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Field-Current Identities, Broken Symmetries, Current-Mixing Model, and the Algebra of Gauge Fields*

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It is shown in general that the algebra of the massive gauge fields can recover the current-algebra relations only for the current-mixing model, and how the possible modifications of the spectral-function sum rules can be made within the framework of the gauge-field theory. It is also shown that the Oakes-Sakurai generalization of the first sum rule is a natural consequence of the current-mixing model.

RECENTLY, we have shown that[1] a modified form of the Weinberg[2] second sum rule in accordance with the Okubo ansatz[3] of symmetry breaking,

$$\int \rho_{\alpha\beta}{}^{(1)}(m^2)dm^2 = A\delta_{\alpha\beta} + Bd_{\alpha\beta 8}, \tag{1}$$

yields several physically interesting results when it is applied to the $U(3)\times U(3)$ symmetry together with the original form of the first sum rule,

$$\int [m^{-2}\rho_{\alpha\beta}{}^{(1)}(m^2) + \rho_{\alpha\beta}{}^{(0)}(m^2)]dm^2 = s\delta_{\alpha\beta}. \tag{2}$$

Here $\rho_{\alpha\beta}{}^{(1)}$ and $\rho_{\alpha\beta}{}^{(0)}$ are the spin-1 and spin-0 spectral functions of the currents $J_\mu{}^\alpha$. More recently Kimel[4] has argued that a gauge-field model, in which the bare masses satisfy a relation of the Gell-Mann-Okubo type so as to warrant (1), does not retain the relations of the current algebra.

In the present note, we would like to show in general that the algebra of gauge fields[5] can recover the current-algebra relations only for the current-mixing model,[6] and in particular how one can construct a gauge-field model which results in (1) and (2) and retains the current-algebra relations at the same time. This is done by considering a generalization of the massive Yang-Mills theory,[7] in which the symmetry-breaking terms are introduced both in the mass and the kinetic terms of the Lagrangian density in quite a general form. This consideration enables us to understand in general what restrictions are to be imposed on the manner of the symmetry-breaking in gauge-field models which guarantee the current-algebra relations. It is felt that the present discussions will be of some use in justifying a certain successful modification of the spectral sum rules in the sense that the algebra of gauge fields is generally believed to be the only theory which explicitly gives equal c-number Schwinger terms for both kinds of currents, provided that we are justified in neglecting a certain product of currents in taking the vacuum expectation.[8] Among other results, we will see from our discussions that (2) or the Oakes-Sakurai extension[9] of (2) to the (8+1) currents, i.e., $s_8\delta_{\alpha\beta}+s_0\delta_{\alpha 0}\delta_{\beta 0}$ instead of $s\delta_{\alpha\beta}$, is a natural consequence of the current-mixing model.

Let us start with the Lagrangian density for a set of massive gauge fields $\phi_\mu{}^\alpha$, α being an internal group index, coupled to the general matter fields represented by ψ,

$$\mathfrak{L} = -\tfrac{1}{4}F_{\mu\nu}{}^\alpha Z_{\alpha\beta}F_{\mu\nu}{}^\beta - \tfrac{1}{2}\phi_\mu{}^\alpha (M^2)_{\alpha\beta}\phi_\mu{}^\beta + \mathfrak{L}_m(\psi, D_\nu\psi), \tag{3}$$

with

$$F_{\mu\nu}{}^\alpha = \partial_\mu\phi_\nu{}^\alpha - \partial_\nu\phi_\mu{}^\alpha - g_0 f_{\alpha\beta\gamma}\phi_\mu{}^\beta\phi_\nu{}^\gamma, \tag{4}$$

$$D_\nu\psi = \partial_\nu\psi + ig_0 T^\alpha\phi_\nu{}^\alpha\psi, \tag{5}$$

where $f_{\alpha\beta\gamma}$ are the internal group-structure constants and T^α is the matrix representation of its Hermitian generators of ψ. The symmetry-breaking is introduced in $Z_{\alpha\beta}$ and $(M^2)_{\alpha\beta}$, both of which are either arbitrary sets of constants or of scalar functions of ψ and $D_\nu\psi$ but symmetric in α and β and positive definite so as to be invertible. Adding a term like $M_{\mu\nu}{}^\alpha F_{\mu\nu}{}^\alpha$, $M_{\mu\nu}{}^\alpha$ being an arbitrary set of tensor functions of ψ only, does not change any of our conclusions. In particular, the case

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[1] Tomoya Akiba and Kyungsik Kang, Phys. Rev. **172**, 1551 (1968).

