Boson Mass Spectrum and Pion Form Factor from $SO(5,2)$. Generalization to $SO(m,2)$*

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The $SO(5,2)$ representation whose basis functions are the solutions of the Bethe-Salpeter equation for two scalar quarks interacting via the exchange of a scalar zero-mass boson to form a bound state of zero mass is written as an infinite-dimensional representation for bosons. A general second-order, infinite-component wave equation is given with a reasonable mass spectrum. The ground-state form factor corresponding to this representation is calculated by identifying the electromagnetic current with the current of the equation, in which case the pion form factor is given by $G(t)=[1-b(\cosh\theta)^2t/2m]/[1-(\cosh\theta)^2t/4m^2]^{5/2}$. The mass spectrum and the form factor can easily be generalized to the group $SO(m,2)$ by a simple substitution, and the mass spectra and form factors obtained before in the framework of the group $SO(4,2)$ appear as special cases.

I. INTRODUCTION

In a previous paper[1] we have considered the Bethe-Salpeter (BS) equation for two scalar quarks interacting via the exchange of a scalar boson of zero mass. It was found that if the mass of the bound state is zero, the set of all solutions of the equation can be accommodated into a single irreducible representation of the noncompact group $SO(5,2)$, while if the mass of the bound state is different from zero, the $SO(5,2)$ representation splits as $\sum \oplus SO(4,2)$. Also, the $SO(5,2)$ representation was written in the canonical basis and its commuting generators were identified with the third component of the isospin, the hypercharge, and a third quantum number. From its I and Y content we find that this representation, as well as the corresponding H-atom representation when written in the canonical basis, does not seem to be realized in nature as internal space representations. In the present case we want to use the $SO(5,2)$ representation completely within the ordinary space time, i.e., we take its basic functions to be eigenfunctions of spin and parity and assign bosons to it. This is a more natural way of looking at the representation, since its basic functions are the wave functions of two spinless quarks which can have nonvanishing relative angular momentum forming a bound state with spin. Of course, the $SO(5,2)$ corresponds to the unrealistic case of massless bound state and we do not expect to have an exact $SO(5,2)$. The ratio of the mass of the bound state to the mass of the quark may be taken as an indication of how "badly" our $SO(5,2)$ is broken. Several experiments[2] have indicated that this ratio is small, and so it seems to be not unreasonable to consider an $SO(5,2)$ group for the mesons.

In Sec. II we give the generators and the Hilbert space of the $SO(m,2)$ representation with the m-dimensional spherical harmonics as basic functions in terms of creation and annihilation operators. For $m=5$, eigenfunctions of spin and parity were obtained to be identified with the wave functions of the particles which belong to the representation. In Sec. III we obtain from a Lagrangian by means of the Euler-Lagrange equation an infinite-component wave equation with a reasonable mass spectrum, i.e., not monotonically decreasing as in the Majorana[3] equation. The requirement that the electromagnetic interactions can be brought in by the minimal substitution uniquely determines the electromagnetic current to be used in the calculation of the form factors. In Sec. IV we calculate the ground-state form factor, which will be identified with the pion form factor, using the electromagnetic current that we obtain from the equation of Sec. III. The mass spectrum and the form factor that we have obtained, as well as other mass spectra and wave equations obtained in the framework of the group $SO(4,2)$, can be generalized to the group $SO(m,2)$.

We should mention that the form factor and the mass spectrum, which were obtained by generalizing to $SO(5,2)$ methods applied to the $SO(4,2)$ group,[4–6] depend only on the infinite-component wave equation that we use, and not on the BS equation. The BS equation was introduced simply because it leads to the same $SO(5,2)$ representation.

II. $SO(5,2)$ REPRESENTATION

We have shown in a previous paper[1] that the set of all m-dimensional spherical harmonics $Y_{i_1\cdots i_t}{}^t$ forms the basis for a single irreducible representation of the group $SO(m,2)$. The harmonic polynomials $F_{i_1\cdots i_t}{}^t$ are expressed in terms of the spherical harmonics by the relation

$$F_{i_1\cdots i_t}{}^t=z^tY_{i_1\cdots i_t}{}^t. \tag{2.1}$$

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[6] A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968).

In a specific basis the set of all linearly independent m-dimensional harmonic polynomials was found to be the following:

$$\begin{aligned}
F_{i_1\cdots i_{2t}}{}^{2t}=&\,m(m+2)\cdots(m+4t-4)(z_1\cdots z_{i_{2t}})-m(m+2)\cdots(m+4t-6)z^2\Sigma(\delta_{i_1i_2}z_{i_3}\cdots z_{i_{2t}})\\
&+\cdots+(-1)^{t-1}m(m+2)\cdots(m+2t-2)z^{2t-2}\Sigma(\delta_{i_1i_2}\cdots\delta_{i_{2t-3}i_{2t-2}}z_{i_{2t-1}}z_{i_{2t}})\\
&+(-1)^t m(m+2)\cdots(m+2t-4)z^{2t}\Sigma(\delta_{i_1i_2}\cdots\delta_{i_{2t-1}i_{2t}})\,,\quad t>1\\
F_{i_1\cdots i_{2t+1}}{}^{2t+1}=&\,m(m+2)\cdots(m+4t-2)(z_1\cdots z_{i_{2t+1}})-m(m+2)\cdots(m+4t-4)z^2\Sigma(\delta_{i_1i_2}z_{i_3}\cdots z_{i_{2t+1}})+\cdots\\
&+(-1)^{t-1}m(m+2)\cdots(m+2t)z^{2t-2}\Sigma(\delta_{i_1i_2}\cdots\delta_{i_{2t-3}i_{2t-2}}z_{i_{2t-1}}z_{i_{2t}}z_{i_{2t+1}})\\
&+(-1)^t m(m+2)\cdots(m+2t-2)z^{2t}\Sigma(\delta_{i_1i_2}\cdots\delta_{i_{2t-1}i_{2t}}z_{i_{2t+1}})\,,\quad t\geqslant 1 \qquad (2.2)\\
F_{i_1i_2}{}^2=&\,mz_{i_1}z_{i_2}-z^2\delta_{i_1i_2}\,,\quad F_i{}^1=z_i\,,\quad F^0=1/(m-2)\,,
\end{aligned}$$

