Geometric Formula for Current-Algebra Commutation Relations*

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A purely algebraic formalism is introduced in order to describe the relation between current algebras and Lagrangian field theory. It is then applied to the description in differential-geometric terms of the equal-time commutation relations for currents defined by Lagrangians derived from Riemannian metrics on internal-symmetry spaces.

I. INTRODUCTION

RECENT work by Bardakci, Frishman, and Halpern,[1,2] Sommerfield,[3] and Sugawara[4] has provided models in which the current algebra commutation relations take a very elegant and symmetric form. Another interesting feature of the models is that they seem to be closely related to the geometry of spaces on which the internal symmetries act as transformation groups.

The primary aim of this paper is to bring this connection to the foreground; the result will be formula (4.21), which expresses the commutation relations in terms of the covariant derivative operation for a Riemannian metric on the space on which the group acts. (The metric arises from the choice of Lagrangian describing the theory.) In order to express the ideas in the clearest mathematical form, we also introduce purely algebraic formalism, independent of Hilbert spaces, for describing the connection between Lagrangian quantum field theory and the theory of current algebras.

II. GAUGE-GROUP CONSTRUCTION

First, we present several general algebraic remarks. Let $\mathbf{G}$ be a Lie algebra over the real numbers. Let F be an associative, commutative algebra over the real numbers. Let $\mathbf{G}_F$ be the tensor product $F\otimes\mathbf{G}$, where the tensor product is taken with respect to the real numbers. $\mathbf{G}_F$ can be made into a real Lie algebra in the following way:

$$[f_1\otimes X_1,\, f_2\otimes X_2]=f_1f_2\otimes[X_1,X_2] \quad \text{for} \quad f_1,\, f_2\in\mathbf{G}. \tag{2.1}$$

It is readily verified that (2.1) makes $\mathbf{G}_F$ into a *bona fide* Lie algebra. (The fact that F is a commutative algebra is crucial here.)

Gell-Mann's "current algebras" are of this form. To see this, specialize F to be the space of real-valued C^∞ functions $f(x)$ of a real 3-vector $x=(x_i)$, $1\leqq i,\ j\leqq 3$. Suppose $\mathbf{G}$ is finite dimensional, with basis (X_α), $1\leqq\alpha,\ \beta\leqq m$, and structure constants $c_{\alpha\beta\gamma}$, i.e.,

$$[X_\alpha,X_\beta]=c_{\alpha\beta\gamma}X_\gamma. \tag{2.2}$$

Then, elements X, X' of $\mathbf{G}_F$ are of the form

$$X=f_\alpha\otimes X_\alpha,\quad X'=f_\alpha'\otimes X_\alpha.$$

(2.1) and (2.2) combine to give the relation

$$[X,X']=f_\alpha f_\beta c_{\alpha\beta\gamma}\otimes X_\gamma. \tag{2.3}$$

(2.3) is, of course, the integrated form of the current-algebra commutation relations proposed by Gell-Mann:

$$[X_\alpha(x),X_\alpha(y)]=\delta(x-y)c_{\alpha\beta\gamma}X_\gamma(x). \tag{2.4}$$

Now, (2.4) arises formally by defining

$$X_\alpha(x)=\delta_x\otimes X_\alpha, \tag{2.5}$$

where δ_x is the Dirac delta function concentrated at the point x, i.e.,

$$\delta_x(y)=\delta(x-y).$$

We would like to use (2.5) to define $X_\alpha(x)$ in a precise mathematical way. There are certain difficulties in doing this in a completely rigorous way—connected with the well-known difficulties in defining the product of distributions—but let us proceed as far as we can in this direction.

Let F_C be the algebra (under multiplication) of C^∞, real-valued function $f(x)$ of compact support. Define the space of Schwartz distributions, as usual,[5] denoted by $\mathfrak{D}$. F_C can be considered as a subspace of $\mathfrak{D}$. Distributions can be multiplied by functions, i.e., $\mathfrak{D}$ is a module over the ring F_C. Let F now be the *free* associative, commutative module over F_C. Recall that it is constructed as the space curves of formal products $\delta_1\cdots\delta_n$ of elements of $\mathfrak{D}$, with the only relations those provided by commutativity, associativity and the required module structure over F_C.