[2] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[3] S. Okubo, Phys. Rev. Letters **15**, 165 (1963). See, also, S. Okubo in *Proceedings of 1967 International Conference on Particles and Fields, Rochester, New York, 1967*, edited by C. R. Hagen *et al.* (Wiley-Interscience, Inc., New York, 1967).

[4] I. Kimel, Phys. Rev. Letters **21**, 177 (1968).

[5] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[6] S. Coleman and H. J. Schnitzer, Phys. Rev. **134**, B863 (1964); N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[7] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[8] Recently an interesting suggestion has been made on this point by J. D. Bjorken and R. Brandt, Phys. Rev. (to be published). They propose the minimal algebra of currents according to which the J^2 terms can be made absent from the equal-time commutators such as (14) by taking a formal limit $g_0\rightarrow 0$, $(M^2)_{\alpha\beta}\rightarrow 0$ such that $(M^2)_{\alpha\beta}/g_0 \rightarrow$ finite constant in the massive Yang-Mills theory. In this case, the c-number Schwinger term in (14) is possibly finite while that in (13) is infinite. However, the α, β dependence of these terms are known and our discussion still holds. We thank Dr. R. Brandt for showing us their results prior to publication.

[9] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters **19**, 1266 (1967).

$(M^2)_{\alpha\beta}=m_0{}^2\delta_{\alpha\beta}$ reduces to the situation discussed by Lee and Zumino.[10]

The field-current identity in the present generalized gauge-field theory is given by

$$J_\mu{}^\alpha=-(1/g_0)(M^2)_{\alpha\beta}\phi_\mu{}^\beta. \tag{6}$$

The equations of motion may be written as

$$Z_{\alpha\beta}\partial_\mu F_{\mu\nu}{}^\beta-(M^2)_{\alpha\beta}\phi_\nu{}^\beta=g_0 s_\nu{}^\alpha, \tag{7}$$

where

$$s_\nu{}^\alpha=f_{\alpha\beta\gamma}Z_{\beta\delta}F_{\mu\nu}{}^\delta\phi_\mu{}^\gamma-i\frac{\delta\mathfrak{L}_m}{\delta D_\nu\psi}T^\alpha\psi. \tag{8}$$

From (6) and (7), it follows that

$$\partial_\mu J_\mu{}^\alpha=\partial_\mu s_\mu{}^\alpha, \tag{9}$$

so that $J_\mu{}^\alpha$ is conserved when $s_\mu{}^\alpha$ is. The canonical momenta conjugate to $\phi_j{}^\alpha$ and ψ are given by

$$\pi_j{}^\alpha=-Z_{\alpha\beta}F_{0j}{}^\beta$$

and (10)

$$\pi_\psi=\delta\mathfrak{L}_m/\delta D_0\psi,$$

respectively. Then one can show from the canonical commutation relations for $\phi_j{}^\alpha$ and ψ that

$$[s_0{}^\alpha(\mathbf{x},t),s_0{}^\beta(\mathbf{x}',t)]=if_{\alpha\beta\gamma}s_0{}^\gamma(\mathbf{x},t)\delta^3(\mathbf{x}-\mathbf{x}'). \tag{11}$$

In addition, the following equal-time commutation relations can be easily verified by making use of (6)–(11):

$$[J_0{}^\alpha(\mathbf{x},t),J_0{}^\beta(\mathbf{x}',t)]=if_{\alpha\beta\gamma}J_0{}^\gamma(\mathbf{x},t)\delta^3(\mathbf{x}-\mathbf{x}') \tag{12}$$

$$\begin{aligned}&[J_0{}^\alpha(\mathbf{x},t),J_j{}^\beta(\mathbf{x}',t)]\\&=ig_0{}^{-2}(M^2)_{\alpha\beta}\partial_j\delta^3(\mathbf{x}-\mathbf{x}')-if_{abc}(M^2)_{\beta b}(M^2)^{-1}{}_{c\gamma}\\&\qquad\times J_j{}^\gamma(\mathbf{x},t)\delta^3(\mathbf{x}-\mathbf{x}'),\end{aligned} \tag{13}$$

