2}{m_a^2}(g_1\hat{g}_1-\tfrac{1}{2}f_1\hat{f}_1)\,.$$

In order to be able to compare with the $\rho\pi\pi$ coupling constant we eliminate f/m in favor of g_1. We choose

[17] J. Wess and B. Zumino (Ref. 8).
[18] S. Gasiorowicz and D. A. Geffen (Ref. 8).

$f/m=g_1/(m_a^2-m_V^2)^{1/2}$ so that f and g_1 have the same sign. Then

$$\begin{aligned}h_1&=g_1\frac{m_a^2-m_\rho^2}{(m_a^2-m_V^2)^{1/2}}\left[1-\frac{m_a^2-m_V^2}{m_a^2}\frac{\gamma}{g_1}\frac{\epsilon m_\rho^2-\epsilon' m_a^2}{m_a^2-m_\rho^2}\right.\\&\quad\left.+\frac{m_a^2-m_V^2}{m_a^2-m_\rho^2}\frac{m_V^2}{m_a^2}\left(2\frac{\hat{g}_1}{g_1}m_\rho^2-\frac{f_1\hat{f}_1}{g_1^2}m_a^2\right)\right],\\h_2&=g_1\frac{1}{(m_a^2-m_V^2)^{1/2}}\left[\delta-\tfrac{1}{2}(\epsilon-\epsilon')\right.\\&\quad\left.+(m_a^2-m_V^2)\frac{m_V^2}{m_a^2}\left(\frac{\hat{g}_1}{g_1}-\frac{f_1\hat{f}_1}{2g_1^2}\right)\right].\end{aligned}\qquad(44)$$

Making the proper normalization of ρ_μ, a_μ', and π with $\sqrt{z_\rho}$, $\sqrt{z_a}$, and $\sqrt{z_\pi}$, we obtain as the final result for the $g_{\rho\pi\pi}$ and $(h_1)_{a\rho\pi}$ and $(h_2)_{a\rho\pi}$ coupling constants

$$\begin{aligned}g_{\rho\pi\pi}&=-\frac{1}{\sqrt{z_\rho}}\frac{m_\rho^2}{m_V^2}g_1\left(1-\frac{1}{2}\frac{m_a^2-m_V^2}{m_a^2}\frac{\gamma}{g_1}+\frac{m_V^4}{m_a^2}\frac{\hat{g}_1}{g_1}+\epsilon\right),\\(h_1)_{a\rho\pi}&=\frac{1}{(z_\rho z_a)^{1/2}}\frac{m_a}{m_V}\frac{m_a^2-m_\rho^2}{(m_a^2-m_V^2)^{1/2}}g_1\left[1-\frac{m_a^2-m_V^2}{m_a^2}\frac{\gamma}{g_1}\right.\\&\quad-\frac{\epsilon m_\rho^2-\epsilon' m_a^2}{m_a^2-m_\rho^2}+\frac{(m_a^2-m_V^2)m_V^2}{(m_a^2-m_\rho^2)m_a^2}\\&\quad\left.\times\left(2\frac{\hat{g}_1}{g_1}m_\rho^2-\frac{f_1\hat{f}_1}{g_1^2}m_a^2\right)\right],\\(h_2)_{a\rho\pi}&=\frac{1}{(z_\rho z_a)^{1/2}}\frac{m_a}{m_V}\frac{1}{(m_a^2-m_V^2)^{1/2}}g_1\left[\delta-\tfrac{1}{2}(\epsilon-\epsilon')\right.\\&\quad\left.+\frac{m_V^2}{m_a^2}(m_a^2-m_V^2)\left(\frac{\hat{g}_1}{g_1}-\frac{f_1\hat{f}_1}{2g_1^2}\right)\right].\end{aligned}\qquad(45)$$

Thus, even under the condition that the "masses" m_a, m_ρ, and m_V together with the normalization constants z_ρ and z_a are known, the three coupling constants still depend on seven constants. Therefore, the correlation of these three coupling constants on the basis of chiral symmetry alone is rather weak in contrast to the chiral-symmetry approach of Zumino and Wess[17,18] or current algebra with single-particle dominance hypothesis,[19] where one encounters a linear relation between $g_{\rho\pi\pi}$, $h_{1a\rho\pi}$, and $h_{2a\rho\pi}$. Only with further assumptions beyond chiral symmetry shall we get the result obtained in Refs. 17 and 18. In these papers the ρ and a fields are introduced as Yang-Mills gauge fields. Then $\gamma=g_1$. Furthermore, we shall assume without any justification that $z_\rho=z_a=1$. Then $\delta=0$, and the parameters m_ρ and m_a are the masses of the ρ and A_1 meson. Furthermore, we assume that $m_V^2=m_\rho^2$; then $f_1=0$, and we put $m_a^2=2m_\rho^2$, which roughly is the experimental relation between the ρ and the A_1 mass. Then we have

$$\begin{aligned}g_{\rho\pi\pi}&=-\left(\tfrac{3}{4}+\tfrac{1}{2}m_\rho^2\frac{\hat{g}_1}{g_1}+\epsilon\right)g_1,\\h_{1a\rho\pi}&=\left(\tfrac{1}{2}-(\epsilon-2\epsilon')+m_\rho^2\frac{\hat{g}_1}{g_1}\right)m_a g_1,\\h_{2a\rho\pi}&=\left(-(\epsilon-\epsilon')+m_\rho^2\frac{\hat{g}_1}{g_1}\right)\frac{1}{m_a}g_1.\end{aligned}\qquad(46)$$

Now, if $\epsilon=\epsilon'=0$, our result agrees with the relation obtained by Zumino and Wess.[17,18] In this case anomalous couplings of ρ_μ, a_μ, and π appear only in connection with the tensors $\rho_{\mu\nu}$ and $a_{\mu\nu}$, but not with $(\rho_\mu\partial_\nu\pi\text{-}\rho_\nu\partial_\mu\pi)$, etc. In Ref. 17 such tensors are excluded because the Lagrangian is constructed on the basis of gauge invariance of the second kind for the fields a_μ and ρ_μ.

