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Axial-Vector Vertex in Spinor Electrodynamics

STEPHEN L. ADLER
Institute for Advanced Study, Princeton, New Jersey 08540

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Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to $\nu_l l$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved. In an Appendix we demonstrate the uniqueness of the triangle diagrams, and discuss a possible connection between our results and the $\pi^0\rightarrow 2\gamma$ and $\eta\rightarrow 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector-current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

INTRODUCTION

THE axial-vector vertex in spinor electrodynamics is of interest because of its connections (i) with radiative corrections to $\nu_l l$ scattering and (ii) with the γ_5 invariance of massless electrodynamics. We will show in this paper, within the framework of perturbation theory, that the axial-vector vertex has anomalous properties which disagree with those found by the formal manipulation of field equations. In particular, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to $\nu_l l$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved.

In Sec. I we derive the usual formulas for the axial-vector divergence and Ward identity, and then show how they are modified by the presence of triangle diagrams. In Sec. II we discuss various consequences of the additional term found in Sec. I. In the Appendix we show that it is *not* possible to redefine the triangle diagram in a physically acceptable way so as to eliminate the anomalous behavior discussed in Secs. I and II. We also discuss in the Appendix a possible connection between our results and the $\pi^0\rightarrow 2\gamma$ and $\eta\rightarrow 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

I. AXIAL CURRENT DIVERGENCE AND WARD IDENTITY

We work in the usual spinor electrodynamics, described by the Lagrangian density[1]

$$\mathcal{L}(x)=\bar{\psi}(x)(i\gamma\cdot\Box-m_0)\psi(x)-\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)-:e_0\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x):\,,\quad(1)$$

$$F_{\mu\nu}(x)\equiv\frac{\partial A_\mu(x)}{\partial x^\nu}-\frac{\partial A_\nu(x)}{\partial x^\mu}\,,\quad \gamma\cdot\Box\equiv\gamma^\mu\frac{\partial}{\partial x^\mu}\,.$$

We define the axial-vector current $j_\mu{}^5(x)$ and the pseudoscalar density $j^5(x)$ by

$$\begin{aligned} j_\mu{}^5(x)&=:\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x):\,,\\ j^5(x)&=:\bar{\psi}(x)\gamma_5\psi(x):\,;\end{aligned}\quad(2)$$

the corresponding vertex parts $\Gamma_\mu{}^5(p,p')$ and $\Gamma^5(p,p')$ are defined by

$$\begin{aligned} &S_F{}'(p)\Gamma_\mu{}^5(p,p')S_F{}'(p')\\ &\quad=-\int d^4xd^4y\,e^{ip\cdot x}e^{-ip'\cdot y}\langle T(\psi(x)j_\mu{}^5(0)\bar{\psi}(y))\rangle_0\,,\\ &S_F{}'(p)\Gamma^5(p,p')S_F{}'(p')\\ &\quad=-\int d^4xd^4y\,e^{ip\cdot x}e^{-ip'\cdot y}\langle T(\psi(x)j^5(0)\bar{\psi}(y))\rangle_0\,.\end{aligned}\quad(3)$$

Using the equations of motion which follow from Eq. (1), the divergence of the axial-vector current may

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[1] We use the notation and metric conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965), pp. 377–390. Note that $\epsilon_{0123}=-\epsilon^{0123}=1$.

easily be calculated to be

$$\frac{\partial}{\partial x_\mu} j_\mu{}^5(x) = 2im_0 j^5(x). \qquad (4)$$

From Eqs. (3) and (4), we obtain the usual axial-vector Ward identity

$$(p-p')^\mu \Gamma_\mu{}^5(p,p') = 2m_0 \Gamma^5(p,p') + S_F'(p)^{-1}\gamma_5 + \gamma_5 S_F'(p')^{-1}. \qquad (5)$$

Our task in this section is to see whether Eqs. (4) and (5), which we have formally derived from the field equations, actually hold in perturbation theory.

To this end, let us rederive Eq. (5) in perturbation theory. It is convenient to write

$$\begin{aligned} \Gamma_\mu{}^5 &= \gamma_\mu\gamma_5 + \Lambda_\mu{}^5, \\ \Gamma^5 &= \gamma_5 + \Lambda^5, \\ S_F'(p)^{-1} &= \boldsymbol{p} - m_0 - \Sigma(p), \end{aligned} \qquad (6)$$

where the vertex corrections $\Lambda_\mu{}^5$ and Λ^5 and the proper self-energy part $\Sigma(p)$ are calculated using $(\boldsymbol{p}-m_0)^{-1}$ as the free propagator. (Use of the bare mass $m_0 = m - \delta m$ in the free propagator automatically includes the mass-renormalization counter terms.) In terms of $\Lambda_\mu{}^5$, Λ^5, and Σ, Eq. (5) becomes

$$(p-p')^\mu \Lambda_\mu{}^5(p,p') = 2m_0 \Lambda^5(p,p') - \Sigma(p)\gamma_5 - \gamma_5\Sigma(p'). \qquad (7)$$

In order to derive Eq. (7), let us divide the diagrams contributing to $\Lambda_\mu{}^5(p,p')$ into two types: (a) diagrams in which the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to the fermion line beginning with external four-momentum p' and ending with external four-momentum p; (b) diagrams in which the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to an internal closed loop [See Figs. 1(a) and 1(b), respectively]. A typical contribution of type (a) has the form

$$\sum_{k=1}^{2n-1} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{\boldsymbol{p}+\boldsymbol{p}_j-m_0}\right] \gamma^{(k)} \frac{1}{\boldsymbol{p}+\boldsymbol{p}_k-m_0} \gamma_\mu\gamma_5 \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_k-m_0} \times \prod_{j=k+1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_j-m_0}\right] \gamma^{(2n)}(\cdots), \qquad (8)$$

where we have focused our attention on the line to which the $\gamma_\mu\gamma_5$ vertex is attached and have denoted the remainder of the diagram by $(\cdots)$. Multiplying Eq. (8) by $(p-p')^\mu$ and making use of the identity

$$\frac{1}{\boldsymbol{p}+\boldsymbol{p}_k-m_0}(\boldsymbol{p}-\boldsymbol{p}')\gamma_5 \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_k-m_0} = \frac{1}{\boldsymbol{p}+\boldsymbol{p}_k-m_0}(2m_0\gamma_5) \times \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_k-m_0} + \frac{1}{\boldsymbol{p}+\boldsymbol{p}_k-m_0}\gamma_5 + \gamma_5 \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_k-m_0} \qquad (9)$$

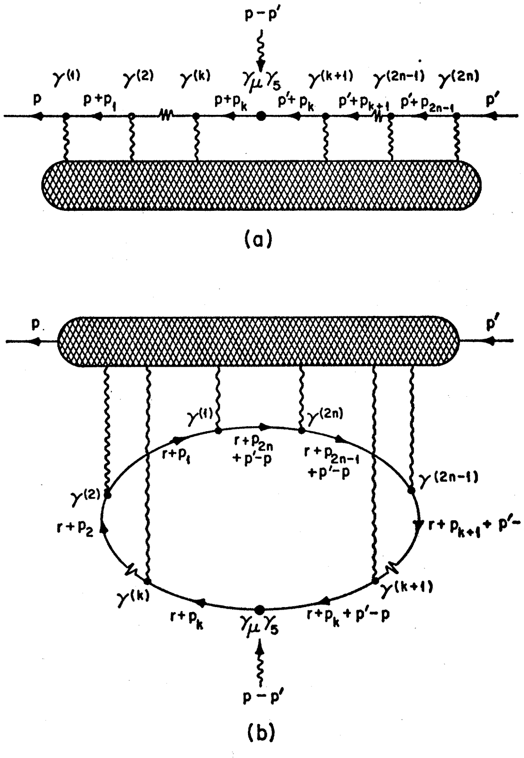

FIG. 1. Diagrams contributing to the axial-vector vertex. (a) The axial-vector vertex is attached to the fermion line beginning with external four-momentum p' and ending with external four-momentum p. (b) The axial-vector vertex is attached to an internal closed loop.