We can now use F as defined to construct $\mathbf{G}_F$ as before, as the "current algebra." Then, $X_\alpha(x)$ as defined in (2.5) makes literal sense as an element of $\mathbf{G}_F$. Of course, instead of (2.4), we have the following relations:

$$[X_\alpha(x),X_\beta(y)]=\delta_x\delta_y c_{\alpha\beta\gamma}\otimes X_\gamma(x). \tag{2.6}$$

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[1] K. Bardakci, Y. Frishman, and M. Halpern, Phys. Rev. **170**, 1353 (1968).

[2] K. Bardakci and M. B. Halpern, Phys. Rev. **172**, 1542 (1968).

[3] C. M. Sommerfield, Phys. Rev. (to be published).

[4] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[5] R. F. Streater and A. S. Wightman, *PCT, Spin, and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).

Since they give the same results in terms of test functions, this is not too serious; for example, we can introduce as new relations in F the relations

$$\delta_x\delta_y=\delta(x-y)\delta_x\,.$$

We can also consider elements of the "universal enveloping algebra"[6] of $\mathbf{G}_F$, as integrals of formal products of elements of $\mathbf{G}_F$. For example, if $\mathbf{G}$ is a compact Lie algebra, and if (X_α) is a basis that is orthonormal with respect to the Killing form[6] of $\mathbf{G}$, then

$$\Delta=\int X_\alpha(x)X_\alpha(x)dx$$

is a "Casimir operator" of $\mathbf{G}_F$. Algebraic methods of constructing linear representations of $\mathbf{G}_F$ and "deformations" of the Lie algebra structure relations (2.1), (2.3), (2.4), i.e., "Schwinger terms," have been given in Ref. 7.

Finally, the formalism enables us to define the "partial derivatives" ∂_i of such symbols as $X(x)$. For example, if $x=(x_j)$,

$$\partial_i X((x_j))=\lim_{\epsilon\to 0}\frac{1}{\epsilon}[X((x_j+\epsilon\delta_{ij}))-X((x_j))]\,. \quad (2.7)$$

If we have something of the form $X(x,y)$, $\partial_i{}^x X(x,y)$ denotes this "partial derivative" with respect to x, with y held constant.

III. LIE ALGEBRA OF CANONICAL FIELD THEORY

Choose indices $1\leqq a,\ b\leqq m$. Let $\mathbf{G}$ be the Heisenberg Lie algebra, defined by symbols $(q_a,p_a,1)$, with the Lie algebra structure defined by

$$0=[p_a,p_b]=[q_a,q_b]\,,\quad [p_a,q_b]=\delta_{ab}\,, \quad (3.1)$$

with δ_{ab} the Kronecker delta, i.e., $\delta_{ab}=0$ if $a\neq b$, $=1$ if $a=b$. Define, as before, $\mathbf{G}_{F_C}$, and $\mathbf{G}_F$. Within $\mathbf{G}_F$, we can define

$$p_a(x)\delta_x\otimes p_a\,,\quad q_a(x)=\delta_x\otimes q_b\,, \quad (3.2)$$

and we have, analogously to (2.4), field-theoretic canonical commutation relations

$$[q_a(x),p_b(y)]=\delta_{ab}\delta(x-y)\,. \quad (3.3)$$

As for any Lie algebra, we can define the universal enveloping algebra of $\mathbf{G}_F$ as the formal "products" of elements of $\mathbf{G}_F$. This enables us to make sense of "functions" of the symbols $q_a(x)$, $p_a(x)$. In particular, we can use the formalism of Lagrangians, for example, to construct "currents." One way of doing this can be described as follows. In addition to the variables q_a, introduce variables $q_{a\mu}$ $(0\leqq\mu\leqq 3)$. Let L be a function $f(q_a,q_{b\mu})$ of the indicated variables. Let

$$p_{a\mu}=\partial L/\partial q_{a\mu}\,. \quad (3.4)$$

Let us suppose that relations (3.4) can be solved for q_{a0} as a function of p_{a0}. [The condition for this is, of course, that $\det(\partial^2/\partial q_{a0}\partial q_{ba})\neq 0$.] This enables us to convert any function $f(q_a,q_{b\mu})$ of the indicated variables into a function of the variables q_a, q_{ai}, p_a by identifying p_a with p_{a0}. In turn, this function can be converted into an element of the universal enveloping algebra of $\mathbf{G}_F$ by identifying q_a with $q_a(x)=\delta_x\otimes q_a$, q_{ai} with $\partial_i q_a(x)$, and p_a with $p_a(x)=\delta_x\otimes q_a$. (Of course, we face the usual problem of taking into account the non-commutativity of products in the universal enveloping algebra of $\mathbf{G}_F$.)