where Σ is an operator which symmetrizes the indices $i_1\cdots i_t$, keeping only distinct permutations of the indices, i.e., the terms $\delta_{i_1i_2}\delta_{i_3i_4}\cdots$ and $\delta_{i_2i_1}\delta_{i_3i_4}\cdots$ are counted as one term. Also, the z_i's are the components of an m-dimensional vector and δ_{ij}, i, $j=1, 2, \cdots, m$, is the Kronecker tensor. The generators $L_{\mu\nu}$, μ, $\nu=1, 2, \cdots, m+2$, of the group $SO(m,2)$ are given by

$$\begin{aligned}
L_{ij}&=M_{ij}\,,\\
L_{i,m+1}&=\tfrac{1}{2}(iV_i-P_i)\,,\\
L_{i,m+2}&=\tfrac{1}{2}(iP_i-V_i)\,, \qquad (2.3)\\
L_{m+1,m+2}&=S\,,\\
L_{\mu\nu}&=-L_{\nu\mu}\,,\qquad i,j=1,2,\ldots,m
\end{aligned}$$

where

$$\begin{aligned}
M_{ij}&=-i\left(z_i\frac{\partial}{\partial z_j}-z_j\frac{\partial}{\partial z_i}\right),\\
S&=z_i\frac{\partial}{\partial z_i}+\tfrac{1}{2}(m-2)\,,\\
P_i&=-i\frac{\partial}{\partial z_i}, \qquad (2.4)\\
V_i&=z_j{}^2\frac{\partial}{\partial z_i}-2z_iz_j\frac{\partial}{\partial z_j}-(m-2)z_i=z_j{}^2\frac{\partial}{\partial z_i}-2z_iS\,.
\end{aligned}$$

In the above expressions summation is assumed over repeated indices with metric δ_{ij}. The operators $L_{\mu\nu}$ satisfy the relations

$$[L_{\mu\nu},L_{\rho\sigma}]=i(g_{\nu\rho}L_{\mu\sigma}+g_{\mu\sigma}L_{\nu\rho}-g_{\mu\rho}L_{\nu\sigma}-g_{\nu\sigma}L_{\mu\rho})\,, \qquad (2.5)$$

with the metric

$$g_{11}=g_{22}=\cdots=g_{mm}=-1\,,\; g_{m+1,m+1}=g_{m+2,m+2}=1\,, \qquad (2.6)$$

so that the operators $L_{\mu\nu}$ form the Lie algebra of the group $SO(m,2)$. Also, they transform the harmonic polynomials among themselves in an irreducible fashion. Indeed, we find

$$\begin{aligned}
M_{kl}F_{i_1\cdots i_t}{}^t&=i(\delta_{ki_n}F_{li_1\cdots i_{n-1}i_{n+1}\cdots i_t}{}^t\\
&\qquad-\delta_{li_n}F_{ki_1\cdots i_{n-1}i_{n+1}\cdots i}{}^t)\,, \qquad (2.7a)\\
SF_{i_1\cdots i_t}{}^t&=[t+\tfrac{1}{2}(m-2)]F_{i_1\cdots i_t}\,, \qquad (2.7b)\\
P_lF_{i_1\cdots i_t}{}^t&=-i[(m+2t-4)\delta_{li_n}F_{i_1\cdots i_{n-1}i_{n+1}\cdots i_t}{}^{t-1}\\
&\qquad-2\delta_{i_ni_k}F_{li_1\cdots i_{n-1}i_{n+1}\cdots i_{k-1}i_{k+1}\cdots i_t}{}^{t-1}]\,, \qquad (2.7c)\\
V_lF_{i_1\cdots i_t}{}^t&=-F_{li_1\cdots i_t}{}^{t+1}\,, \qquad (2.7d)
\end{aligned}$$

where the indices i_n and i_k take all values $i_1\cdots i_t$, i.e., summation is understood, with the restriction that in the second term on the right-hand side of Eq. (2.7c) we have $n<k$.

The importance of the above representation lies in the fact that it is realized by several physical systems and their generalization to higher dimensions.[1] For example, solutions of the "Schrödinger" equation of the $m-1$ dimensional analog of the H atom, and more generally of the "Schrödinger" equation with $O(m)$ invariant potentials, are the m-dimensional spherical harmonics forming a representation of the group $SO(m,2)$. The same thing happens for the m-dimensional rigid rotator. In this section we shall be interested in the $SO(5,2)$ representation which is realized by the BS equation as argued in the Introduction. The group $SO(5)$ has been considered before[7] as an internal-symmetry group for hadrons. In our case, however, we shall use an $SO(5)$ representation which can be extended to $SO(5,2)$ completely within the ordinary space time.

Let us introduce the creation and annihilation operators $a_i{}^\dagger$, a_i, $i=1, 2, \cdots, m$, which satisfy the commutation relations

$$[a_i,a_j{}^\dagger]=\delta_{ij}\,,\quad [a_i,a_j]=[a_i{}^\dagger,a_j{}^\dagger]=0\,.$$

We can express the generators of the group $SO(m,2)$ given by Eqs. (2.3) and (2.4) in terms of the operators $a_i{}^\dagger$ and a_i if we make the replacement

$$z_i\to a_i{}^\dagger\,,\quad \partial/\partial z_i\to a_i\,. \qquad (2.8)$$

[7] (a) A. Salam and T. C. Ward, Nuovo Cimento **20**, 1228 (1961) [observe that $S_p(2)\approx O(5)$]; (b) R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).