Other three-meson couplings of interest are the $a\sigma\pi$ and $\sigma\pi\pi$ couplings. For the first of these two, we derive from the Lagrangian $\hat{L}$

$$L_{a\sigma\pi}=g_1\left[\mathbf{a}_\mu'\cdot\partial_\mu\boldsymbol{\pi}\,\sigma-\mathbf{a}_\mu'\cdot\boldsymbol{\pi}\,\partial_\mu\sigma+\frac{\hat{g}_1}{g_1}\left(1+zg_1\frac{m}{f}\right)\mathbf{a}_{\mu\nu}'\cdot\partial_\mu\boldsymbol{\pi}\,\partial_\nu\sigma\right],\qquad(47)$$

which is equal to

$$L_{a\sigma\pi}=g_1\mathbf{a}_\mu'(\partial_\mu\boldsymbol{\pi}\sigma-\boldsymbol{\pi}\partial_\mu\sigma)+2\hat{g}_1\frac{m_V^2}{m_a^2}\mathbf{a}_{\mu\nu}'\cdot\partial_\mu\boldsymbol{\pi}\,\partial_\nu\sigma.\qquad(48)$$

On the mass shell and for renormalized pions [$\pi_r=(\sqrt{z_\pi})\pi$], this reduces to

$$L_{a\sigma\pi_r}=\mathbf{a}_\mu'\cdot\partial_\mu\boldsymbol{\pi}_r\,\sigma 2g_1\left(1+\frac{\hat{g}_1}{g_1}m_V^2\right)\frac{m_a}{m_V}.\qquad(49)$$

$L_{\sigma\pi\pi}$ has the following contributions:

$$L_{\sigma\pi\pi}=g_1\frac{(m_a^2-m_V^2)^{1/2}}{m_a^2}\left(\frac{m_V^2}{m_a^2}\sigma\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}-\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}\partial_\mu\sigma-\frac{4m_a^2}{g_1^2}\lambda\sigma\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right).\qquad(50)$$

On the mass shell and for renormalized pions, the Lagrangian is

$$L_{\sigma\pi_r\pi_r}=g_1\frac{(m_a^2-m_V^2)^{1/2}}{m_V^2}\left(\frac{m_V^2}{m_a^2}(\tfrac{1}{2}m_\sigma^2-m_\pi^2)+\tfrac{1}{2}m_\sigma^2-\frac{4\lambda}{g_1^2}m_a^2\right)\sigma\boldsymbol{\pi}_r\cdot\boldsymbol{\pi}_r.\qquad(51)$$

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[19] See H. J. Schnitzer and S. Weinberg (Ref. 3); S. G. Brown and G. B. West (Ref. 3); K. Fabricius (Ref. 3).

For zero-mass pions and

$$m_\sigma^2=8\lambda m^2/f^2=(8\lambda/g_1^2)(m_a^2-m_V^2)$$

and

$$2m_\rho^2=2m_V^2=2m_a^2\,,$$

this reduces to

$$L\sigma_{\pi_r\pi_r}=-\tfrac{1}{4}(m_\sigma^2/m_\rho)g_1\sigma\boldsymbol{\pi}_r\cdot\boldsymbol{\pi}_r\,. \quad (52)$$

We see that the $\sigma\pi\pi$ coupling in the original σ model is reduced by a factor of 4 through the extra couplings originating from the introduction of the ρ and a fields. A part of this reduction, however, is lost by the proper renormalization of the pion field $\pi\rightarrow\pi_r$.

A similar analysis can be carried out with that part of the meson Lagrangian which couples the η, δ, and d fields. To eliminate a $d_\mu\partial_\mu\eta$ coupling, one defines in analogy to the a-π problem a new field d_μ' to be associated with the D meson by

$$d_\mu=d_\mu'+\zeta\partial_\mu\eta\,.$$

Then the mass of the d_μ' is

$$m_{d'}^2=m_d^2+f_3^2m^2/f^2$$

and

$$\zeta=\pm(m_d'^2-m_d^2)^{1/2}/m_d'^2\,,$$

where the two signs are related to the relative sign of f and f_3. The normalization of the η field is changed by

$$z_\eta=(4m_d'^2-3m_d^2)/m_d'^2\,,$$

which changes also the mass of the η compared to the δ mass. We have

$$m_\eta^2=m_\delta^2/\sqrt{z_\eta}\,.$$

Therefore, $m_\eta^2>\frac{1}{2}m_\delta^2$, which is not satisfied by $\eta(549)$ and $\pi_V(1016)$.[15] But there are two other candidates for the η: $\eta'(958)$ and $E(1420)$,[15] which makes further analysis rather complicated. On the other hand, f_3 which determines all these masses, can be correlated with the decay $d\rightarrow\delta+\pi$, which seems to be the dominant mode of $D(1285)$.[15]