For example, let us consider the Lagrangian for free Klein-Gordon particles, namely,

$$L=\tfrac{1}{2}g_{\mu\nu}g_{a\mu}q_{a\nu}+m^2q_aq_a\,, \quad (3.5)$$

where $g_{\mu\nu}$ is the usual Lorentz metric tensor ($g_{\mu\nu}=0$ if $\mu\neq\nu$, 1 if $\mu=\nu=0$, -1 if $1\leqq\mu=\nu\leqq 3$). Then,

$$\partial L/\partial q_{a\mu}=g_{\mu\nu}q_{a\nu}\,;$$

hence,

$$p_a=q_{a0}\,. \quad (3.6)$$

The "energy-momentum tensor" $\theta_{\mu\nu}$ can be constructed as usual:

$$\theta_{\mu\nu}=\tfrac{1}{2}p_{a\mu}q_{a\nu}-g_{\mu\nu}\,. \quad (3.7)$$

The generators of the Lie algebra of the Poincaré group can then be written as follows, as elements of the universal enveloping algebra of $\mathbf{G}_F$:

$$M_{\mu\nu}=\int(x_\mu\theta_{\nu 0}-x_\nu\theta_{\mu 0})d^3x\,;$$
$$P_\mu=\int\theta_{0\mu}d^3x\,. \quad (3.8)$$

[Of course, in order that $M_{\mu\nu}$, P_μ defined by (3.8), really satisfy the commutation relations of the Poincaré group, the Lagrangian L must be Lorentz-invariant.] For example, with relations (3.5) and (3.6), we have

$$P_0=\int\theta_{00}(x)d^3x$$
$$=\int[\tfrac{1}{2}\partial_i q_a(x)\partial_i q_a(x)+m^2q_a(x)q_a(x)]d^3x\,, \quad (3.9)$$

which is, of course, the usual formula for the "energy" for Klein-Gordon free particles.

We now turn to the consideration of more complicated theories of this type mentioned in Refs. 1–4.

[6] R. Hermann, *Lie Groups for Physicists* (W. A. Benjamin, Inc., New York, 1966).

[7] R. Hermann, University of California, Berkeley, report (unpublished).

IV. CURRENTS GENERATED BY GROUPS OF INTERNAL SYMMETRIES

Let (x,q,p) be the variables used in Sec. III. Suppose that **L** is a Lie algebra of vector fields on q space.[8] Each $X\in\mathbf{L}$ can be realized then as a first-order differential operator

$$X\equiv A_a(q)(\partial/\partial q_a). \tag{4.1}$$

Associate with X the classical function

$$f_X=A_a(q)p_a. \tag{4.2}$$

It is then well known from classical mechanics that the assignment $X\rightarrow f_X$ as a homomorphism between **L** (considered as a Lie algebra under Poisson bracket) and **G** the Lie algebra (under Poisson bracket) of functions of p and q. We can also associate to X the following element of the universal enveloping algebra fo $\mathbf{G}_F$:

$$X(x)=A_a(q(x))p_a(x). \tag{4.3}$$

It is readily verified that "current algebra" commutation relations are satisfied:

$$[X(x),X'(y)]=\delta(x-y)[X,X'](x),\quad \text{for } X, X'\in\mathbf{L}. \tag{4.4}$$

Let $L(q_a,q_{b\mu})$ be a Lagrangian as explained in Sec. III. Given $X\in\mathbf{L}$, one can then define the "current" associated with X, as follows:

$$V_\mu{}^X=A_aL_{a\mu},$$

or

$$V_0{}^X(x)=A_a(q(x))p_a(x); \tag{4.5}$$

$$V_i{}^X=A_a(q(x))L_{ai}. \tag{4.6}$$

In view of (4.3), we have

$$V_0{}^X=X(x), \tag{4.7}$$

i.e., the zeroth components of the "current" generated by elements of **L** satisfy the "current algebra" commutation relations (4.4).

Our job is now to compute the commutation relations between $V_\mu{}^X$ and $V_\nu{}^{X'}$ for X, $X'\in\mathbf{L}$. Rather than working in complete generality, let us suppose that L is of the following form:

$$L=\tfrac{1}{2}g_{\mu\nu}g_{ab}(q)\partial_\mu q_a\partial_\nu q_b, \tag{4.8}$$

where $(g_{\mu\nu})$ is the Lorentz metric tensor—constant, of course—and $(g_{ab}(q))$ is a "metric tensor" for the "internal symmetry" q space.