$$\begin{aligned}&[\partial_0J_j{}^\alpha(\mathbf{x},t)-\partial_jJ_0{}^\alpha(\mathbf{x},t),J_k{}^\beta(\mathbf{x}',t)]\\&=-ig_0{}^{-2}(M^2Z^{-1}M^2)_{\alpha\beta}\delta_{jk}\delta^3(\mathbf{x}-\mathbf{x}')\\&\quad+if_{\alpha\beta b}(M^2)_{\alpha a}(M^2)^{-1}{}_{b\gamma}J_j{}^\gamma(\mathbf{x},t)\partial_k\delta^3(\mathbf{x}-\mathbf{x}')\\&\quad-ig_0{}^2f_{abc}f_{def}(M^2)_{\alpha a}(M^2)^{-1}{}_{bd}(M^2)^{-1}{}_{c\delta}(M^2)_{\beta e}\\&\qquad\times(M^2)^{-1}{}_{\gamma f}J_k{}^\gamma(\mathbf{x},t)J_j{}^\delta(\mathbf{x},t)\delta^3(\mathbf{x}-\mathbf{x}').\end{aligned} \tag{14}$$

Now we can show that the usual relations of the current algebra can be recovered from (12) and (13) *only* for the current-mixing model. Here, by the current-mixing model we mean a diagonal M^2, while Z can be, in general, nondiagonal so that the eigenvalues for the bare mass squared are not necessarily common for all $\phi_\mu{}^\alpha$. To see that within the framework of the present generalization of the algebra of gauge fields, the current-mixing model is the only one to retain the current-algebra relations, consider, for example, $\alpha=6$ and $\beta=7$ in (13) and

$$(M^2)_{\alpha\beta}=m\delta_{\alpha\beta}+m_1\delta_{\alpha0}\delta_{\beta0}+m_2(\delta_{\alpha0}\delta_{\beta8}+\delta_{\alpha8}\delta_{\beta0}), \tag{15}$$

which is a typical case of the mass-mixing model.[11] Then one can easily verify that so long as $m_2\neq0$, (13) does not reduce to the usual current-algebra relation between $J_0{}^6$ and $J_j{}^7$ unless $(M^2)_{30}$ is nonzero. But $(M^2)_{30}\neq0$ is inconsistent with (15). Thus we must have $m_2=0$ and consequently only the current-mixing model is compatible with the current-algebra relations.

The most important aspect of (13) and (14) may be that the Schwinger terms are explicitly known c numbers whose precise values are given to the extent that M^2 and Z are known in a given gauge-field model. Following the usual argument[5] of taking the vacuum expectation values of (13) and (14), we find that

$$\int[m^{-2}\rho_{\alpha\beta}{}^{(1)}(m^2)+\rho_{\alpha\beta}{}^{(0)}(m^2)]dm^2=\frac{1}{g_0{}^2}(M^2)_{\alpha\beta} \tag{16}$$

and

$$\int\rho_{\alpha\beta}{}^{(1)}(m^2)dm^2=\frac{1}{g_0{}^2}(M^2Z^{-1}M^2)_{\alpha\beta}, \tag{17}$$

assuming, as usual,[4,5] that we are permitted to neglect an ambiguous term[8] arising from the vacuum expectation value of the last term in (14).

It is then clear from (16) that (2) can be realized for the current-mixing model in which $(M^2)_{\alpha\beta}=m_0{}^2\delta_{\alpha\beta}$. In this case, (17) reduces to

$$\int\rho_{\alpha\beta}{}^{(1)}(m^2)dm^2=\frac{m_0{}^4}{g_0{}^2}(Z^{-1})_{\alpha\beta}, \tag{18}$$

which still leaves a great deal of arbitrariness in the symmetry-breaking interactions. In particular, the modified second rule (1) can be derived from a gauge-field model in which the symmetry-breaking interactions are specified by $(Z^{-1})_{\alpha\beta}=\delta_{\alpha\beta}+ad_{\alpha\beta8}$ together with $(M^2)_{\alpha\beta}=m_0{}^2\delta_{\alpha\beta}$ in the Lagrangian density. We have shown in Ref. 1 that (1) and (2) give the $SU(6)$ mass relation and mixing angles for the vector mesons and fairly rigorous estimate of the decay constants of the pseudoscalar mesons. Another interesting scheme of symmetry breaking with current-mixing is the gauge-field model in which $Z_{\alpha\beta}=\delta_{\alpha\beta}+a\,d_{\alpha\beta8}$ with $(M^2)_{\alpha\beta}=m_0{}^2\delta_{\alpha\beta}$. This model gives a mass relation of the Gell-Mann–Okubo type for the inverse mass squared of the vector mesons.