So far, we have derived only consequences from our chiral-symmetric Lagrangian with the pure symmetry assumption. To obtain results for a finite pion mass, we must introduce a symmetry-breaking term in the Lagrangian which we choose proportional to $\boldsymbol{\pi}^2$:

$$L_{\mathrm{sb}}=-\tfrac{1}{2}c\boldsymbol{\pi}^2\,. \quad (53)$$

Then this term accounts for the mass of the pion. The divergence of the chiral current j_μ^5 is

$$\partial_\mu\mathbf{j}_\mu^5=\partial L/\partial\boldsymbol{\alpha}=c\sigma'\boldsymbol{\pi}\,. \quad (54)$$

Thus the Goldberger-Treiman coupling constant g_π which determines the weak decay of the pion π_r is

$$g_\pi=-\frac{1}{\sqrt{z_\pi}}c\frac{m}{f}=-\frac{m_Vm}{m_af}m_\pi^2=-\frac{m_V(m_a^2-m_V^2)^{1/2}}{m_ag_1}m_\pi^2\,, \quad (55)$$

where $m_\pi^2=c/z_\pi$ is the actual pion mass. f is also linked to the pion-nucleon coupling constant $g_{\pi n}$. In our Lagrangian, the pion-nucleon coupling $L_{\pi n}$ has the following form:

$$L_{\pi_rn}=(z_\pi)^{-1/2}\bar{\psi}\boldsymbol{\tau}\cdot(fi\gamma_5\boldsymbol{\pi}_r+zf_\rho i\gamma_5\gamma_\mu\partial_\mu\boldsymbol{\pi}_r)\psi\,. \quad (56)$$

For nucleons on the mass shell the coupling is

$$L_{\pi_rn}=g_{\pi n}\bar{\psi}\boldsymbol{\tau}\cdot\boldsymbol{\pi}_ri\gamma_5\psi\,, \quad (57)$$

with

$$g_{\pi n}=\frac{m_a}{m_V}f\left(1-\frac{2f_\rho g_1m^2}{f^2m_a^2}\right); \quad (58)$$

with ρ universality, $2f_\rho=g_1$, we would have

$$g_{\pi n}=(m_V/m_a)f\,.$$

IV. SEVERAL LOW-ENERGY THEOREMS

In this section, we shall calculate low-energy parameters such as s-wave scattering length for various processes as $\pi N\rightarrow\pi N$, $\pi\pi\rightarrow\pi\pi$, $\gamma N\rightarrow\pi N$, and $\pi N\rightarrow\sigma N$. It will be assumed that the low-energy parameters can be obtained from lowest-order perturbation theory based on the chiral-symmetric Lagrangian with symmetry breaking. This assumption may be questionable for cases where the unitarity plays a major role as in low-energy s-wave NN scattering or in the p-wave parameters in πN scattering. Therefore, we shall concentrate on the parameters and reactions mentioned above, for which unitarity effects are expected to be small.

In π-N scattering, we have contributions from nucleon exchange to be calculated from the interaction Hamiltonian H_{int}:

$$H_{\mathrm{int}}=-\frac{m_a}{m_V}f\bar{\psi}\boldsymbol{\tau}\cdot i\gamma_5\left(\boldsymbol{\pi}_r-\frac{f_\rho g_1m}{f^2m_a^2}\gamma_\mu\partial_\mu\boldsymbol{\pi}_r\right)\psi\,, \quad (59)$$

σ exchange, where the appropriate interaction is

$$H_{\mathrm{int}}=-f\bar{\psi}\psi\sigma+g_1\frac{(m_a^2-m_V^2)^{1/2}}{m_V^2}\left(\frac{4m_a^2\lambda}{g_1^2}\sigma\boldsymbol{\pi}_r\cdot\boldsymbol{\pi}_r+\boldsymbol{\pi}_r\cdot\partial_\mu\boldsymbol{\pi}_r\partial_\mu\sigma-\frac{m_V^2}{m_a^2}\sigma\partial_\mu\boldsymbol{\pi}_r\cdot\partial_\mu\boldsymbol{\pi}_r\right)\,, \quad (60)$$

and ρ exchange, with the interaction Hamiltonian

$$H_{\mathrm{int}}=-f_\rho\bar{\psi}i\gamma_\mu\boldsymbol{\tau}\psi\cdot\boldsymbol{\rho}_\mu+\tilde{G}_1\partial_\mu\boldsymbol{\pi}_r\cdot\boldsymbol{\rho}_\mu\times\boldsymbol{\pi}_r+\tilde{G}_2\partial_\mu\boldsymbol{\pi}_r\times\partial_\mu\boldsymbol{\pi}_r\cdot\boldsymbol{\rho}_{\mu\nu}+\frac{g_\rho}{4m}\bar{\psi}\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]\boldsymbol{\tau}\psi\cdot\boldsymbol{\rho}_{\mu\nu}\,, \quad (61)$$

where

$$\tilde{G}_1=\frac{m_\rho^2}{m_V^2}g_1\,,\quad \tilde{G}_2=\frac{m_a^2-m_V^2}{2m_a^2m_V^2}\gamma-\frac{m_V^2}{m_a^2}\hat{g}_1-\epsilon g_1\frac{1}{m_a^2}\,,$$

the magnetic coupling of the ρ which is not related to other known coupling constants. Since we are only interested in the s-wave scattering length, we calculate the forward amplitude $T_0=T(s=(m+m_\pi)^2,\ t=0)$ at threshold which is related to the scattering length by

$$a=-\frac{1}{4\pi}\frac{m}{m+m_\pi}T_0. \tag{62}$$

T_0 is easily obtained from the invariant amplitudes A and B, which are defined as usual:

$$T=\bar{u}(p')[-A+i\gamma(q'+q)/2B]u(p), \tag{63}$$

so that

$$T_0=-(A+m_\pi B). \tag{64}$$

Of course, we consider the t-channel $I=0$, 1 amplitudes separately, usually defined by

$$T_{\alpha\beta}=\delta_{\alpha\beta}T^{(+)}+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]T^{(-)}. \tag{65}$$