Then,

$$L_{a\mu}=g_{\mu\nu}g_{ab}q_{b\nu};$$

hence,

$$p_a=L_{a0}=g_{ab}q_{b0},$$

or

$$q_{b0}=g^{-1}{}_{ab}p_b,$$

where $(g^{-1}{}_{ab})$ is the inverse matrix to (g_{ab}), i.e.,

$$g^{-1}{}_{ab}g_{bc}=\delta_{ac}.$$

Hence;

$$L_{ai}=-g_{ab}q_{bi},$$

or, following the rules for converting functions of q_a, $q_{b\mu}$ into elements of the universal enveloping algebra of $\mathbf{G}_F$,

$$V_i{}^X(x)=-g_{ab}(q(x))\partial_iq_b(x)A_a(q(x)). \tag{4.9}$$

We see that

$$[V_i{}^X(x),V_j{}^{X'}(y)]=0,\quad \text{for } X, X'\in\mathbf{L}. \tag{4.10}$$

$$\begin{aligned}[V_0{}^X(x),V_i{}^{X'}(y)]
&=[A_ap_a(x),g_{cb}\partial_iq_bA_c'(y)]\\
&=-A_a(q(x))[p_a(x),g_{cb}(q(y))\partial_iq_b(y)A_c'(q(y))]\\
&=-A_a(q(x))\partial_ag_{cb}\partial_iq_bA_c'\delta(x-y)\\
&\quad+g_{cb}(q(y))[p_a(x),\partial_iq_b(y)]A_c'(q(y))\\
&\quad+g_{cb}(q(y))\partial_iq_b(y)\partial_aA_c'(q(y))\delta(x-y).\end{aligned} \tag{4.11}$$

(∂_a applied to a function of q denotes the partial derivative $\partial/\partial q_a$.) Now,

$$\begin{aligned}[p_a(x),\partial_iq_b(y)]&=[\delta_x\otimes p_a,\ \partial_i\delta_y\otimes q_b]\\
&=\delta_x\partial_i\delta_y\delta_{ab}\\
&=\left[f\otimes p_a,\ -\int\partial_if'(y)q_b(y)dy\right]\\
&=-[f\otimes p_a,\ \partial_if'\otimes q_b]\\
&=-f\partial_if'\delta_{ab}.\end{aligned} \tag{4.12}$$

Now,

$$\int f(x)f'(y)\partial_i{}^y\delta(x-y)dxdy=-\int f(x)\partial_if'(y)\delta(x-y)dxdy=-f\partial_if'. \tag{4.13}$$

Hence, (4.12) can be written as

$$[p_a(x),\partial_iq_b(y)]=\partial_i{}^y\delta(x-y)\delta_{ab}. \tag{4.14}$$

Combining (4.11) with (4.14) gives

$$\begin{aligned}[V_0{}^X(x),V_i{}^{X'}(y)]=-A_a(q(x))[&\partial_ag_{cb}\partial_iq_bA_c'(q(y))\\
&\times\delta(x-y)+g_{cb}(q(y))A_c'(q(y))\delta_{ab}\partial_i{}^y\delta(x-y)\\
&+g_{cb}\partial_iq_b\partial_aA_c'(q(y))\delta(x-y)].\end{aligned} \tag{4.15}$$

Now, one sees directly from (4.9) that $V_i{}^X$ and $V_0{}^X$ are independent of the coordinate system (q_a). Hence, the right-hand side of (4.15) should be expressible in terms of invariant quantities. Let

$$\langle X,X'\rangle=g_{ab}A_aA_b'. \tag{4.16}$$

It is the inner product of the vector fields X, X' with

[8] R. Hermann, *Differential Geometry and the Calculus of Variation* (Academic Press Inc., New York, 1968).

respect to the Riemannian metric

$$ds^2=g_{ab}dq_a dq_b. \tag{4.17}$$

Let us fix attention at a point q^0. We can then choose (q_a) as Riemannian normal coordinates at q^0.[8] In these coordinates, we have, at the point q^0,

$$g_{ab}=\delta_{ab};\quad \partial g_{ab}/\partial q_c=0. \tag{4.18}$$

Evaluating the right-hand side of (4.15) at q^0, using relations (4.16) to (4.18), gives

$$\langle X,X'\rangle\partial_i{}^y\delta(x-y)-A_a\partial_a A_c{}'\partial_i q_b\delta(x-y). \tag{4.19}$$