gives, after a little algebraic rearrangement,

$$\begin{aligned} &\sum_{k=1}^{2n-1} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{\boldsymbol{p}+\boldsymbol{p}_j-m_0}\right] \gamma^{(k)} \frac{1}{\boldsymbol{p}+\boldsymbol{p}_k-m_0} 2m_0\gamma_5 \\ &\quad\times \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_k-m_0} \prod_{j=k+1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_j-m_0}\right] \gamma^{(2n)}(\cdots) \\ &\quad -(\cdots) \prod_{j=1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\boldsymbol{p}+\boldsymbol{p}_j-m_0}\right] \gamma^{(2n)}\gamma_5 \\ &\quad -\gamma_5 \prod_{j=1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\boldsymbol{p}'+\boldsymbol{p}_j-m_0}\right] \gamma^{(2n)}(\cdots). \end{aligned} \qquad (10)$$

The first, second, and third terms in Eq. (10) are, respectively, the type-(a) piece of Λ^5, and the pieces of $-\Sigma(p)\gamma_5$ and $-\gamma_5\Sigma(p')$ corresponding to the type-(a) piece of $\Lambda_\mu{}^5$ in Eq. (8). Summing over all type-(a) contributions to $\Lambda_\mu{}^5$, we get

$$(p-p')^\mu \Lambda_\mu{}^{5(\mathrm{a})}(p,p') = 2m_0 \Lambda^{5(\mathrm{a})}(p,p') - \Sigma(p)\gamma_5 - \gamma_5\Sigma(p'). \qquad (11)$$

We turn next to contributions to $\Lambda_\mu{}^5$ of type (b). A typical term is

$$\int d^4r\,\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}\gamma_\mu\gamma_5\right.$$
$$\left.\times\frac{1}{r+p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}$$
$$\times(\cdots).\quad(12)$$

Multiplying by $(p-p')^\mu$ and using Eq. (9) gives

$$\int d^4r\,\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}2m_0\gamma_5\right.$$
$$\left.\times\frac{1}{r+p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}$$
$$\times(\cdots)+\int d^4r\,\mathrm{tr}\left\{\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\right.$$
$$\left.-\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}(\cdots).\quad(13)$$

The first term in Eq. (13) is the type-(b) contribution to Λ^5 corresponding to Eq. (12), while making the change of variable $r\rightarrow r+p'-p$ in the integration in the second term causes the second and third terms to cancel. This gives, when we sum over all type-(b) contributions,

$$(p-p')^\mu\Lambda_\mu{}^{5(b)}(p,p')=2m_0\Lambda^{5(b)}(p,p').\quad(14)$$

The Ward identity of Eq. (7) is finally obtained by adding Eqs. (11) and (14).

Clearly, the only step of the above derivation which is not simply an algebraic rearrangement is the *change of integration variable* in the second term of Eq. (13). This will be a valid operation provided that the integral is at worst superficially logarithmically divergent, a condition that is satisfied by loops with four or more photons, that is, loops with $n\geq 2$. However, when the loop is a triangle graph with only two photons emerging (See Fig. 2) we have $n=1$, and the integral in Eq. (13) appears to be quadratically divergent. Actually, since

$$\mathrm{tr}\{\gamma_5\gamma^{(1)}r\gamma^{(2)}r\}=0,\quad(15)$$

the integral in the $n=1$ case is superficially *linearly* divergent. Since it is well known that translation of a linearly divergent integral is not necessarily a valid operation,[2] we must check carefully to see whether Eq. (14) holds for the triangle graph.

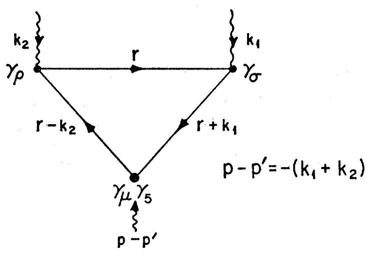

FIG. 2. The axial-vector triangle graph. There is a second diagram, with the photon four-momenta and polarization indices interchanged, which makes a contribution equal to that of the diagram pictured.

To do this we make use of an explicit expression for the triangle graph calculated by Rosenberg.[3] The sum of the diagram illustrated in Fig. 2 and the corresponding diagram with the two photons interchanged is

$$\frac{-ie_0{}^2}{(2\pi)^4}R_{\sigma\rho\mu}\equiv 2\int\frac{d^4r}{(2\pi)^4}(-1)\mathrm{tr}\left\{\frac{i}{r+k_1-m_0}(-ie_0\gamma_\sigma)\right.$$
$$\left.\times\frac{i}{r-m_0}(-ie_0\gamma_\rho)\frac{i}{r-k_2-m_0}\gamma_\mu\gamma_5\right\}.\quad(16)$$

Evaluation of Eq. (16) by the usual regulator techniques leads to the following expression for $R_{\sigma\rho\mu}$ [A_j denotes $A_j(k_1,k_2)$]:

$$\begin{aligned}R_{\sigma\rho\mu}(k_1,k_2)=&A_1k_1{}^\tau\epsilon_{\tau\sigma\rho\mu}+A_2k_2{}^\tau\epsilon_{\tau\sigma\rho\mu}\\&+A_3k_{1\rho}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\sigma\mu}+A_4k_{2\rho}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\sigma\mu}\\&+A_5k_{1\sigma}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\rho\mu}+A_6k_{2\sigma}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\rho\mu},\\A_1=&k_1\cdot k_2A_3+k_2{}^2A_4,\\A_2=&k_1{}^2A_5+k_1\cdot k_2A_6,\\A_3(k_1,k_2)=&-A_6(k_2,k_1)=-16\pi^2I_{11}(k_1,k_2),\\A_4(k_1,k_2)=&-A_5(k_2,k_1)=16\pi^2[I_{20}(k_1,k_2)-I_{10}(k_1,k_2)],\end{aligned}\quad(17)$$

where

$$I_{st}(k_1,k_2)=\int_0^1dx\int_0^{1-x}dy\,x^sy^t[y(1-y)k_1{}^2+x(1-x)k_2{}^2+2xyk_1\cdot k_2-m_0{}^2]^{-1}.\quad(18)$$

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[2] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1955), pp. 458–461.

[3] L. Rosenberg, Phys. Rev. **129**, 2786 (1963). In Eq. (16) and Fig. 2, we have labeled the legs of the triangle in accordance with Rosenberg's notation, which differs from the labeling convention used in Eqs. (12) and (13). Because the integral defining the triangle graph is linearly divergent, the value of the triangle graph is ambiguous and depends on the labeling convention and the method of evaluation of the integral. For example, if Eq. (16) is evaluated by symmetric integration about the origin in r space, the value of $R_{\sigma\rho\mu}$ so obtained satisfies the usual axial-vector Ward identity (but is not gauge-invariant with respect to the vector indices). If, on the other hand, Eq. (16) is evaluated by symmetric integration around some other point in r space, say $r=k_1$ [or, alternatively, if we integrate symmetrically around $r=0$ but label the triangle using the convention of Eqs. (12) and (13)], then the result has an anomalous axial-vector Ward identity. The value in Eq. (17) which we have assigned to $R_{\sigma\rho\mu}$ is the unique value which is gauge-invariant with respect to the vector indices. Further discussion of the ambiguity in the definition of Eq. (16), and a justification of the specific choice in Eq. (17), are given in the Appendix.