For convenience we display the invariant amplitudes for the three exchanges:

(a) N exchange:

$$\begin{aligned}
A^{(+)}&=4mg_{\rm pv}^2+4g_{\rm ps}g_{\rm pv},\\
A^{(-)}&=0,\\
B^{(+)}&=-(g_{\rm ps}+2mg_{\rm pv})^2\left(\frac{1}{s-m^2}-\frac{1}{u-m^2}\right),\\
B^{(-)}&=-(g_{\rm ps}+2mg_{\rm pv})^2\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right)-2g_{\rm pv}^2,
\end{aligned} \tag{66}$$

where

$$g_{\rm ps}=-\frac{m_a}{m_V}f,\quad g_{\rm pv}=\frac{(m_a^2-m_V^2)^{1/2}}{m_am_V}f_\rho.$$

(b) σ exchange:

$$A^{(+)}=\frac{m_a^2-m_V^2}{m_V^2}\frac{2f^2}{m}\frac{1}{m_\sigma^2-t}\left[\frac{4m_a^2\lambda}{g_1^2}-\frac{t}{2}\left(1+\frac{m_V^2}{m_a^2}\right)+m_\pi^2\frac{m_V^2}{m_a^2}\right], \tag{67}$$

where $g_1m=f(m_a^2-m_V^2)^{1/2}$ has been used. Because of $m_\sigma^2=(8\lambda/g_1^2)(m_a^2-m_V^2)$ we can write $A^{(+)}$ as

$$A^{(+)}=-\frac{f^2}{m}\frac{1}{m_\sigma^2-t}\left\{\frac{m_a^2-m_V^2}{m_V^2}\left[t\left(1+\frac{m_V^2}{m_a^2}\right)-2m_\pi^2\frac{m_V^2}{m_a^2}\right]-\frac{m_a^2m_\sigma^2}{m_V^2}\right\}. \tag{68}$$

(c) ρ exchange:

$$\begin{aligned}
B^{(-)}&=\frac{2}{m_\rho^2-t}(\tilde{G}_1-t\tilde{G}_2)(f_\rho+g_\rho),\\
A^{(-)}&=\frac{1}{m_\rho^2-t}(\tilde{G}_1-t\tilde{G}_2)g_\rho\frac{u-s}{2m}.
\end{aligned} \tag{69}$$

The final result for the two scattering lengths $a^{(+)}=\frac{1}{3}(a_{1/2}+2a_{3/2})$ and $a^{(-)}=\frac{1}{3}(a_{1/2}-a_{3/2})$ is

$$a^{(+)}=-\frac{1}{16\pi}\frac{m_\pi^2}{m^2(m+m_\pi)}\left[\frac{(g_{\rm ps}+2mg_{\rm pv})^2}{1-m_\pi^2/4m^2}-\frac{8f^2m^2}{m_\sigma^2}\frac{m_a^2-m_V^2}{m_a^2}\right], \tag{70}$$

$$a^{(-)}=\frac{1}{2\pi}\frac{m_\pi m}{m+m_\pi}\left[\frac{g_1f_\rho}{m_V^2}+\frac{1}{4m^2-m_\pi^2}(g_{\rm ps}+2mg_{\rm pv})^2-g_{\rm pv}^2\right]. \tag{71}$$

If we retain only the leading term in m_π/m and substitute the definitions for $g_{\rm ps}$ and $g_{\rm pv}$ and use $f^2/m^2=g_1^2/(m_a^2-m_V^2)$, we obtain

$$a^{(-)}=\frac{m_\pi}{8\pi}\frac{m_a^2}{m_V^2(m_a^2-m_V^2)}g_1^2. \tag{72}$$

We notice that the ρ contribution is cancelled by the pseudovector part of the π-N interaction terms. The current-algebra formula for $a^{(-)}$,

$$a^{(-)}=m_\pi^5/8\pi g_\pi^2, \tag{73}$$

is obtained from Eq. (71) if $g_{\rm ps}=0$, $2f_\rho=g_1$, and $g_\pi=-m_\pi^2m_\rho/\sqrt{2}g_1$. The latter formula, the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation,[20] is obtained from Eq. (55) if $m_a=\sqrt{2}m_V$. Of course, terms proportional to $g_{\rm ps}$ in the second term in (71) do not arise in the current-algebra derivations, where only pseudovector couplings appear. Naturally, in our model, we must work with the complete formula. Furthermore, we notice that the current-algebra formula also follows from the last result, Eq. (72), if $m_a=(2m_V)^{1/2}$ and $g_\pi=-m_\pi^2m_\rho/(2g_1)^{1/2}$.