We find

$$
\begin{aligned}
L_{ij}&=-i(a_i^\dagger a_j-a_j^\dagger a_i)\,,\\
L_{i,m+1}&=\tfrac{1}{2}i[a_j^\dagger a_j^\dagger a_i-2a_i^\dagger a_j^\dagger a_j-(m-2)a_i^\dagger+a_i]\,,\\
L_{i,m+2}&=-\tfrac{1}{2}[a_j^\dagger a_j^\dagger a_i-2a_i^\dagger a_j^\dagger a_j-(m-2)a_i^\dagger-a_i]\,,\\
L_{m+1,m+2}&=a_j^\dagger a_j+\tfrac{1}{2}(m-2)\,.
\end{aligned}
\tag{2.9}
$$

The above operators satisfy Eqs. (2.5) with the metric of Eqs. (2.6). The replacement $z_i\to a_i^\dagger$ translates the space of the harmonic polynomials $F_{i_1\cdots i_t}{}^t$ of Eqs. (2.2) into a Hilbert space if we assume that the operators that we obtain by this translation operate on the vacuum state $|0\rangle$. We have, for example,

$$F_{i_1i_2}{}^2\to(ma_{i_1}{}^\dagger a_{i_2}{}^\dagger-a_i^\dagger a_j^\dagger\delta_{i_1i_2})|0\rangle\,. \tag{2.10}$$

The exact definition of the scalar product which is based on the relation $\langle a_i|a_j\rangle=\delta_{ij}$ will be given in Sec. III.

Consider the $SO(5,2)$ group. We usually label the basic vectors of a representation by the weights which in the case of $SO(5)$ are the eigenvalues of the operators[1] $H_1=(1/\sqrt{5})L_{12}$ and $H_2=(1/\sqrt{5})L_{34}$. Instead, we shall use here the eigenvalues of the operators

$$
\begin{aligned}
M_3&=\tfrac{1}{2}(L_{12}+L_{34})\,,\\
\Lambda_3&=\tfrac{1}{2}(L_{12}-L_{34})\,.
\end{aligned}
\tag{2.11}
$$

However, within an irreducible representation, in addition to the weights we need[8] $\frac{1}{2}(r-3l)$ more numbers, where r is the number of generators of the group and l is the rank of the group. In the present case the two additional numbers[9] are chosen to be the eigenvalues of the $SO(4)$ commuting spins $\mathbf{M}=(M_1,M_2,M_3)$ and $\mathbf{\Lambda}=(\Lambda_1,\Lambda_2,\Lambda_3)$, where

$$
\begin{aligned}
M_1&=\tfrac{1}{2}(L_{23}+L_{14})\,,\quad M_2=\tfrac{1}{2}(L_{31}+L_{24})\,,\\
\Lambda_1&=\tfrac{1}{2}(L_{23}-L_{14})\,,\quad \Lambda_2=\tfrac{1}{2}(L_{31}-L_{24})\,.
\end{aligned}
\tag{2.12}
$$

Our Hilbert space is the space of the five-dimensional spherical harmonics in the basis in which the operators M^2, M_3, Λ^2, and Λ_3 are diagonal. We find

$$
\begin{aligned}
|t;M,\Lambda,M_3=\pm M,\Lambda_3=\pm M\rangle&=(1\pm i2)^{2M}55\cdots5\,,\\
0\leqslant M\leqslant\tfrac{1}{2}t\,,\quad M&=\Lambda\,,
\end{aligned}
\tag{2.13}
$$

where there are $(t-2M)$ 5 indices. The index t, which is the degree of homogeneity in the $a^\dagger$'s, distinguishes the different irreducible representations. In the above notation only the coefficients and the tensor indices of the tensor terms of $|t;M,\Lambda,\pm M,\pm M\rangle$ are shown. For example,

$$|3;1111\rangle=F_{115}{}^3\pm2iF_{125}{}^3-F_{225}{}^3\,,$$

where the F^3's have been given before. The states with $M_3\neq\pm M$, $\Lambda_3\neq\pm M$ are obtained from the above states by application of the rising or lowering operators

$$
\begin{aligned}
M_\pm&=\tfrac{1}{2}(L_{23}+L_{14}\pm iL_{31}\pm iL_{24})\,,\\
\Lambda_\pm&=\tfrac{1}{2}(L_{23}-L_{14}\pm iL_{31}\mp iL_{24})\,.
\end{aligned}
\tag{2.14}
$$

We combine the commuting operators $\mathbf{M}$ and $\mathbf{\Lambda}$ to obtain the spin $\mathbf{J}$:

$$\mathbf{J}=\mathbf{M}+\mathbf{\Lambda}\,, \tag{2.15}$$

which will be identified with the spin of the particle. The states $|t,J,J_3\rangle$ are given by

$$
\begin{aligned}
|t;M,\Lambda,J,J_3\rangle\equiv|t,J,J_3\rangle&=(-1)^{-J_3}\sum_{M_3,\Lambda_3}(2J+1)^{1/2}\\
&\times\begin{pmatrix}M&\Lambda&J\\M_3&\Lambda_3&-J_3\end{pmatrix}|t;M,\Lambda,M_3,\Lambda_3\rangle\,.
\end{aligned}
\tag{2.16}
$$

Since $M=\Lambda$, only integral spins can be obtained, and so only bosons can be assigned to the representation.