We will also need an expression for the triangle graph with $\gamma_\mu\gamma_5$ replaced by $2m_0\gamma_5$. Defining

$$\frac{-ie_0{}^2}{(2\pi)^4}2m_0R_{\sigma\rho}\equiv 2\int\frac{d^4r}{(2\pi)^4}(-1)\mathrm{tr}\left\{\frac{i}{\boldsymbol{r}+\boldsymbol{k}_1-m_0}(-ie_0\gamma_\sigma)\right.$$
$$\left.\times\frac{i}{\boldsymbol{r}-m_0}(-ie_0\gamma_\rho)\frac{i}{\boldsymbol{r}-\boldsymbol{k}_2-m_0}2m_0\gamma_5\right\},\quad(19)$$

we find that

$$R_{\sigma\rho}=k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\sigma\rho}B_1\,,$$
$$B_1=8\pi^2m_0I_{00}(k_1,k_2)\,.\quad(20)$$

We are now ready to calculate the divergence of the axial-vector triangle diagram. If the Ward identity holds, we should find

$$-(k_1+k_2)^\mu R_{\sigma\rho\mu}=2m_0R_{\sigma\rho}\,,\quad(21)$$

but from Eqs. (16)–(20) we find, instead,

$$-(k_1+k_2)^\mu R_{\sigma\rho\mu}=2m_0R_{\sigma\rho}+8\pi^2k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\sigma\rho}\,.\quad(22)$$

We see that *the axial-vector Ward identity fails in the case of the triangle graph.* The failure is a result of the fact that the integration variable in a linearly divergent Feynman integral cannot be freely translated.

The breakdown of the axial-vector Ward identity which we have just found is related to another anomalous property of the triangle graph. To see this, let us consider the behavior of the general axial-vector loop diagram with $2n$ photon vertices (See Fig. 3), as the $2n-1$ independent photon momenta $k_1,\cdots,k_{2n-1}$ approach infinity simultaneously in the manner

$$k_j=\xi q_j\,,\quad j=1,\cdots,2n-1\,;$$
$$q_j\text{ fixed}\,,\quad\xi\to\infty\,,\quad(23)$$

while the momentum $p-p'$ carried by the axial-vector current is held fixed. According to Weinberg's theorem,[4] the asymptotic behavior of the loop graph in this limit is

$$\xi^\alpha(\ln\xi)^\beta\,,\quad(24)$$

where β is undetermined by Weinberg's analysis and where α is the maximum of the superficial divergences[5] $\alpha(g)$ of the subgraphs[5] g linking the $2n$ photon lines (i.e., linking the momenta which are becoming infinite). For the diagram of Fig. 3 there are two such subgraphs, illustrated in Fig. 4, with superficial divergences $\alpha(1)=-2n+1$ and $\alpha(2)=-2n+3$. Thus, the asymptotic coefficient α is $\alpha(2)=-2n+3$, and comes from the subgraph in which all propagators in the loop are involved. Now Weinberg's theorem always tells us what the maximal asymptotic *power* of a graph is, but it does not guarantee that the *coefficient* of the maximal term is nonvanishing. In fact, in the case of the axial-vector loop diagram we will show that the coefficient of the $\xi^{-2n+3}(\ln\xi)^\beta$ term *does* vanish, so that the leading asymptotic behavior is $\xi^{-2n+2}(\ln\xi)^{\beta'}$, one power lower than is predicted by naive power counting. Let us denote by $L(p-p',m_0;p_1,\cdots,p_{2n-1})$ the graph illustrated in Fig. 3,

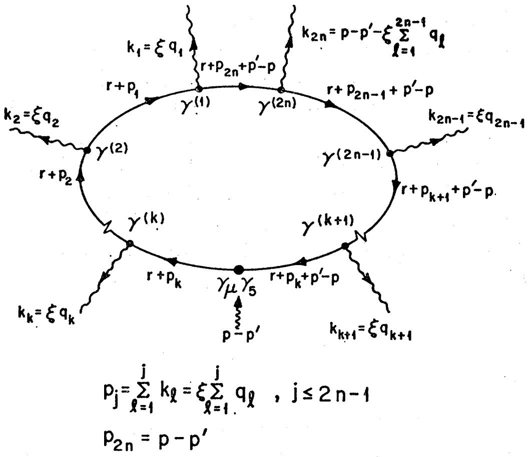

FIG. 3. Diagram for calculation of the asymptotic behavior of the general axial-vector loop.

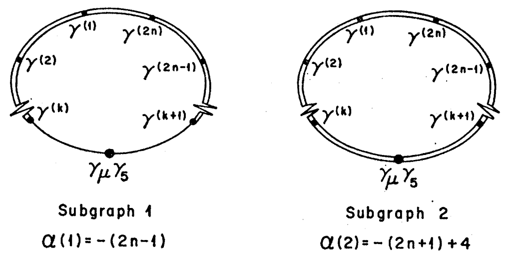

FIG. 4. Subgraphs (doubled lines) which determine the asymptotic behavior of Fig. 3.

$$L(p-p',m_0;p_1,\cdots,p_{2n-1})$$
$$=\int d^4r\,\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{\boldsymbol{r}+\boldsymbol{p}_j-m_0}\right]\right.$$
$$\times\gamma^{(k)}\frac{1}{\boldsymbol{r}+\boldsymbol{p}_k-m_0}\gamma_\mu\gamma_5\frac{1}{\boldsymbol{r}+\boldsymbol{p}_k+\boldsymbol{p}'-\boldsymbol{p}-m_0}$$
$$\left.\times\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{\boldsymbol{r}+\boldsymbol{p}_j+\boldsymbol{p}'-\boldsymbol{p}-m_0}\right]\right\}.\quad(25)$$

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[4] S. Weinberg, Phys. Rev. **118**, 838 (1960). For a simplified exposition of Weinberg's results, see J. D. Bjorken and S. D. Drell, Ref. 1, pp. 317–330 and pp. 364–368. Weinberg's theorem applies for arbitrary *spacelike* four-vectors q_j. There can also be powers of ln lnξ, ln ln lnξ, etc., in Eq. (24), which we do not indicate explicitly.

[5] The superficial divergence of the subgraph is obtained, as usual, by adding -1 for each internal fermion line, -2 for each internal boson line, and $+4$ for each internal integration. For the precise definition of subgraph in the general case, see Ref. 4.

Clearly we can write

$$\begin{aligned} &L(p-p',m_0;p_1,\cdots,p_{2n-1})\\ (\mathrm{A})\quad &=L(p-p',m_0;p_1,\cdots,p_{2n-1})\\ &\qquad -L(0,m_0;p_1,\cdots,p_{2n-1}) \qquad (26)\\ (\mathrm{B})\quad &+L(0,m_0;p_1,\cdots,p_{2n-1})-L(0,0;p_1,\cdots,p_{2n-1})\\ (\mathrm{C})\quad &+L(0,0;p_1,\cdots,p_{2n-1}). \end{aligned}$$

Because differencing the loop graph with respect to either the axial-vector current four-momentum $p-p'$ or the fermion mass m_0 decreases the degree of divergence by one, terms (A) and (B) on the right-hand side of Eq. (26) have $\alpha(2)=-2n+2$, and therefore behave asymptotically as $\xi^{-2n+2}(\ln\xi)^{\beta'}$. Term (C) on the right-hand side of Eq. (26) can be rewritten as

$$\begin{aligned} &L(0,0;p_1,\cdots,p_{2n-1})\\ &=\int d^4r\,\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j}\right]\gamma^{(k)}\frac{1}{r+p_k}\gamma_\mu\gamma_5\right.\\ &\qquad\qquad\left.\times\frac{1}{r+p_k}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j}\right]\right\}\\ &=\int d^4r\,\mathrm{Tr}\left\{\gamma_5\frac{\partial}{\partial r^\mu}\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j}\right]\right\}. \end{aligned} \qquad (27)$$

Integrating by parts with respect to r gives

$$L(0,0;p_1,\cdots,p_{2n-1})=0,$$

proving that the asymptotic behavior of the loop graph is one power better than given by Weinberg's theorem.