Next we calculate the s-wave scattering length for π-π scattering. The π-π scattering amplitude is usually written in the following form:

$$T=A\delta_{\alpha\beta}\delta_{\gamma\delta}+B\delta_{\alpha\gamma}\delta_{\beta\delta}+C\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{74}$$

where α and β are isospin quantum numbers of the incoming pions and γ and δ are the labels for the pions

[20] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); see also the papers of Ref. 19.

in the final state. The amplitudes with specific isospin in the s channel $A_s{}^I$ are determined by A, B, and C in the following way:

$$A_s{}^0=3A+B+C,\quad A_s{}^1=B-C,\quad A_s{}^2=B+C. \tag{75}$$

Contributions to the π-π scattering amplitude come from ρ exchange, σ exchange, and a direct $(\boldsymbol{\pi}^2)^2$-interaction term proportional to λ. This term can be combined with a term originating from the σ-exchange contribution using the relation $m_\sigma{}^2=(8\lambda/g_1{}^2)(m_a{}^2-m_V{}^2)$. This way all terms not proportional to $m_\pi{}^2$ cancel. The final result for the amplitude $A(s,t,u)=A(s,u,t)$ is

$$A=-(\tilde{G}_1-t\tilde{G}_2)^2\frac{s-u}{m_\rho{}^2-t}-(\tilde{G}_1-u\tilde{G}_2)^2\frac{s-t}{m_\rho{}^2-u}$$
$$+\frac{g_1{}^2}{m_V{}^2(m_a{}^2-m_V{}^2)}\Bigg\{s(m_a{}^4-2m_V{}^4)-4m_\pi{}^2m_V{}^2(m_a{}^2-m_V{}^2)$$
$$-\frac{m_V{}^4}{m_a{}^4(m_\sigma{}^2-s)}[sm_V{}^2+2m_\pi{}^2(m_a{}^2-m_V{}^2)]^2\Bigg\}. \tag{76}$$

The other amplitudes can be determined with the crossing relations $B(s,t,u)=A(t,s,u)$ and $C(s,t,u)=A(u,t,s)$. With this, the s-wave scattering length for $I=0$ and $I=2$ and the p-wave scattering length for $I=1$ are easily calculated. These scattering length, called a_{IJ}, where the two subscripts refer to isospin and angular momentum, are $[a_{IJ}=-(1/32\pi m_\pi)T^{IJ}]$

$$a_{00}=\frac{m_\pi}{4\pi}\left[\frac{2\tilde{G}_1{}^2}{m_\rho{}^2}-\frac{g_1{}^2}{m_V{}^2}\left(1-\frac{29m_\pi{}^2}{4m_\sigma{}^2}\right)\right], \tag{77a}$$

$$a_{20}=\frac{m_\pi}{4\pi}\left[-\frac{\tilde{G}_1{}^2}{m_\rho{}^2}+\frac{g_1{}^2}{m_V{}^2}\left(1+\frac{m_\pi{}^2}{4m_\sigma{}^2}\right)\right], \tag{77b}$$

$$a_{11}=\frac{1}{8\pi m_\pi}\left(\frac{\tilde{G}_1{}^2}{m_\rho{}^2}-\frac{2}{3}\frac{g_1{}^2}{m_V{}^2}\right), \tag{77c}$$

where we have used $m_a{}^2=2m_V{}^2$ and neglected terms of higher order in $m_\pi{}^2/m_\sigma{}^2$. These scattering lengths differ from the results obtained with current algebra even if the σ contribution, which is the second term in Eqs. (77), is neglected. Eventually, we shall assume that $m_V=m_\rho$; then $\tilde{G}_1=g_1$, so that the ρ and σ contributions can be added together.

Now we derive low-energy expressions for the photoproduction amplitudes. As usual, we write the complete photoproduction amplitude f in the barycentric system for photoproduction of pions on nucleons as a matrix element between initial and final Pauli spinors:

$$f=(\chi_f{}^\dagger,F\chi_i).$$

The F is expanded in a complete set of rotational invariants in Pauli spin space. We adopt the notation of Chew, Goldberger, Low, and Nambu (CGLN).[21] Then for pion momentum $|\mathbf{q}|\to 0$, only the function F_1 is not vanishing, from which the electric s-wave multipole E_{0+} can be easily calculated. We have

$$E_{0+}=\lim_{|\mathbf{q}|\to 0}F_1=\lim_{|\mathbf{q}|\to 0}\frac{[(E_1+m)(E_2+m)]^{1/2}}{8\pi\sqrt{s}}\{[(\sqrt{s})-m]A$$
$$+[(\sqrt{s})-m]^2D-(t-\mu^2)(C-D)\}, \tag{78}$$

where A, B, C, and D are the CGLN invariant amplitudes in Dirac space and E_1 and E_2 are the initial and final nucleon energies. We have calculated the invariant amplitudes for the three isotopic-spin configurations and then determined $E_{0+}{}^{(\pm,0)}$ from them. We have neglected terms of higher order than $(m_\pi/m)^2$. The result is

$$E_{0+}{}^{(0)}=E_{0+}{}^{(+)}=-\frac{1}{4\pi}\frac{2m+m_\pi}{2(m+m_\pi)}\frac{eg_{\pi n}}{2m}\frac{m_\pi}{2m}\left(1-\frac{m_\pi}{2m}\right),$$
$$E_{0+}{}^{(-)}=\frac{1}{4\pi}\frac{2m+m_\pi}{2(m+m_\pi)}\frac{eg_{\pi n}}{2m}, \tag{79}$$

where $g_{\pi n}$ was defined in Eq. (58). The expression for $E_{0+}{}^{(-)}$ agrees with the Kroll-Ruderman theorem, as has to be expected, whereas $E_{0+}{}^{(0,+)}$ agrees with the current-algebra prediction of Balachandran, Gundzik, Narayanaswami, and Nicodemi (BGNN),[22] but disagrees with the pole approximation in the static model as worked out by CGLN.[21] This is somewhat surprising, since our results come mostly also from nucleon exchange. From pure nucleon exchange with pseudoscalar coupling we derive for $E_{0+}{}^{(+)}$ instead of (79)