We can define parity in the following two ways:

$$\text{(a)}\ \ a_i\to-a_i\,,\quad a_5\to a_5\,,\quad i=1,\cdots,4 \tag{2.17a}$$

$$\text{(b)}\ \ a_i\to-a_i\,,\ a_4\to a_4\,,\ a_5\to a_5\,,\ i=1,\cdots,3\,. \tag{2.17b}$$

The second definition of parity may be more appropriate in view of the fact that in treating the BS equation we have made the identification[1]

$$z_i=\frac{2p_5{}^2p_i}{\mathbf{p}^2+p_5{}^2}\,,\quad z_5=\frac{(-\mathbf{p}^2+p_5{}^2)p_5}{\mathbf{p}^2+p_5{}^2}\,,\quad i=1,\ldots,4 \tag{2.18}$$

where p_i is an energy-momentum 4-vector and p_5 is the mass of the quark. Under parity the generators of the group $SO(5,2)$ behave as follows $(i,j=1,2,3)$:

	L_{ij}	L_{i4}	L_{i5}	L_{45}	L_{67}	L_{i6}	L_{i7}	L_{46}	L_{47}	L_{56}	L_{57}
(a)	+	+	−	−	+	−	−	−	−	+	+
(b)	+	−	−	+	+	−	−	+	+	+	+

If the parity P is defined as in (2.17a), we get

$$PM_\pm P^{-1}=M_\pm\,,\quad P\Lambda_\pm P^{-1}=\Lambda_\pm\,.$$

Then from Eqs. (2.13) and (2.16) we find that the states $|t;M,\Lambda,J,J_3\rangle$ are eigenstates of the parity with eigenvalue

$$P=(-1)^{2M}\epsilon_{\text{int}}\,,$$

where ϵ_{int} is the intrinsic parity of the vacuum. So we find the following spin parity content of the representation for $\epsilon_{\text{int}}=-1$:

$$
\begin{aligned}
&t=0\,,\quad J^P=0^-;\\
&t=1\,,\quad J^P=0^-,\quad(0^+,1^+);\\
&t=2\,,\quad J^P=0^-,\quad(0^+,1^+),\quad(0^-,1^-,2^-);
\end{aligned}
$$

etc. To find the parity of the states in case (b), one observes that

$$PM_\pm P^{-1}=\Lambda_\pm\,.$$

Then from the symmetry properties of the $3-j$ symbols and the fact that $M=\Lambda$, we find from Eqs.

[8] G. Racah, CERN Report No. CERN-61-68, 1961 (unpublished).

[9] We recall that an $O(5)$ spherical harmonic is labeled by four numbers, i.e., as Y_{Nnlm}.

(2.13) and (2.16) that the states $|t;M,\Lambda,J,J_3\rangle$ are again eigenstates of the parity with eigenvalue

$$P=(-1)^J\epsilon_{\text{int}}.$$

Thus the spin-parity content of the representation is the following for $\epsilon_{\text{int}}=-1$:

$$\begin{aligned}&t=0, \quad J^P=0^-;\\ &t=1, \quad J^P=0^-, \quad (0^-,1^+);\\ &t=2, \quad J^P=0^-, \quad (0^-,1^+), \quad (0^-,1^+,2^-);\end{aligned}$$

etc. The choice (2.17b) for parity implies that the 1^- mesons do not exist in the same $SO(5,2)$ representation with the 0^- mesons, and so we must consider a parity doubling.[10]

For both definitions of the parity we may choose as Lorentz group one of the following subgroups:

$$\begin{aligned}&\text{(i)} \quad (1,2,3,6),\\ &\text{(ii)} \quad (1,2,3,7),\end{aligned}$$

i.e., the subgroup with generators $L_{\mu\nu}$, μ, $\nu=1$, 2, 3, 6 in case (i) and μ, $\nu=1$, 2, 3, 7 in case (ii). In case (i) above we make the following identification:

L_{i6}, $i=1$, 2, 3—Lorentz boosters,

$\Gamma_\mu=(L_{67},L_{i7})$, $\Gamma_\mu'=(L_{65},L_{i5})$—Lorentz 4-vectors,

L_{56}, L_{57}—scalars under parity, and under the $O(3)$ subgroup of the Lorentz group which may be used as mixing operators.

In case (II) we have

L_{i7}, $i=1$, 2, 3—Lorentz boosters,

$\Gamma_\mu=(L_{76},L_{i6})$, $\Gamma_\mu'=(L_{75},L_{i5})$—Lorentz vector,

L_{56}, L_{57}—mixing operators.

The above identification is valid if parity is defined as in (2.17a) or as in (2.17b). However, if parity is defined as in (2.17b), we have the additional mixing operators L_{46} and L_{47}.

III. MASS SPECTRUM

To proceed with the calculation of physical quantities, we must define a scalar product with respect to which the generators are Hermitian. We introduce the kernel $\lambda(S)$ which is a function of the diagonal generator $S=L_{m+1,m+2}$ and we require $L_{\mu\nu}$ to be Hermitian with respect to the scalar product $\langle F'|\lambda(S)L_{\mu\nu}|F\rangle$, i.e., we require

$$\langle F'|L_{\mu\nu}^\dagger\lambda(S)|F\rangle=\langle F'|\lambda(S)L_{\mu\nu}|F\rangle. \tag{3.1}$$

We find that the operators L_{ij}, i, $j=1$, 2, $\cdots$, m, and S are Hermitian. Also, since the functions $F_{i_1\cdots i_t}{}^t$ are harmonic polynomials, we have

$$a_ia_i|F\rangle=\langle F|a_i^\dagger a_i^\dagger=0.$$

Using the above equation we find that Hermiticity of the operators $L_{i,m+1}$ and $L_{i,m+2}$ implies

$$\lambda(S)=\frac{2^{(m-2)/2}\Gamma(\frac{1}{2}m-1)}{2^S\Gamma(S)}, \tag{3.2}$$

which was normalized such that $\lambda(S)|0\rangle=|0\rangle$. The operator $\lambda(S)$ serves as a metric which makes the operators $L_{i,m+1}$ and $L_{i,m+2}$ Hermitian.