The only nonalgebraic step in this proof is the integration by parts with respect to r, an operation which is valid provided that the integration variable in

$$\int d^4r\,\mathrm{Tr}\left\{\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j}\right]\right\} \qquad (28)$$

can be freely translated. This is the same condition as we found above for validity of the axial-vector Ward identity. Thus again, our proof is valid for $n\geq 2$, but we expect possible trouble in the case of the triangle graph ($n=1$). From the explicit expression for the triangle graph in Eqs. (17) and (18), we see that if we write $k_1=\xi q$, $k_2=-\xi q+p'-p$, then as $\xi\rightarrow\infty$ we find

$$R_{\sigma\rho\mu}(k_1,k_2)\rightarrow -8\pi^2\xi q^\tau\epsilon_{\tau\sigma\rho\mu}+O(\ln\xi). \qquad (29)$$

In other words, the asymptotic power is $\alpha=1=-2n+3$, as given by Weinberg's rules, rather than one power lower, as is the case for the loop graphs with $n\geq 2$. It is easy to check that when Eq. (29) is multiplied by $-(k_1+k_2)^\mu$, the term with the anomalous asymptotic behavior agrees, for large ξ, with the term in Eq. (22) which violates the Ward identity. Thus, the breakdown of the axial-vector Ward identity in the triangle graph and the anomalous asymptotic behavior of the triangle graph are basically the same phenomenon.

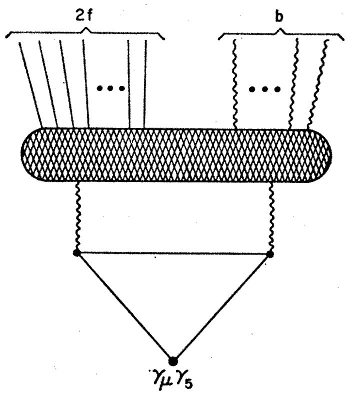

FIG. 5. Contribution of the triangle diagram to the general axial-vector vertex. We have not drawn the second diagram in which the photon lines emerging from the triangle are crossed.

It is clear that the breakdown of the Ward identity for the basic triangle graph will also cause failure of the Ward identity for any graph of the type illustrated in Fig. 5, in which the two photon lines coming out of the triangle graph join onto a "blob" from which $2f$ fermion and b boson lines emerge. From Eq. (22) for the divergence of the basic triangle graph, it is possible to show that the breakdown of the axial-vector Ward identity in the general case is simply described by replacing Eq. (4) for the axial-vector-current divergence (which we have shown to be incorrect) by

$$\frac{\partial}{\partial x_\mu}j_\mu{}^5(x)=2im_0j^5(x)+\frac{\alpha_0}{4\pi}{:}F^{\xi\sigma}(x)F^{\tau\rho}(x){:}\,\epsilon_{\xi\sigma\tau\rho}. \qquad (30)$$

[Equation (30) is easily verified by using the Feynman rules for the vertices of $j_\mu{}^5$, j^5, and $(\alpha_0/4\pi){:}F^{\xi\sigma}F^{\tau\rho}{:}\,\epsilon_{\xi\sigma\tau\rho}$, which are given in Fig. 6.] For example, if we define $\bar{F}(p,p')$ by

$$\begin{aligned} S_F{}'(p)\bar{F}(p,p')S_F{}'(p')=&-\int d^4xd^4y\,e^{ip\cdot x}e^{-ip'\cdot y}\\ &\times\langle T(\psi(x){:}F^{\xi\sigma}(0)F^{\tau\rho}(0){:}\,\epsilon_{\xi\sigma\tau\rho}\bar{\psi}(y))\rangle_0, \end{aligned} \qquad (31)$$

then the axial-vertex Ward identity of Eq. (5) is modified to read

$$\begin{aligned} (p-p')^\mu\Gamma_\mu{}^5(p,p')=&\,2m_0\Gamma^5(p,p')-i(\alpha_0/4\pi)\bar{F}(p,p')\\ &+S_F{}'(p)^{-1}\gamma_5+\gamma_5S_F{}'(p')^{-1}. \end{aligned} \qquad (32)$$

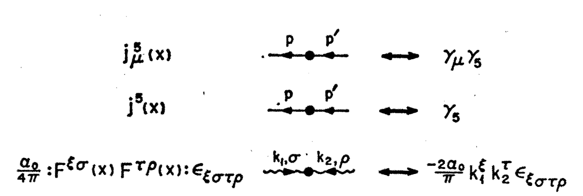

FIG. 6. Feynman rules for the vertices appearing in Eq. (30).

Equation (30), which is the principal result of this section, states the surprising fact that *the axial-vector-current divergence, as calculated in perturbation theory, contains a well-defined extra term which is not obtained when the axial-vector divergence is calculated by formal use of the equations of motion.*[6]

II. CONSEQUENCES OF THE EXTRA TERM

In this section we investigate the consequences of the extra term which we have found in the axial-vector-current divergence [Eq. (30)] and in the axial-vector-current Ward identity [Eq. (32)]. We consider, in particular, the questions of (A) renormalization of the axial-vector vertex, (B) radiative corrections to $\nu_l l$ scattering, and (C) the connection between γ_5 invariance and a conserved axial-vector current in massless quantum electrodynamics.

A. Renormalization of the Axial-Vector Vertex

Recently, Preparata and Weisberger[7] have proved the following theorem: If a local current, constructed as a bilinear product of fermion fields, is conserved apart from mass terms, then the vertex parts of both the current and its divergence are made finite by multiplication by the wave-function renormalization constants of the fields from which the current is constructed. *If* Eq. (4) correctly described the divergence of the axial-vector current in spinor electrodynamics, then the theorem of Preparata and Weisberger would apply in this case. However, we have seen that the divergence is actually given by Eq. (30), and involves an additional term which is *not* a mass term. The effect of this extra term, we shall see, is to cause the Preparata-Weisberger argument to break down.