$$E_{0+}{}^{(+)}=\frac{1}{4\pi}\frac{2m+m_\pi}{2(m+m_\pi)}\frac{eg_{\mathrm{ps}}}{2m}\frac{m_\pi}{2m}\left(1-\frac{m_\pi}{2m}+\kappa_V\right), \tag{80}$$

where $\kappa_V=(\kappa_p-\kappa_n)$ is the isovector anomalous magnetic moment. In our approach the anomalous magnetic moment term in the interaction of the photon field with the nucleons can be generated only from interactions with the vector mesons. Thus the isovector anomalous moment interaction can be obtained from the magnetic coupling of the ρ mesons:

$$L_{n\rho}=\bar{\psi}\tfrac{1}{2}i[\gamma_\mu,\gamma_\mu]\boldsymbol{\tau}\psi\cdot\boldsymbol{\varrho}_{\mu\nu}(\kappa\epsilon_1 m/f), \tag{81}$$

contained in L_n [Eq. (21)]. Then κ_V will be proportional to the coupling constant of this term:

$$e\kappa_V/4m=-c_0(2\kappa\epsilon_1 m/f), \tag{82}$$

with a proportionality constant c_0 to be determined in a dynamical calculation of the anomalous magnetic

[21] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[22] A. P. Balachandran, M. G. Gundzik, P. Narayanaswami, and F. Nicodemi (Ref. 3).

TABLE I. π-N scattering lengths.

Parameter	Chiral Lagrangian	Current algebra	Experiment
$a^{(-)}$	$\frac{m_\pi}{8\pi}\frac{m_a^2}{m_\rho^2(m_a^2-m_\rho^2)}g_1^2=0.082$	$\frac{m_\pi^5}{8\pi g_\pi^2}=0.091$	0.088 ± 0.003
$a^{(+)}$	$-\frac{m_\pi}{16\pi}\frac{m_\pi}{m^3}\left(g_{\pi n}^2-8f^2\frac{m^2(m_a^2-m_\rho^2)}{m_\sigma^2m_a^2}\right)=0.0033$	0	-0.003 ± 0.004

moments. Of course, here we make the assumption that the entire anomalous magnetic moment of the nucleons is generated from the ρ and the ω meson. The interaction term (81) contributes to $\pi N\to\rho N$ or $\pi N\to\gamma N$ through nucleon exchange. But, in addition to this term, we have a direct-interaction term for $\pi N\to\rho N$ in our Lagrangian which is also contained in L_n. It has the following form:

$$L_{n\rho\pi}=\bar{\psi}\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]i\gamma_5\psi\boldsymbol{\varrho}_{\mu\nu}\cdot\boldsymbol{\pi}_r\kappa\epsilon_1 m_a/m_V. \tag{83}$$

This term generates a direct interaction with the photon field proportional to c_0, which leads to the following contribution to $E_{0+}{}^{(+)}$:

$$\begin{aligned}(E_{0+}{}^{(+)})_{\mathrm{direct}}&=-\frac{1}{4\pi}\frac{2m+m_\pi}{2(m+m_\pi)}2\kappa\epsilon_1c_0\frac{m_am_\pi}{m_V}\\&=-\frac{1}{4\pi}\frac{2m+m_\pi}{2(m+m_\pi)}\frac{eg_{\mathrm{ps}}\kappa_Vm_\pi}{4m^2},\end{aligned}\tag{84}$$

and cancels the second term in Eq. (80) coming from nucleon exchange. There is no such term if the pion-nucleon coupling is pseudovector. In this case one immediately arrives at Eq. (79), with $g_{\pi n}=2mg_{\mathrm{pv}}$, which has essentially the same form as derived in current algebra directly.[22]

The gauge invariance of the photoproduction amplitudes is guaranteed by the usual gauge-invariant construction of the interaction with the photon field. This produces an interaction term in first order from the pseudovector part of the pion-nucleon interaction. This term is needed to cancel terms from the nucleon exchange caused by the pseudovector interaction. These terms sometimes destroy the gauge invariance of the current-algebra results for pion photoproduction.[22]

TABLE II. π-π scattering length.

Parameter	Chiral Lagrangian	Current algebra
a_{00}	$\frac{m_\pi}{4\pi}\frac{g_1^2}{m_\rho^2}\left(1+\frac{29m_\pi^2}{m_\sigma^2}\right)=0.15$	$\frac{m_\pi^5}{4\pi g_\pi^2}=0.18$
a_{10}	0	0
a_{20}	$\frac{m_\pi^3g_1^2}{16\pi m_\rho^2m_\sigma^2}=0.00057$	$-\frac{m_\pi^5}{8\pi g_\pi^2}=-0.088$
a_{11}	$\frac{g_1^2}{24\pi m_\pi m_\rho^2}=0.014$	$\frac{m_\pi^3}{16\pi g_\pi^2}=0.044$