We shall now return to the group $SO(5,2)$. Let us choose the Lorentz subgroup (1,2,3,6). The mass spectrum of the particles will be obtained from the solution (diagonalization) of a wave equation. Consider a wave equation of the form

$$(L_\sigma p^\sigma-\mu L_{57}-\nu)\tilde{\Psi}=H\tilde{\Psi}=0, \tag{3.3}$$

where L_σ is a Lorentz 4-vector and L_{57} is a scalar under $SO(3,1)$ and parity, which is necessary in order to be able to diagonalize the equation in the rest frame. Other terms of the form $\tau L_{\rho\sigma}$ may be added if they are scalars under $SO(3,1)$ and parity. Equation (3.3) gives rise to the conserved current $\tilde{\Psi}^\dagger\lambda L_\sigma\tilde{\Psi}$, where λ is the operator of Eq. (3.2). Let us assume that Eq. (3.3) can be obtained from a Lagrangian $\mathcal{L}$ by means of the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial(\tilde{\Psi}^\dagger\lambda)}-\frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial(\partial\tilde{\Psi}^\dagger\lambda/\partial x_\mu)}=0, \tag{3.4}$$

and that the electromagnetic (EM) interactions can be brought in by the minimal substitution $p^\sigma\to p^\sigma-eA^\sigma$. Since the interaction Lagrangian is of the form

$$\mathcal{L}_{\text{EM}}=-A^\sigma J_\sigma, \tag{3.5}$$

where J_σ is the electromagnetic current, the above assumptions imply that the electromagnetic current to be used in the calculation of form factors is the current of the equation multiplied by the charge, i.e.,

$$J_\sigma=e\tilde{\Psi}^\dagger\lambda L_\sigma\tilde{\Psi}. \tag{3.6}$$

Current conservation has been applied before to fix the mass spectrum.[5,6] We prefer to obtain the mass spectrum by diagonalizing the wave equation, since its solutions satisfy the current-conservation equation, while the current-conservation equation does not involve the operators $H-L_\sigma p^\sigma$ of Eq. (3.3), and for each one of its solutions we must find a wave equation which gives the same mass spectrum (if this is possible). It is much simpler to diagonalize a wave equation than to solve the current-conservation equation.

[10] Barut (Ref. 5) has found that the states of the meson tower of $SO(4,2)$ are (0^-), $(0^-,1^+)$, $(0^-,1^+,2^-)$, $\cdots$. Thus a parity doubling is necessary to include the vector mesons.

Consider the Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}a\tilde{\Psi}^\dagger\lambda i\overleftrightarrow{\partial}^\sigma\Gamma_\sigma\tilde{\Psi}+b(\partial^\sigma\tilde{\Psi}^\dagger)\lambda\partial_\sigma\tilde{\Psi}+c(\partial^\sigma\tilde{\Psi}^\dagger)\lambda L_{57}(\partial_\sigma\tilde{\Psi})-\mu\tilde{\Psi}^\dagger\lambda L_{57}\tilde{\Psi}-\nu\tilde{\Psi}^\dagger\lambda\tilde{\Psi}+\mathcal{L}_{\mathrm{EM}}\,, \quad (3.7)$$

$$\mathcal{L}_{\mathrm{EM}}=-eA^\sigma(a\tilde{\Psi}^\dagger\lambda\Gamma_\sigma\tilde{\Psi}-\tfrac{1}{2}b\tilde{\Psi}^\dagger\lambda i\overleftrightarrow{\partial}_\sigma\tilde{\Psi}-\tfrac{1}{2}c\tilde{\Psi}^\dagger\lambda i\overleftrightarrow{\partial}_\sigma L_{57}\tilde{\Psi})\,.$$

From $\mathcal{L}-\mathcal{L}_{\mathrm{EM}}$ we obtain the equation

$$[(a\Gamma_\sigma+bp_\sigma+cp_\sigma L_{57})p^\sigma-\mu L_{57}-\nu]\tilde{\Psi}=0\,. \quad (3.8)$$

Assuming that p^σ is a timelike 4-vector and going to the rest system, we can diagonalize the above equation by the rotation $e^{i\theta_t L_{56}}$. We find

$$am_t\tanh\theta_t+cm_t^2=\mu\,,\qquad (am_t/\cosh\theta_t)(t+\tfrac{3}{2})+bm_t^2=\nu\,, \quad (3.9)$$

from which we obtain the mixing angle θ_t:

$$\sinh\theta_t=\frac{\mu-cm_t^2}{\nu-bm_t^2}(t+\tfrac{3}{2})\,, \quad (3.10)$$

and if b and c are not both zero, the mass spectrum

$$m_t^2=\frac{1}{2}\left(c^2+\frac{b^2}{(t+\frac{3}{2})^2}\right)^{-1}\left\{a^2+2c\mu+\frac{2b\nu}{(t+\frac{3}{2})^2}\pm\left[\left(a^2+2c\mu+\frac{2b\nu}{(t+\frac{3}{2})^2}\right)^2-4\left(c^2+\frac{b^2}{(t+\frac{3}{2})^2}\right)\times\left(\mu^2+\frac{\nu^2}{(t+\frac{3}{2})^2}\right)\right]^{1/2}\right\}. \quad (3.11)$$

For $b=c=0$ we get the mass spectrum

$$m_t^2=\frac{1}{a^2}\left(\mu^2+\frac{\nu^2}{(t+\frac{3}{2})^2}\right). \quad (3.12)$$