First let us review how the Preparata-Weisberger result could be derived if Eq. (4), and the corresponding Ward identify of Eq. (5), were true. Since both $j_\mu{}^5$ and j^5 are local bilinear products of fermion fields, the vertex parts $\Gamma_\mu{}^5$ and Γ^5 are *multiplicatively renormalizable.* Thus we can write

$$\begin{aligned} \Gamma_\mu{}^5(p,p') &= Z_A{}^{-1}\tilde{\Gamma}_\mu{}^5(p,p'), \\ \Gamma^5(p,p') &= Z_D{}^{-1}\tilde{\Gamma}^5(p,p'), \\ S_F'(p) &= Z_2\tilde{S}_F'(p), \end{aligned} \tag{33}$$

where the tilde quantities are finite (cutoff-independent) and where Z_A, Z_D, and Z_2 are cutoff-dependent renormalization constants. Substituting Eq. (32) into Eq. (5) we get

$$\begin{aligned} (p-p')^\mu\tilde{\Gamma}_\mu{}^5(p,p') &= (2m_0Z_A/Z_D)\tilde{\Gamma}^5(p,p') \\ &\quad +(Z_A/Z_2)[\tilde{S}_F'(p)^{-1}\gamma_5+\gamma_5\tilde{S}_F'(p')^{-1}], \end{aligned} \tag{34}$$

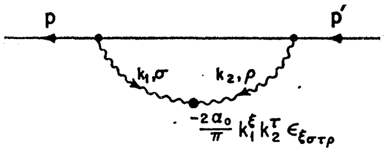

FIG. 7. Diagram giving the lowest-order contribution of the extra term in Eq. (32). The heavy dot denotes the vertex of $(\alpha_0/4\pi){:}F^{\xi\sigma}F^{\tau\rho}{:}\,\epsilon_{\xi\sigma\tau\rho}$.

and varying the cutoff gives

$$\begin{aligned} 0 = \delta(2m_0Z_A/Z_D)\tilde{\Gamma}^5(p,p')+\delta(Z_A/Z_2) \\ \times[\tilde{S}_F'(p)^{-1}\gamma_5+\gamma_5\tilde{S}_F'(p')^{-1}]. \end{aligned} \tag{35}$$

Putting p,p', or both on mass shell then implies that

$$\delta(2m_0Z_A/Z_D)=\delta(Z_A/Z_2)=0\,, \tag{36}$$

which means that both $2m_0Z_A/Z_D$ and Z_A/Z_2 are cutoff-independent, and hence finite. Thus, if Eqs. (4) and (5) were correct, multiplication by the wave-function renormalization constant Z_2 would make $\Gamma_\mu{}^5$ and Γ^5 fiinte.

Let us now consider the actual situation, in which the divergence of the axial-vector current is given by Eq. (30) and the axial-vector Ward identity by Eq. (32). The extra term in Eq. (32) first appears in order $\alpha_0{}^2$ of perturbation theory. [See Fig. 7.] This lowest-order contribution is already logarithmically divergent; introducing a cutoff by replacing the photon propagator $1/(q^2+i\epsilon)$ with $[1/(q^2+i\epsilon)][-\Lambda^2/(-\Lambda+q^2+i\epsilon)]$, we find that

$$\begin{aligned} -i(\alpha_0/4\pi)\bar{F}(p,p') = -\tfrac{3}{4}(\alpha_0/\pi)^2\ln(\Lambda^2/m^2)(p-p)^\mu \\ \times\gamma_\mu\gamma_5+\alpha_0{}^2\times\text{finite}+O(\alpha_0{}^3). \end{aligned} \tag{37}$$

We will also need part of the expression for $\Gamma^5(p,p')$ to order α_0,

$$\begin{aligned} \Gamma^5(p,p') &= \gamma_5[1+O(\alpha_0)]+(\alpha_0/2\pi)m_0 \\ &\quad \times I(p,p')(p-p')^\mu\gamma_\mu\gamma_5+O(\alpha_0{}^2), \\ I(p,p') &= \int_0^1 dx\int_0^{1-x} dy[x(1-x)p^2+y(1-y)p'^2 \\ &\quad -2xyp\cdot p'-(x+y)m_0{}^2]^{-1}. \end{aligned} \tag{38}$$

Comparing Eqs. (37) and (38), we see that *it is impossible to cancel away the divergence in Eq. (37) by adding to it a constant multiple of Eq. (38)*: A constant counter term of order $\alpha_0{}^2$ multiplying the leading γ_5 term in Eq. (38) cannot cancel the divergence in Eq. (37), because the latter is proportional to $(p-p')^\mu\gamma_\mu\gamma_5$, while a constant counter term of order α_0 multiplying the $(p-p')^\mu\gamma_\mu\gamma_5$ term in Eq. (38) cannot cancel the divergence in Eq. (37) because of the nontrivial functional dependence of $I(p,p')$ on p and p'. In other words, the axial-vector divergence with the extra term included,

$$2m_0\Gamma^5(p,p')-i(\alpha_0/4\pi)\bar{F}(p,p')\,, \tag{39}$$

is *not multiplicatively renormalizable.*

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[6] We show in the Appendix that this extra term cannot be eliminated by redefining the triangle graph.

[7] G. Preparata and W. I. Weisberger, Phys. Rev. 175, 1965 (1968), Appendix C.

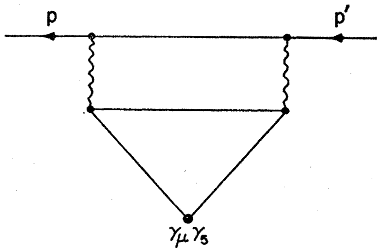

FIG. 8. Lowest-order contribution of the triangle diagram to the axial-vector vertex. We have not drawn the diagram in which the photon lines are crossed.

Since multiplicative renormalizability of the divergence was essential to the Preparata-Weisberger argument outlined above, this argument no longer applies. We expect, then, that even after multiplication by Z_2, there will still be logarithmically divergent terms in the axial-vector vertex. Such terms first appear in order α_0^2 of perturbation theory, as a result of the diagram shown in Fig. 8; the divergence of Fig. 8 is just a consequence of the anomalous asymptotic behavior of the triangle graph pointed out in Sec. I. Introducing a cutoff in the photon propagator as above, we find that

$$Z_2\Gamma_\mu{}^5(p,p')=\gamma_\mu\gamma_5[1-\tfrac{3}{4}(\alpha_0/\pi)^2\ln(\Lambda^2/m^2)] +\alpha_0\times\text{finite}+\alpha_0^2\times\text{finite}+O(\alpha_0^3)\,. \quad (40)$$

Equation (40) shows explicitly that the axial-vector vertex, while still multiplicatively renormalizable, is not simply made finite by multiplication by the wave-function renormalization constant Z_2. Rather, we have [see Eq. (33)]

$$Z_A=Z_2[1+\tfrac{3}{4}(\alpha_0/\pi)^2\ln(\Lambda^2/m^2)+O(\alpha_0^3)]\,. \quad (41)$$

B. Radiative Corrections to $\nu_l l$ Scattering

As an application of Eq. (40), let us consider the radiative corrections to $\nu_l l$ scattering, where l is a μ or an e. According to the usual local current-current theory, the leptonic weak interactions are described by the effective Lagrangian

$$\mathcal{L}_{\text{eff}}=(G/\sqrt{2})j_\lambda{}^\dagger j^\lambda\,, \quad (42)$$

where $G\approx10^{-5}/M_{\text{proton}}{}^2$ is the Fermi constant and where[8]

$$j^\lambda=\bar{\nu}_\mu\gamma^\lambda(1-\gamma_5)\mu+\bar{\nu}_e\gamma^\lambda(1-\gamma_5)e \quad (43)$$

is the leptonic current. In addition to the usual terms describing muon decay, Eq. (42) contains the terms

$$(G/\sqrt{2})[\bar{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu\bar{\nu}_\mu\gamma^\lambda(1-\gamma_5)\mu +\bar{e}\gamma_\lambda(1-\gamma_5)\nu_e\bar{\nu}_e\gamma^\lambda(1-\gamma_5)e]\,, \quad (44)$$

which describe elastic neutrino-lepton scattering. In order to study radiative corrections to the basic $\nu_l l$ scattering process, it is convenient to use a Fierz transformation to rewrite Eq. (44) in the form (the so-called "charge retention ordering")

$$(G/\sqrt{2})[\bar{\mu}\gamma_\lambda(1-\gamma_5)\mu\bar{\nu}_\mu\gamma^\lambda(1-\gamma_5)\nu_\mu +\bar{e}\gamma_\lambda(1-\gamma_5)e\bar{\nu}_e\gamma^\lambda(1-\gamma_5)\nu_e]\,. \quad (45)$$

The radiative corrections to Eq. (45) may then be obtained simply by calculating the radiative corrections to the charged lepton currents $\bar{\mu}\gamma_\lambda(1-\gamma_5)\mu$ and $\bar{e}\gamma_\lambda(1-\gamma_5)e$, without any reference to the neutrino currents.