As the last example, we give the results for the threshold behavior for the σ-production amplitude. The T matrix for the process $\pi_\alpha+N\to\sigma+N'$ has the isotopic-spin structure

$$T_\alpha=T\tau_\alpha.$$

The scattering amplitude f for forward production at threshold, which is related to T by

$$f=-[m/4m(m+m_\sigma)]T, \tag{85}$$

is given by nucleon exchange in the s and u channels and by π and a exchange in the t channel. In the forward direction only the spin-nonflip amplitude is nonzero and has, with terms of the order m_π^2/m_σ^2 neglected, the following form:

$$\begin{aligned}f=-\frac{1}{8\pi}\left(\frac{mm_\sigma}{(m+m_\sigma)^3}\right)^{1/2}\Bigg[&fg_{\pi n}\left(\frac{m_V^2}{m_a^2}\frac{m+m_\sigma}{m^2}-\frac{1}{m}\right)\\&-2f_\rho g_1\left(\frac{m_a}{m_V}\frac{(m+m_\sigma)^2}{m_a^2(m+m_\sigma)+m_\sigma^2m}-\frac{m+m_\sigma}{m_am_V}\right)\\&-4f_\rho\hat{g}_1\frac{m_\sigma^4}{m_a^3}\frac{m_Vm(2m+m_\sigma)}{m_a^2(m+m_\sigma)+m_\sigma^2m}\Bigg].\end{aligned}\tag{86}$$

Thus f would be of order m_π^2/m_σ^2 like the isospin-symmetric part of the π-N scattering amplitude, if the expression in large parentheses in Eq. (86) vanishes. Unfortunately we do not know enough about all the coupling constants contained in this equation to solve for m_σ. Indications however are that f_ρ is small compared to $g_{\pi n}$. Neglecting the last two terms in Eq. (86) gives the following equation for m_σ:

$$m_\sigma=[(m_a^2-m_V^2)/m_V^2]m. \tag{87}$$

Unfortunately, there exists no experimental information on σ production near threshold. Since it can be detected only by its 2π decay, it must compete with uncorrelated 2π production and with 2π production coming from the ρ. But the contribution from the ρ^0 in reactions like $\pi^-+p\to\pi^+\pi^-n$ is favored by isospin and angular momentum factors. Therefore, information about the threshold behavior of σ production can be gained only

TABLE III. Threshold electric dipole amplitudes.

Parameter	Chiral Lagrangian	Current algebra	Experiment
$E_{0+}{}^{(0,+)}$	$\dfrac{-eg_{\pi n}}{16\pi}\dfrac{m_\pi}{m^2}=-0.155\times10^{-2}$	$\dfrac{-eg_A m_\pi{}^3}{16\pi m g_\pi}=-0.158\times10^{-2}$	$E_{0+}{}^{(0)}=(-0.105\pm0.024)\times10^{-2}$
			$E_{0+}{}^{(+)}=(-0.104\pm0.057)\times10^{-2}$
$E_{0+}{}^{(-)}$	$\dfrac{eg_{\pi n}}{4\pi m_\pi}\dfrac{m_\pi}{2m}=+2.260\times10^{-2}$	...	$E_{0+}{}^{(-)}=(2.122\pm0.024)\times10^{-2}$

if the behavior of all contributions leading to $\pi^+\pi^-$ within the mass of the $\pi^+\pi^-$ system in the vicinity of the σ mass is calculated and compared to experiment. Another possibility is information about the $n\pi^0\pi^0$ final state where ρ does not contribute. But here one has large contributions from $\Delta(1238)$ production. But it might be interesting to derive the threshold properties for $\pi N\to\rho N$ and $\pi N\to 2\pi N$ with our Lagrangian. For $\pi N\to 2\pi N$ we expect results similar to those of the Weinberg and Schwinger Lagrangians as calculated recently by Olsson and Turner.[23]

V. NUMERICAL RESULTS AND COMPARISON WITH EMPIRICAL DATA

In this section we briefly collect the numerical results for the meson decays and threshold parameters for the various reactions and state the assumptions about the coupling constants that we use as inputs. Furthermore, we compare the results for the low-energy parameters and decay widths with the experimental values and with the current-algebra predictions. As input we use the pion-decay coupling constant g_π, the pion-nucleon coupling constant $g_{\pi n}$, and various masses. The empirical values for these parameters are $g_\pi/m_\pi{}^2=0.10m$, $g_{\pi n}=13.5$ ($g_{\pi n}{}^2/4\pi=14.6$), $m_\rho{}^2=31m_\pi{}^2$, $m_a{}^2=59m_\pi{}^2$, and $m_\pi{}^2=0.019$ GeV^2. According to Eq. (33), the symmetry mass m_V and the ρ mass m_ρ are not equal. The difference is determined by the coupling constant f_1. Information about f_1 could come from relations between π-ρ and π-a scattering or ρ and a production by pions on nucleons, using L_n' defined in Eq. (22). Experiments indicate that even at low energies $\pi N\to\rho N$ is approximately described by one-pion exchange. Therefore, we assume that f_1 is small, and then we have $m_V=m_\rho$. Then we calculate from Eq. (55) values for f and g_1. The result is $f=7.25$, $g_1=5.7$. m_σ is chosen so that Eq. (87) is satisfied. This yields $m_\sigma=0.905m=0.85$ GeV. Actually, there is no information available about the σ production at threshold. On the other hand, the empirical evidence for the existence of the σ referred to in the Introduction is not unambiguous enough to determine the mass of the resonance. But the value we choose for m_σ is consistent with the results of the π-π phase-shift analyses. With these parameters we calculated the π-N, π-π, and photoproduction low-energy parameters from Eqs. (70), (72), and (77). The results together with the experimental data and the current-algebra predictions are exhibited in Tables I–III. In Table III, the formulas have only the leading powers in m_π/m. The complete formulas can be found in the text, but the latter have been used to produce the numbers in Table III. The value for the parameter $a^{(-)}$ essentially agrees with the current-algebra prediction and with experiment. The s-wave scattering length $a^{(+)}$ differs from zero, but is very small. The experimental value is taken from a recent letter of Hamilton[24] in which he discusses possible values for $a^{(-)}$ and $a^{(+)}$ depending on the source of information that one uses. The values quoted in Table I are derived from his "charge-independent" value $a_1-a_3=0.265\pm0.009$ and the π^+p scattering length $a_3=-0.091\pm0.005$. Pure experimental values obtained from $\pi^\pm p$ scattering experiments are $a^{(-)}=0.090\pm0.003$ and $a^{(+)}=0.001\pm0.003$.[24] These values are also consistent with calculations based on forward or partial-wave dispersion relations. Lovelace, for example, quotes $a^{(-)}=0.0886\pm0.0027$ and $a^{(+)}=0.0186\pm0.0036$.[25] Of course, small changes due to unitarity corrections are to be expected for our theoretical values for $a^{(+)}$. Therefore, the "agreement" with the experimental values is satisfactory.