Introduce the covariant derivative operation $(X,X')\rightarrow\nabla_X X'$ for vector fields associated with the metric (4.17). Since (q_a) are normal coordinates at q^0, we have

$$\nabla_X X'=A_a\partial_a(A_b)(\partial/\partial q_b). \tag{4.20}$$

Putting this all together, we have

$$[V_0{}^X(x),V_i{}^{X'}(y)]=\langle X,X'\rangle(q(x))\partial_i{}^x\delta(x-y)+V_i{}^{\nabla_X X'}(q(x))\delta(x-y). \tag{4.21}$$

This is the general formula for the current-algebra commutations relations.

Now, these commutation relations may be considered to be of optimally simple form if the relations (4.21) do not depend on the fields $q(x)$. One can then read off from (4.21) a set of conditions which will give such conditions. Suppose that **L**, **L**′ are Lie algebras of vector fields on q space such that

$$\langle X,X'\rangle=\text{const for } X\in\mathbf{L},\ X'\in\mathbf{L}'. \tag{4.22}$$

For $X\in\mathbf{L}$, $X'\in\mathbf{L}'$, $\nabla_X X'$ is a linear combination of

elements of **L**′ with constant coefficients. (4.23)

It is then an exercise in differential geometry to work out the conditions on the metric $g_{ab}dq_a dq_b$ which are implied by (4.22) and (4.23). This will be done in a later work. For the moment, it suffices to mention the most obvious solution of (4.22) and (4.23). Suppose (q_a) are variables on a compact Lie group L, with the metric chosen as the unique one invariant under left and right translation, and with $\mathbf{L}=\mathbf{L}'=$Lie algebra of left-invariant vector fields on L.[8,9] This leads [e.g., for $L=SU(2)$, $SU(3)$, or $SU(3)\times SU(3)$] to the situations considered in Refs. 1–4.

We can also readily work out the energy-momentum tensor $\theta_{\mu\nu}$. For example, consider the energy operator

$$\begin{aligned}\theta_{00}&=\tfrac{1}{2}p_{a0}q_{a0}-L\\&=\tfrac{1}{2}p_a q_{a0}-L\\&=\tfrac{1}{2}p_a g_{ab}{}^{-1}p_b-\tfrac{1}{2}g_{ab}g_{ac}{}^{-1}g_{bd}{}^{-1}p_c p_d+\tfrac{1}{2}g_{ab}\partial_i q_b\\&=\tfrac{1}{2}g_{ab}\partial_i q_a\partial_i q_b.\end{aligned} \tag{4.24}$$

Suppose that **L** is a Lie algebra of vector fields on q space, with $X^\alpha=A_a{}^\alpha(\partial/\partial q_a)$, $1\leqq\alpha,\ \beta\leqq n$, a basis of **L**. Using (4.9), let us postulate a relation of the form

$$\theta_{00}=h_{\alpha\beta}V_i{}^\alpha V_i{}^\beta=h_{\alpha\beta}g_{ac}\partial_i q_a A_c{}^\alpha g_{bd}\partial_i q_b A_d{}^\beta. \tag{4.25}$$

Comparing (4.24) and (4.25) gives the relation

$$h_{\alpha\beta}g_{ac}A_c{}^\alpha g_{bd}A_d{}^\beta=\tfrac{1}{2}g_{ab}. \tag{4.26}$$

Again, let us work in normal coordinates at a point. (4.25) takes the form

$$\tfrac{1}{2}\delta_{ab}=h_{\alpha\beta}A_a{}^\alpha A_b{}^\beta. \tag{4.27}$$

(4.25) can be written in coordinate free form as follows:

$$\tfrac{1}{2}\langle X,X\rangle=h_{\alpha\beta}\langle X,X^\alpha\rangle\langle X^\beta,X\rangle \quad \text{for each vector field } X \text{ on } q \text{ space}. \tag{4.28}$$

Again, solutions of this form impose conditions on the metric that will be investigated systematically in a later paper. One may readily verify that, in case the metric is the biinvariant one on L, and in case (X^α) is a basis for **L** (the Lie algebra of left-invariant vector fields on L) that is orthonormal with respect to the Killing form,[8,9] that relation (4.27) holds, with

$$h_{\alpha\beta}=\tfrac{1}{2}\delta_{\alpha\beta}.$$

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[9] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press Inc., New York, 1962).