Now, application of standard electrodynamic perturbation theory shows that the effect of the radiative corrections to the charged lepton currents is to replace the matrix elements $\bar{\mu}\gamma_\lambda(1-\gamma_5)\mu$, $\bar{e}\gamma_\lambda(1-\gamma_5)\,e$ (we use μ, e to denote spinors here) by

$$\bar{\mu}Z_2{}^{(\mu)}[\Gamma_\lambda{}^{(\mu)}-\Gamma_\lambda{}^{5(\mu)}]\mu\,,\quad \bar{e}Z_2{}^{(e)}[\Gamma_\lambda{}^{(e)}-\Gamma_\lambda{}^{5(e)}]e\,. \quad (46)$$

In Eq. (46), $\Gamma_\lambda{}^{(\mu,e)}$ and $\Gamma_\lambda{}^{5(\mu,e)}$ denote the proper vector and axial-vector vertices, while the wave-function renormalization factors $Z_2{}^{(\mu,e)}$ come from self-energy insertions on the external lepton lines which run into and out of the proper vertices. From the usual electrodynamic Ward identity for the vector part, we know that $Z_2{}^{(\mu)}\Gamma_\lambda{}^{(\mu)}$ and $Z_2{}^{(e)}\Gamma_\lambda{}^{(e)}$ are finite. On the other hand, Eq. (40) tells us that

$$Z_2{}^{(\mu,e)}\Gamma_\lambda{}^{5(\mu,e)}=\gamma_\lambda\gamma_5[1-\tfrac{3}{4}(\alpha_0/\pi)^2\ln(\Lambda^2/m^2)] +\alpha_0\times\text{finite}+\alpha_0^2\times\text{finite}+O(\alpha_0^3)\,, \quad (47)$$

which means that, on account of the presence of axial-vector triangle diagrams, *the radiative corrections to $\nu_e e$ and $\nu_\mu\mu$ scattering diverge in the fourth order of perturbation theory.* This result contrasts sharply with the fact that the radiative corrections to muon decay or to the scattering reaction $\nu_\mu+e\rightarrow\nu_e+\mu$ are finite to all orders in perturbation theory.[7] The crucial difference between the two cases, of course, is that because of separate muon and electron-number conservation, the current $\bar{\mu}\gamma_\lambda(1-\gamma_5)e$ cannot couple into closed electron or muon loops, and thus the troublesome triangle diagram is not present.

Two points of view can be taken towards the divergent radiative corrections in $\nu_l l$ scattering. One viewpoint is that we know, in any case, that the local current-current theory of leptonic weak interactions cannot be correct, since this theory leads at high energies to nonunitary matrix elements, and since it gives divergent results for higher-order weak-interaction effects.[9] Thus, it is entirely possible that the modifications in Eq. (44) necessary to give a satisfactory weak-interaction theory will also cure the disease of infinite radiative corrections in $\nu_l l$ scattering. The other viewpoint is that we should try to make the radiative corrections to $\nu_l l$ scattering finite, within the framework of a local weak-interaction theory. It turns out that this

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[8] We omit the normal ordering signs.

[9] For recent discussions of the sicknesses of the local current-current theory and their possible remedies, see N. Christ, Phys. Rev. **176**, 2086 (1968); and M. Gell-Mann, M. L. Goldberger, N. M. Kroll, and F. E. Low, Phys. Rev. (to be published).

is possible, if we introduce $\nu_e\mu$ and $\nu_\mu e$ scattering terms into the effective Lagrangian, so that Eq. (44) is replaced by

$$(G/\sqrt{2})[\bar{\mu}\gamma_\lambda(1-\gamma_5)\mu-\bar{e}\gamma_\lambda(1-\gamma_5)e] \times[\bar{\nu}_\mu\gamma^\lambda(1-\gamma_5)\nu_\mu-\bar{\nu}_e\gamma^\lambda(1-\gamma_5)\nu_e]. \quad (48)$$

This works because the troublesome extra term in Eq. (30) is independent of the bare mass m_0, so that it cancels between the muon and electron terms in Eq. (48), giving[10]

$$\frac{\partial}{\partial x_\lambda}[\bar{\mu}\gamma_\lambda\gamma_5\mu-\bar{e}\gamma_\lambda\gamma_5 e]=2im_0^{(\mu)}\bar{\mu}\gamma_5\mu-2im_0^{(e)}\bar{e}\gamma_5 e. \quad (49)$$

Application of the Preparata-Weisberger argument to Eq. (49) then shows that the radiative corrections to Eq. (48) are finite in all orders of perturbation theory. Experimentally, it will be possible to distinguish between Eq. (48) and Eq. (44) by looking for elastic scattering of muon neutrinos from electrons.

C. Connection Between γ_5 Invariance and a Conserved Axial-Vector Current in Massless Electrodynamics

Finally, let us discuss the effects of the axial-vector triangle diagram in the case of massless spinor electrodynamics [Eq. (1) with $m_0=0$]. We will find that the triangle diagram leads to a breakdown of the usual connection between symmetries of the Lagrangian and conserved currents. As in our previous discussions, we begin by describing the standard theory, which holds in the absence of singular phenomena.[8] Let $\{\Phi(x)\}=\{\Phi_1(x),\Phi_2(x),\cdots\}$ and $\{\partial_\lambda\Phi\}$ be a set of canonical fields and their space-time derivatives, and let us consider the field theory described by the Lagrangian density

$$\mathcal{L}(x)\equiv\mathcal{L}[\{\Phi\},\{\partial_\lambda\Phi\}]. \quad (50)$$

To establish the connection between invariance properties of $\mathcal{L}$ and conserved currents, we make the infinitesimal, local gauge transformation on the fields,

$$\Phi_j(x)\to\Phi_j(x)+\Lambda(x)G_j[\{\Phi(x)\}], \quad (51)$$

and define the associated current J^α by

$$J^\alpha=-\delta\mathcal{L}/\delta(\partial_\alpha\Lambda). \quad (52)$$

Then, by using the Euler-Lagrange equations of motion of the fields, we easily find[11] that the divergence of the current is given by

$$\partial_\alpha J^\alpha=-\delta\mathcal{L}/\delta\Lambda. \quad (53)$$

In particular, if the gauge transformation of Eq. (51), with *constant* gauge function Λ, leaves the Lagrangian invariant, then $\delta\mathcal{L}/\delta\Lambda=0$ and the current J^α is conserved. Thus, to any continuous invariance transformation of the Lagrangian there is associated a conserved current. It is also easily verified that the charge $Q(t)=\int d^3x\, J^0(\mathbf{x},t)$ associated with the current J^α has the properties

$$dQ(t)/dt=0, \quad (54a)$$

$$[Q,\Phi_j(x)]=iG_j(x). \quad (54b)$$

Equation (54b) states that Q is the generator of the gauge transformation in Eq. (51), for constant Λ.