There is no direct experimental information about π-π scattering lengths. With a forward dispersion relation it was found that $a_{11}=0.044$,[26] which agrees better with the current-algebra value than with the prediction from the chiral Lagrangian. We remark that the formulas for the s-wave π-π scattering lengths based on current algebra do not agree with those of other authors,[3] although the signs of the numbers are identical and their orders of magnitude are about the same. Our current-algebra version is equivalent to pure ρ exchange in all three channels.

The experimental numbers for the electric dipole amplitudes are from Adamovich *et al.*[27] They agree reasonably well with the theoretical predictions. Inclusion of ρ- and ω-exchange contributions which are of order $m_\pi/m_\rho{}^2$ might improve the agreement. $E_{0+}{}^{(-)}$ cannot be obtained in current algebra, since the basic

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[27] M. I. Adamovich, V. G. Larionova, A. I. Lebedev, S. P. Kharlamov, and F. R. Yagudina, Yadern. Fiz. **2**, 135 (1965) [English transl.: Soviet J. Nucl. Phys. **2**, 95 (1966)].

formula is not gauge-invariant. Of course, $E_{0+}^{(-)}$ is already given by gauge invariance alone; thus no current-algebra formula is needed in this case.

With the coupling constants and masses given above, we can calculate the decay width of the σ. Using Eq. (51) for the coupling of σ to 2π, we obtain $\Gamma_\sigma = 595$ MeV for a σ mass of 850 MeV. This result is consistent with the experimental information referred to in Refs. 11 and 12. It was pointed out in Sec. III that our Lagrangian approach yields a rather weak correlation between the coupling constants of $\rho \rightarrow \pi\pi$, $a \rightarrow \rho\pi$, and $a \rightarrow \sigma\pi$. The four coupling constants which determine these three decays actually depend on seven parameters. Only if further assumptions are made can two of the decays be calculated. For example, starting from Eqs. (46) and (49), we can assume that ϵ and ϵ' are negligible to all the other constants. Then we obtain from $\Gamma_\rho = 120 \pm 20$ MeV [15] that $|g_{\rho\pi\pi}| = 5.36 \pm 0.46$, and with $g_1 = 5.7$ we compute $m_\rho^2 \hat{g}_1 / g_1 = 0.38 \pm 0.16$, using the first of Eqs. (46). Then h_1, h_2, and the $a\sigma\pi$ coupling are determined, and yield $\Gamma_{a\rho\pi} = 14.5 \pm 2.9$ MeV and $\Gamma_{a\sigma\pi} = 21.8 \pm 5.4$ MeV. Even these small numbers for the A_1 widths are not inconsistent with the experimental data.[15] Another possibility is $\hat{g}_1 = \epsilon' = 0$; then we have $\epsilon = 0.19 \pm 0.08$, and obtain $\Gamma_{a\rho\pi} = 35.4 \pm 9.8$ MeV and $\Gamma_{a\sigma\pi} = 11.4$ MeV.

Finally, we compare our meson-nucleon coupling constants with some of the analyses of low-energy nucleon-nucleon scattering. The three coupling constants determined in our approach are the σ-N coupling constant $f = 7.25$, the ρ-N coupling constant $f_\rho = -2.1$, calculated from Eq. (58), and the A_1-N coupling constant, which is equal to f_ρ. We notice that f_ρ has the opposite sign compared to g_1. Thus ρ universality,[28] which states that $g_1 = 2f_\rho$, is badly violated. Actually, there exist many analyses based on the one-boson-exchange model[14] which do not agree very well with each other, because different unitarization schemes are used for the low partial waves. Therefore, we compare only with some of the recent analyses based on higher partial waves where unitarity is unimportant or on forward dispersion relations, or a combination of both. For example, Köpp and Söding,[29] who combined the dispersion relations for the forward spin-averaged amplitude with an analysis of some higher partial waves, obtain $f^2/4\pi = 3.5$ with $m_\sigma = 700$ MeV and $f_\rho^2/4\pi = 1.2$ compared to our $f^2/4\pi = 4.2$ and $f_\rho^2/4\pi = 0.35$, whereas Bugg,[30] who uses only forward dispersion relations but includes n-p data, claims that $f^2/4\pi = 14.1$ with $m_\sigma = 613$ MeV and $f_\rho^2/4\pi = 1.88$. The difference in $f^2/4\pi$ obtained in Refs. 29 and 30 might come from the fact that in Ref. 29 the authors assume a second pole with a mass of 400 MeV as an approximation of the two-pion continuum in addition to the σ resonance. It appears that further work is necessary before a meaningful comparison with the information coming from nucleon-nucleon scattering can be made.

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