Suppose that we have written the $SO(m,2)$ representation in such a way that linear combinations of its basis functions have definite spin and parity, i.e., particles can be assigned to the representation. Let us write the equation

$$[(a\Gamma_\sigma+bp_\sigma+cp_\sigma L_{m,m+2})p^\sigma-\mu L_{m,m+2}-\nu]\tilde{\Psi}=0\,,$$

where $\Gamma_\sigma=(L_{m+1,m+2},L_{i,m+2})$, $i=1,\ \ldots,\ 3$. The mass spectrum that we obtain if we diagonalize this equation is given by Eq. (3.11), in which we have made the substitution

$$t+\tfrac{3}{2}\to t+\tfrac{1}{2}(m-2)\,. \quad (3.13)$$

For $m=4$ we obtain the mass spectrum of Ref. 5. All mass spectra obtained before for the group $SO(4,2)$[4,11] can be generalized to the group $SO(m,2)$ with the substitution[12]

$$n\to t+\tfrac{1}{2}(m-2)\,. \quad (3.14)$$

For $t\to\infty$ we obtain for all $SO(m,2)$ groups from Eqs. (3.11) and (3.13) the saturation mass

$$m_\infty=[a^2+2c\mu\pm a(a^2+4c\mu)^{1/2}]/2c^2\,.$$

The existence of a saturation mass for bosons justifies somehow, according to our arguments in the Introduction, the use of an $SO(5,2)$ representation, if we assume that the saturation mass is much smaller than the mass of the quarks.

Equation (3.8) has, in addition, spacelike solutions that are obtained by diagonalizing it under the assumption that p^σ is spacelike. Equations without spacelike solutions[4b,11] can be written using generators of the group $SO(m,2)$ in a similar fashion.

We may proceed to assign the mesons with the same internal quantum numbers to an $SO(5,2)$ representation, determining at the same time the constants of Eq. (3.11) in such a way that their mass is approximately given by this expression. Having assigned the particles to the representation, we may calculate decay rates. Since, however, the number of known mesons is relatively small, we shall postpone such an assignment.

IV. GROUND-STATE FORM FACTOR

Let us identify the electromagnetic current L_μ with

$$L_\mu=e(a\Gamma_\mu-bi\overleftrightarrow{\partial}_\mu-ci\overleftrightarrow{\partial}_\mu L_{57})\,, \quad (4.1)$$

where $\Gamma_\mu=(L_{67},L_{i7})$, the Lorentz boosters with L_{i6}, and the mixing operator with L_{56}. We observe that Eq. (3.8) gives rise to the above electromagnetic current. This equation can be diagonalized by the rotation $e^{i\theta L_{56}}$, and since we want to identify the solutions of the equation with the physical states, we take the mixing operator to be L_{56}. Then the form factor $G(t)$ of the ground state corresponding to a spin-zero particle is given in terms of the vertex function J_μ by

$$J_\mu=\langle 0|e^{-i\theta L_{56}\dagger}\lambda(S)L_\mu e^{-i\xi L_{36}}e^{i\theta L_{56}}|0\rangle=G(t)(p_f+p_i)_\mu/2m\,, \quad (4.2)$$

$$\sinh\xi=|\mathbf{p}|/m\,,\quad \cosh\xi=E/m\,, \quad (4.3)$$

where one of the particles is in its rest frame and the other is boosted in the z direction to momentum p_i. We find, commuting $e^{-iL_{56}}$ and $\lambda(S)$, that

$$J_\mu=\langle 0|e^{-i\theta L_{56}}L_\mu e^{-i\xi L_{36}}e^{i\theta L_{56}}|0\rangle\,. \quad (4.4)$$

In order to find $G(t)$, we shall calculate one of the components of J_μ, say, J_0. Details of this calculation, which is based on a method developed by Nambu,[4b]

[11] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).

[12] The index n used in Refs. 4–6 is the eigenvalue of the operator L_{56} of the group $SO(4,2)$, so that in our notation n becomes $t+1$.

are given in the Appendix. We find

$$J_0=e\{\tfrac{3}{2}a\cosh\theta+2bm[1+(\sinh\tfrac{1}{2}\xi\cosh\theta)^2]+3cm\sinh\theta\}\times\frac{(\cosh\frac{1}{2}\xi)^2}{[1+(\cosh\theta\sinh\frac{1}{2}\xi)^2]^{5/2}}. \quad (4.5)$$

Also, we have

$$\frac{E+m}{2m}=1-\frac{t}{4m^2}=(\cosh\tfrac{1}{2}\xi)^2,\qquad (\sinh\tfrac{1}{2}\xi)^2=-\frac{t}{4m^2}. \quad (4.6)$$

From Eqs. (4.2), (4.5), and (4.6) we get

$$G(t)=e\frac{\frac{3}{2}a\cosh\theta+2bm+3cm\sinh\theta-b(\cosh\theta)^2t/2m}{[1-(\cosh\theta)^2t/4m^2]^{5/2}}. \quad (4.7)$$

Charge normalization implies

$$G(0)=e(\tfrac{3}{2}a\cosh\theta+2bm+3cm\sinh\theta)=e, \quad (4.8)$$

and so our final expression for the ground-state form factor obtained from the current of Eq. (4.1) is

$$G(t)=e\frac{1-b(\cosh\theta)^2t/2m}{[1-(\cosh\theta)^2t/4m^2]^{5/2}}. \quad (4.9)$$

We observe that we get the same from factor if we use the current $L_\mu'=e(a\Gamma_\mu-bi\overleftrightarrow{\partial}_\mu)$, while the mass spectrum of Eq. (3.11) does depend on the constant c. The above result can easily be generalized to the group $SO(m,2)$. If we consider the electromagnetic current of Eq. (4.1) with $\Gamma_\mu=(L_{m+1,m+2},L_{i,m+2})$, the mixing operators $L_{m,m+1}$, and the Lorentz boosters $L_{i,m+1}$, and proceed as before, we find the same expression for the ground-state form factor except that the exponent in the denominator is now $\frac{1}{2}m$. The ground-state form factor of the meson representation has been calculated by Barut[5] in the framework of the group $SO(4,2)$. The expression of the form factor that he found has the exponent 2 instead of $\frac{5}{2}$ in the denominator.