Let us now specialize to the case of massless electrodynamics, with Eq. (51) the gauge transformation

$$\psi(x)\to[1+i\gamma_5\Lambda(x)]\psi(x). \quad (55)$$

When Λ is a constant and $m_0=0$, this transformation leaves the Lagrangian of Eq. (1) invariant, so that according to Eq. (53), the associated current J^α should be conserved. But calculating J^α, we find

$$J^\alpha=-\delta\mathcal{L}/\delta(\partial_\alpha\Lambda)=\bar{\psi}\gamma^\alpha\gamma_5\psi, \quad (56)$$

which according to Eq. (30) has the divergence

$$\partial_\alpha J^\alpha=(\alpha_0/4\pi)F^{\xi\sigma}(x)F^{\tau\rho}(x)\epsilon_{\xi\sigma\tau\rho}. \quad (57)$$

Thus, Eq. (53), which was obtained by formal calculation using the equations of motion, breaks down in this case. We see that because of the presence of the axial-vector triangle diagram, *even though the Lagrangian (and all orders of perturbation theory) of massless electrodynamics are γ_5 invariant, the axial-vector current associated with the γ_5 transformation is not conserved.*

However, it is amusing that even though there is no conserved current connected with the γ_5 transformation, there is still a generator $\bar{Q}^5$ with the properties of Eq. (54). To see this, let us consider the quantity $\bar{j}^5$ defined by

$$\bar{j}_\mu{}^5(x)=j_\mu{}^5(x)-\frac{\alpha_0}{\pi}A^\xi(x)\frac{\partial A^\tau(x)}{\partial x_\rho}\epsilon_{\xi\mu\tau\rho}; \quad (58)$$

referring to Eq. (30), we see that

$$\frac{\partial}{\partial x_\mu}\bar{j}_\mu{}^5(x)=0. \quad (59)$$

Although $\bar{j}_\mu{}^5$ is conserved, it is explicitly *gauge-dependent* and therefore *is not an observable current operator.* But the associated charge

$$\bar{Q}^5=\int d^3x\, \bar{j}_0{}^5(x) =\int d^3x\left[\psi^\dagger(x)\gamma_5\psi(x)+\frac{\alpha_0}{\pi}\mathbf{A}\cdot\nabla\times\mathbf{A}\right] \quad (60)$$

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[10] What is happening here is that the muon triangle diagram and the electron triangle diagram contribute with opposite sign, and so regularize each other.

[11] For details, see S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968), pp. 15–18.

is gauge-invariant and therefore observable. According to Eq. (59), $\bar{Q}^5$ is time-independent, and its commutator with $\psi(x)$ (calculated formally by use of the canonical commutation relations) is

$$[\bar{Q}^5,\psi(x)]=-\gamma_5\psi(x)=i[i\gamma_5\psi(x)]. \qquad (61)$$

Comparison with Eq. (59) then shows that $\bar{Q}^5$ is the conserved generator of the γ_5 transformations.[12]

After this manuscript was completed, we learned that Bell and Jackiw[13] had independently studied the anomalous properties of the axial-vector triangle graph, in the context of the σ model. In the Appendix we discuss certain questions raised both by the paper of Bell and Jackiw and in conversations with Professor S. Coleman.

Note added in proof. (1) All field quantities appearing in the paper denote *unrenormalized* fields, with the one exception that in Eqs. (A29), (A30), and (A34), ϕ_{π^0} and ϕ_η denote, respectively, the renormalized pion and η fields.

(2) It is our claim that Eq. (30) is an *exact* result, valid to all orders in electromagnetism, and similarly that the σ-model analog, Eq. (A22), is exact to all orders in both the electromagnetic and strong couplings. These conclusions follow in our diagrammatic analysis from the fact that electromagnetic or strong radiative corrections to the basic triangle always involve axial-vector loops with more than three vertices, which satisfy the normal axial-vector Ward identities. A more detailed discussion of this question will be given by the author and W. A. Bardeen (to be published).

(3) Field-theoretic derivations of Eq. (30) have been given by C. R. Hagen (to be published), R. Jackiw and K. Johnson (to be published), B. Zumino (to be published), and R. A. Brandt (to be published). Jackiw and Johnson point out that the essential features of the field-theoretic derivation, in the case of external electromagnetic fields, are contained in J. Schwinger, Phys. Rev. **82**, 664 (1951).

(4) In Eq. (A1) we state that the general form of the triangle diagram is $R_{\sigma\rho\mu}$, Rosenberg's gauge-invariant expression, plus an arbitrary multiple of $\epsilon_{\tau\sigma\rho\mu}(k_1-k_2)^\tau$; we infer this form for the extra term by studying how the triangle graph is changed by shifts in the integration variable. It is easy to see that this is the *only allowed form* for the ambiguity, by noting that the extra term must satisfy the following conditions. (i) The extra term must have the dimensions of a mass; (ii) the extra term must be a three-index ($\sigma\rho\mu$) Lorentz pseudotensor; (iii) the extra term must be symmetric under interchange of the photon variables (k_1,σ) and (k_2,ρ); (iv) the extra term must have *no singularities* in any of the variables k_1^2, k_2^2, $k_1\cdot k_2$ and m_0, since the discontinuities of the triangle diagram across its singularities involve no linear divergences and hence are unambiguously contained in Rosenberg's expression $R_{\sigma\rho\mu}$.

(5) The statement in Ref. 20, that the simultaneous presence of isoscalar and isovector vector mesons affects the $\pi^0 \rightarrow 2\gamma$ prediction, is not correct. There will, of course, be an extra term of the form

$$\partial B^\xi(I=1)/\partial x_\sigma \partial B^\tau(I=0)/\partial x_\rho \epsilon_{\xi\sigma\tau\rho}$$

in the PCAC equation. However, the matrix element of this term relevant to the $\pi^0 \rightarrow 2\gamma$ low-energy theorem, when expressed in terms of Fourier transforms of the vector-meson fields, is proportional to

$$\int d^4k \langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|B_{k+k_1+k_2}{}^\xi(I=1)B_{-k}{}^\tau(I=0)|0\rangle \\ \times(k_1+k_2)^\sigma k^\tau \epsilon_{\xi\sigma\tau\rho}.$$

Because of photon gauge invariance, the matrix element

$$\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|B_{k+k_1+k_2}{}^\xi(I=1)B_{-k}{}^\tau(I=0)|0\rangle$$

is proportional to k_1k_2, and so the two-vector meson term is of order $k_1k_2(k_1+k_2)$. Since the low-energy theorem involves only terms of order k_1k_2, the two-vector meson contribution is of higher order and does *not* affect our result. This also means that the extra terms in the PCAC equation proposed recently by R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Letters **27B**, 657 (1968), do not in fact lead to a non-null PCAC prediction for $\pi^0 \rightarrow 2\gamma$.

ACKNOWLEDGMENTS

I am grateful to Professor S. B. Treiman for many interesting discussions. I wish to thank Dr. R. J. Eden and Professor A. B. Pippard for the hospitality of the Cavendish high energy group and Clare Hall, where the work reported here was begun. The calculations were completed and the manuscript was written during the 1968 National Accelerator Laboratory summer study at Aspen. I wish to thank Dr. Bell and Dr. Jackiw for an interesting correspondence about their results.

APPENDIX

We discuss here the following questions raised both by the recent paper of Bell and Jackiw and in conversations with Professor S. Coleman: (1) Is the expression $R_{\sigma\rho\mu}$ [see Eq. (17)] which we have used for the triangle graph unique, or is it possible to redefine $R_{\sigma\rho\mu}$ by a subtraction in such a way as to eliminate the anomalies discussed in the text? (2) What is the connection between our results and the σ-model discussion of Bell and Jackiw, and between our results and the physical $\pi^0 \rightarrow 2\gamma$ and $\eta \rightarrow 2\gamma$ decays?