It should also be noted from (16) that the Oakes-Sakurai extension of the first sum rule to the (8+1) currents is a natural consequence of the current-mixing model in which $(M^2)_{\alpha\beta}=m_0{}^2\delta_{\alpha\beta}+m_1{}^2\delta_{\alpha0}\delta_{\beta0}$. In this case,

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[10] T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).

[11] S. Okubo, Phys. Rev. Letters **5**, 165 (1963); S. L. Glashow, Phys. Rev. Letters **11**, 48 (1963); J. J. Sakurai, Phys. Rev. **132**, 434 (1963). See also, Kroll, Lee, and Zumino in Ref. 6.

(17) becomes

$$\int \rho_{\alpha\beta}^{(1)}(m^2)dm^2 = \frac{m_0^4}{g_0^2}(Z^{-1})_{\alpha\beta} + \frac{m_0^2 m_1^2}{g_0^2}[\delta_{\alpha 0}(Z^{-1})_{\beta 0}+\delta_{\beta 0}(Z^{-1})_{\alpha 0}] + \frac{m_0^4}{g_0^2}\delta_{\alpha 0}\delta_{\beta 0}(Z^{-1})_{\alpha\beta}, \quad (19)$$

which, in particular, for a model with $(Z^{-1})_{\alpha\beta}=\delta_{\alpha\beta}+b\delta_{\alpha 0}\delta_{\beta 0}$ gives

$$\int \rho_{\alpha\beta}^{(1)}(m^2)dm^2 = s_2\delta_{\alpha\beta}+s_2'\delta_{\alpha 0}\delta_{\beta 0},$$

and for

$$(Z^{-1})_{\alpha\beta}=\delta_{\alpha\beta}+b\delta_{\alpha 0}\delta_{\beta 0}+c(\delta_{\alpha 0}\delta_{\beta 8}+\delta_{\alpha 8}\delta_{\beta 0}) \quad (20)$$

gives

$$\int \rho_{\alpha\beta}^{(1)}(m^2)dm^2 = A_2\delta_{\alpha\beta}+A_2'\delta_{\alpha 0}\delta_{\beta 0} + A_2''(\delta_{\alpha 0}\delta_{\beta 8}+\delta_{\beta 0}\delta_{\alpha 8}). \quad (21)$$

Both (20) and (21) are not very interesting because the former yields the degenerate masses and the latter gives $m_\rho^2=m_{K^*}^2$ when they are applied to the nonet vector mesons together with the first sum rule and the usual saturation assumption. It is also possible to derive (1) from (19).

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Note added in proof. Kimel[4] has shown that the particular Lagrangian model corresponding to the special case, $(M^2)_{\alpha\beta}=m_0^2\delta_{\alpha\beta}(1+D'd_{8\alpha\beta})$ and $Z_{\alpha\beta}=\delta_{\alpha\beta}$ in Eq. (3), does not retain the usual current-algebra relations for the once-integrated commutators $[Q^\alpha, J_i^\beta(x)]$. What we have shown here is that any gauge-field model of Eq. (3) with mass-mixing in the terminology of Kroll, Lee, and Zumino[6] does not recover the relations of current algebra for the current-current commutators Eq. (13). Moreover our proof has an advantage over the discussions given in Ref. 4 as the necessity of disappearance of the off-diagonal term, i.e., $m_2=0$ in (15), would directly imply that Weinberg's first sum rule should be given by that of Oakes and Sakurai. We do not therefore need a separate proof for this latter point unlike the situation in Ref. 4 where the Jacobi identity $[J_0^0(x),[Q^6,J_i^7(y)]]$ and the commutativity of the charge Q^6 and the unitary singlet current density J_0^0 are used to show it. We note from (12) that the time-time commutation relations obey the usual current-algebra relations whether or not the symmetry breaking is introduced in either the mass or the kinetic term in (3). A similar point has also been noticed by Sugawara[12] who has proposed to test a theory of $SU(3)\times SU(3)$ currents with the symmetry breaking given by $(M^2)_{\alpha\beta}=m_0^2(\delta_{\alpha\beta}+\epsilon d_{\alpha\beta 8})$, another example of a mass-mixing model, via Weinberg's first sum rule.

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[12] H. Sugawara, Phys. Rev. Letters **21**, 772 (1968); Weinberg's first sum rule for the mass-mixing model has also been considered by D. P. Majumdar, Nuovo Cimento **57A**, 170 (1968).