We shall identify the $G(t)$ of Eq. (4.9) with the form factor of the pion. This form factor has been measured[13] recently in the electroproduction reaction $e+p\to e+n+\pi^+$ for three values of the momentum transfer: -0.039, -0.117, and -0.234 $(\mathrm{BeV}/c)^2$. Its value at -0.234 $(\mathrm{BeV}/c)^2$ seems to deviate significantly from the prediction of the vector-meson-dominance model. However, the data and the theory on which the calculation was based are not accurate enough to permit definite conclusions.[14] The evidence rather favors a

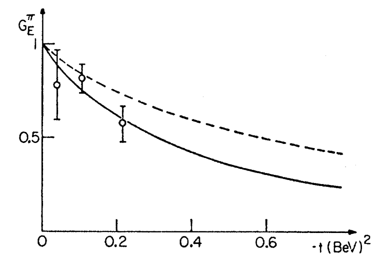

FIG. 1. Calculated form factor of the pion for $[(\cosh\theta)/2m]^2=1.5$ BeV^{-2}, $mb=\frac{1}{2}$ (solid line), and comparison with the form factor obtained from vector-meson dominance, $G_\pi(t)=(1-t/m_\rho^2)^{-1}$ (dashed line), and the experimental points.

pion form factor decreasing more rapidly, as $-t$ increases, than the one predicted by the vector-meson-dominance model, and not very different from the $G_E{}^p$ of the proton. The lack of experimental information does not allow us to fix the constants b and θ in Eq. (4.9). However, a form factor decreasing more rapidly than the one predicted by the vector-meson-dominance model can be easily obtained, as we see in Fig. 1, where we have chosen $[(\cosh\theta)/2m]^2=1.5$ BeV^{-2} and $mb=\frac{1}{2}$.

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APPENDIX: CALCULATION OF FORM FACTOR

We want to calculate J_0 of Eq. (4.4). Since

$$e^{-i\theta L_{56}}[aL_{67}+b(m+E)+c(m+E)L_{57}]e^{i\theta L_{56}}=[a\cosh\theta+2cm\sinh\theta\,(\cosh\tfrac{1}{2}\xi)^2]L_{67}+2bm(\cosh\tfrac{1}{2}\xi)^2+[\alpha\sinh\theta+2cm(\cosh\tfrac{1}{2}\xi)^2\cosh\theta]L_{57}, \quad (A1)$$

we get

$$J_0=e[\tfrac{3}{2}a\cosh\theta+2bm(\cosh\tfrac{1}{2}\xi)^2+3cm(\cosh\tfrac{1}{2}\xi)^2\sinh\theta]\langle 0|N(\xi,\theta)|0\rangle+[a\sinh\theta+2cm(\cosh\tfrac{1}{2}\xi)^2\cosh\theta]\times\langle 0|L_{57}N(\xi,\theta)|0\rangle, \quad (A2)$$

where

$$N(\xi,\theta)=e^{-i\theta L_{56}}e^{-i\xi L_{36}}e^{i\theta L_{56}}=e^{-i\xi(L_{36}\cosh\theta+L_{35}\sinh\theta)}. \quad (A3)$$

To evaluate the above expression consider the operators

$$G_{12}=\tfrac{1}{2}(L_{35}+L_{67}),\qquad G_{23}=\tfrac{1}{2}(L_{36}+L_{57}),$$
$$G_{31}=\tfrac{1}{2}(-L_{37}+L_{56}),\qquad G_{12}'=\tfrac{1}{2}(L_{35}-L_{67}), \quad (A4)$$
$$G_{23}'=\tfrac{1}{2}(L_{36}-L_{57}),\qquad G_{31}'=\tfrac{1}{2}(L_{37}+L_{56}),$$

which form an $O(2,1)\otimes O(2,1)$ algebra $[g_{\mu\nu}=(--+)]$. We get

$$L_{35}=G_{12}+G_{12}',\quad L_{36}=G_{23}+G_{23}', \quad (A5)$$

and so $N(\xi,\theta)$ becomes

$$N(\xi,\theta)=e^{-i\xi(\cosh\theta\,G_{23}+\sinh\theta\,G_{12})}\times e^{-i\xi(\cosh\theta\,G_{23}'+\sinh\theta\,G_{12}')}. \quad (A6)$$

[13] C. W. Akerlof, W. W. Ash, K. Berkelman, C. A. Lichtenstein, A. Ramanauskas, and R. H. Siemann, Phys. Rev. **163**, 1482 (1967).

[14] S. D. Drell and D. J. Silverman, Phys. Rev. Letters **20**, 1325 (1968).

Each factor of the above product is an $O(2,1)$ element and can be written[15] in the form

$$e^{-i\beta M_{12}}e^{-i\gamma M_{31}}e^{-i\delta M_{12}}, \qquad \text{(A7)}$$

where M_{12} and M_{31} are $O(2,1)$ generators.

To calculate the coefficients β, γ, and δ we shall use the spinor representation of $O(2,1)$ whose generators are

$$M_{12}=\tfrac{1}{2}\sigma_3\,,\quad M_{23}=\tfrac{1}{2}i\sigma_1\,,\quad M_{31}=\tfrac{1}{2}i\sigma_2. \qquad \text{(A8)}$$

Substituting (A8) in the first factor of the expression (A6) and in (A7), we calculate the Euler angles β, γ, and δ, in terms of ξ and θ. We find

$$\begin{aligned}\sinh\tfrac{1}{2}\gamma&=\pm\sinh\tfrac{1}{2}\xi\cosh\theta\,,\\ \cos\beta&=\pm(\sinh\tfrac{1}{2}\xi\sinh\theta)/\cosh\tfrac{1}{2}\gamma\,,\\ \sin\beta&=\mp(\cosh\tfrac{1}{2}\xi)/\cosh\tfrac{1}{2}\gamma\,,\\ \sin\delta&=-\sin\beta\,,\quad \cos\delta=-\cos\beta\,.\end{aligned} \qquad \text{(A9)}$$

We find the same angles β, γ, and δ for the second factor of the expression (A6). So we have

$$N(\xi,\theta)=e^{-i\beta L_{35}}e^{-i\gamma L_{56}}e^{-i\delta L_{35}}. \qquad \text{(A10)}$$

Also, we find

$$L_{57}e^{-i\beta L_{35}}=e^{-i\beta L_{35}}(L_{57}\cos\beta+L_{37}\sin\beta). \qquad \text{(A11)}$$

From Eqs. (2.3), (2.7), (2.9), (A10), and (A11) we get

$$\begin{aligned}\langle 0|N(\xi,\theta)|0\rangle&=\langle 0|e^{-i\gamma L_{56}}|0\rangle\,,\\ \langle 0|L_{57}N(\xi,\theta)|0\rangle&=\tfrac{1}{2}\langle a_5|e^{-i\gamma L_{56}}|0\rangle.\end{aligned} \qquad \text{(A12)}$$

We shall calculate $\langle 0|e^{-i\gamma L_{56}}|0\rangle$ and $\langle a_5|e^{-i\gamma L_{56}}|0\rangle$ using a method developed by Nambu.[16] From

$$\langle 0|L_{56}e^{-i\gamma L_{56}}|0\rangle=\langle 0|e^{-i\gamma L_{56}}L_{56}|0\rangle$$

we get, using Eqs. (2.9),

$$\langle a_5|e^{-i\gamma L_{56}}|0\rangle=-3\langle 0|e^{-i\gamma L_{56}}|a_5\rangle. \qquad \text{(A13)}$$

Also, from

$$\langle 0|L_{57}e^{-i\gamma L_{56}}|0\rangle=\langle 0|e^{-i\gamma L_{56}}(\cosh\gamma\, L_{57}-\sinh\gamma\, L_{67})|0\rangle$$

we get

$$\langle a_5|e^{-i\gamma L_{56}}|0\rangle=3\cosh\gamma\,\langle 0|e^{-i\gamma L_{56}}|a_5\rangle -3\sinh\gamma\,\langle 0|e^{-i\gamma L_{56}}|0\rangle. \qquad \text{(A14)}$$

[15] V. Bargmann, Ann. Math. **48**, 598 (1947), p. 595.

[16] Reference 4(b), Appendix B.

From Eqs. (A13) and (A14) we get

$$\langle a_5|e^{-i\gamma L_{56}}|0\rangle=-3\tanh\tfrac{1}{2}\gamma\,\langle 0|e^{-i\gamma L_{56}}|0\rangle. \qquad \text{(A15)}$$

We have

$$\begin{aligned}\langle 0|e^{-i\gamma L_{56}}|0\rangle&=\langle 0|\exp[-\tfrac{1}{2}\gamma(a_5^{\dagger 2}a_5+3a_5^{\dagger}-a_5)]|0\rangle\\ &=\exp\left[-\tfrac{1}{2}\gamma\left(z_5^2\frac{\partial}{\partial z_5}-\frac{\partial}{\partial z_5}+3z_5\right)\right]1|_{z_5=0}.\end{aligned} \qquad \text{(A16)}$$

Introducing the variable p,

$$p=\tfrac{1}{2}\ln\frac{1-z_5}{1+z_5},\quad p|_{z_5=0}=0, \qquad \text{(A17)}$$

we find that the exponent of Eq. (A16) becomes

$$\begin{aligned}z_5^2\frac{\partial}{\partial z_5}-\frac{\partial}{\partial z_5}+3z_5&=\frac{\partial}{\partial p}-3\tanh p\\ &=e^{3\ln(\cosh p)}\frac{\partial}{\partial p}e^{-3\ln(\cosh p)}.\end{aligned} \qquad \text{(A18)}$$

Since any function f of the operator $U^{-1}OU$ can be written in the form $f(U^{-1}OU)=U^{-1}f(O)U$, we get from Eqs. (A16) and (A18)

$$\begin{aligned}&\exp\left[-\tfrac{1}{2}\gamma\left(z_5^2\frac{\partial}{\partial z_5}-\frac{\partial}{\partial z_5}+3z_5\right)\right]1|_{z_5=0}\\ &=\exp\left[-\tfrac{1}{2}\gamma\left(e^{3\ln(\cosh p)}\frac{\partial}{\partial p}e^{-3\ln(\cosh p)}\right)\right]|_{p=0}\\ &=e^{3\ln(\cosh p)}e^{-\frac{1}{2}\gamma\partial/\partial p}e^{-3\ln(\cosh p)}=e^{3\ln(\cosh p)}\\ &\quad\times e^{-3\ln[\cosh(p-\frac{1}{2}\gamma)]}|_{p=0}=(\cosh\tfrac{1}{2}\gamma)^{-3}.\end{aligned} \qquad \text{(A19)}$$

From Eqs. (A15), (A16), and (A19) we find

$$\langle 0|e^{-i\gamma L_{56}}|0\rangle=(\cosh\tfrac{1}{2}\gamma)^{-3}, \qquad \text{(A20)}$$

$$\langle a_5|e^{-i\gamma L_{56}}|0\rangle=-3(\sinh\tfrac{1}{2}\gamma)/(\cosh\tfrac{1}{2}\gamma)^4. \qquad \text{(A21)}$$

From the above calculation we see that the exponent of $\cosh\frac{1}{2}\gamma$ in Eq. (A20) becomes $-m+2$ for the group $SO(m,2)$. From Eqs. (A2), (A9), (A12), (A20), and (A21) we find Eq. (4.5).