A. Uniqueness of the Triangle Graph

The expression for $R_{\sigma\rho\mu}$ in Eq. (17) is obtained from Eq. (16) by the regulator technique of subtracting from

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[12] Because of an implicit photon field dependence of $j_0^5(x)$ implied by Eq. (30), $\bar{Q}^5$ *does* commute with all the photon field variables. The details of showing this are complicated, and will be given elsewhere.

[13] J. S. Bell and R. Jackiw (unpublished).

Eq. (16) a loop with m_0 replaced by M, performing the r integration, and then letting $M \rightarrow \infty$. Clearly, any *mass-independent* terms in Eq. (16) will be lost in this process. That a mass-independent term is present can be seen from the fact that when we make the change of integration variable $r \rightarrow r+ak_1+bk_2$ in Eq. (16), the result is not left invariant, but rather is changed by multiples of $\epsilon_{\tau\sigma\rho\mu}k_1^\tau$ and $\epsilon_{\tau\sigma\rho\mu}k_2^\tau$. If we are careful to preserve symmetry with respect to the photon variables, the change will be proportional to $\epsilon_{\tau\sigma\rho\mu}(k_1-k_2)^\tau$. The noninvariance of the triangle graph under changes of integration variable is of course just a result of the linear divergence in Eq. (16), and means that in a nonregulator calculation the results obtained for the triangle graph will depend on how the external momenta k_1 and k_2 are taken to run through the internal lines. We may express this ambiguity formally by writing that the general expression for the triangle graph is

$$R_{\sigma\rho\mu}[\zeta]=R_{\sigma\rho\mu}+\zeta\epsilon_{\tau\sigma\rho\mu}(k_1-k_2)^\tau, \qquad \text{(A1)}$$

with $R_{\sigma\rho\mu}$ the regulator value in Eq. (17).

We easily find the following properties of $R_{\sigma\rho\mu}[\zeta]$:

(i) vector index divergence:

$$\begin{aligned} k_1^\sigma R_{\sigma\rho\mu}[\zeta]&=-\zeta k_1^\sigma k_2^\tau\epsilon_{\tau\sigma\rho\mu}, \\ k_2^\rho R_{\sigma\rho\mu}[\zeta]&=\zeta k_2^\rho k_1^\tau\epsilon_{\tau\sigma\rho\mu}; \end{aligned} \qquad \text{(A2)}$$

(ii) axial-vector index divergence:

$$\begin{aligned} &-(k_1+k_2)^\mu R_{\sigma\rho\mu}[\zeta] \\ &\qquad =2m_0R_{\sigma\rho}+(8\pi^2-2\zeta)k_1^\xi k_2^\tau\epsilon_{\xi\tau\sigma\rho}; \end{aligned} \qquad \text{(A3)}$$

(iii) asymptotic behavior: Writing $k_1=\xi q$, $k_2=-\xi q+p'-p$, as $\xi\rightarrow\infty$

$$R_{\sigma\rho\mu}[\zeta]\rightarrow-\xi(8\pi^2-2\zeta)q^\tau\epsilon_{\tau\sigma\rho\mu}; \qquad \text{(A4)}$$

(iv) axial-vector meson to two-photon matrix element: If $l\cdot(k_1+k_2)=\epsilon_1\cdot k_1=\epsilon_2\cdot k_2=k_1^2=k_2^2=0$, then[3]

$$l^\mu\epsilon_1^\sigma\epsilon_2^\rho R_{\sigma\rho\mu}[\zeta]=\zeta l^\mu\epsilon_1^\sigma\epsilon_2^\rho(k_1-k_2)^\tau\epsilon_{\tau\sigma\rho\mu}; \qquad \text{(A5)}$$

(v) large m_0 behavior:

$$\lim_{m_0\rightarrow\infty}R_{\sigma\rho\mu}[\zeta]=\zeta\epsilon_{\tau\sigma\rho\mu}(k_1-k_2)^\tau. \qquad \text{(A6)}$$

Referring first to Eqs. (A2)–(A4), we see that when $\zeta=0$, which is the case discussed in the text, the triangle graph is gauge-invariant with respect to the photon indices but has an anomalous axial-vector Ward identity and anomalous asymptotic behavior. By contrast, when $\zeta=4\pi^2$ there is no longer gauge invariance with respect to the photon indices, but the axial-vector Ward identity and the asymptotic behavior as $\xi\rightarrow\infty$ are normal. Since the formal proof of gauge invariance for the triangle graph suffers from the same difficulties as does the formal proof of the axial-vector Ward identity, there is no *a priori* reason to demand gauge invariance with respect to the photon indices as opposed to a normal axial-vector Ward identity, or, for that matter, to demand either. In other words, as long as we consider only the divergence properties of $R_{\sigma\rho\mu}[\zeta]$, there is no requirement fixing ζ.

There are, however, two additional restrictions on $R_{\sigma\rho\mu}$ which force us to choose $\zeta=0$. First of all, we recall[14] that two real photons can never be in a state with total angular momentum 1, which means that the matrix element for an axial-vector meson to decay into two photons must vanish. In order for our triangle graph to satisfy this requirement, we must have $l^\mu\epsilon_1^\sigma\epsilon_2^\rho R_{\sigma\rho\mu}[\zeta]=0$ when l is an axial-vector meson polarization vector satisfying $l\cdot(k_1+k_2)=0$ and when the photon variables satisfy $\epsilon_1\cdot k_1=\epsilon_2\cdot k_2=k_1^2=k_2^2=0$. Referring to Eq. (A5), we see that this requirement forces us to choose $\zeta=0$. [To check that, even with the constraints on l, ϵ, etc., the expression $l^\mu\epsilon_1^\sigma\epsilon_2^\rho(k_1-k_2)^\tau\epsilon_{\tau\sigma\rho\mu}$ is in general nonvanishing, choose $k_1=(-1,1,0,0)$, $\epsilon_1=(0,0,1,0)$, $k_2=(-2,0,2,0)$, $\epsilon_2=(0,1,0,0)$, $k_1+k_2=(-3,1,2,0)$, $l=(0,0,0,1)$, $k_1-k_2=(1,1,-2,0)$.] Secondly, it is physically unreasonable that a loop diagram such as our triangle graph should influence *low-energy phenomena* in the limit as the mass of the loop fermion becomes infinite. In other words, we expect

$$\lim_{m_0\rightarrow\infty}R_{\sigma\rho\mu}[\zeta]=0, \quad k_1,\, k_2 \text{ fixed} \qquad \text{(A7)}$$

which according to Eq. (A6) again requires $\zeta=0$. Thus, *there are strong physical restrictions which uniquely select the regulator value for the triangle graph*; in particular, it is not permissible to make the choice $\zeta=4\pi^2$ which eliminates the anomalies discussed in the text.

B. Connection with Bell and Jackiw and with $\pi^0\rightarrow 2\gamma$ and $\eta\rightarrow 2\gamma$ Decay

In a recent paper, Bell and Jackiw discuss $\pi^0\rightarrow 2\gamma$ in the σ model; they find and attempt to resolve a paradox arising from the presence of triangle diagrams. We briefly summarize their work, and then discuss our own interpretation of the paradox, which differs from theirs.[15] Bell and Jackiw use a truncated version of the σ model, in which the charged pion and the neutron fields are omitted. Letting ψ, ϕ, and σ be, respectively, the fields of the proton, the neutral pion, and the scalar meson, the Lagrangian density is[8]

$$\begin{aligned} \mathfrak{L}=\bar\psi[i\gamma\cdot\Box-m_0+g_0(\sigma+i\phi\gamma_5)]\psi+\tfrac{1}{2}[(\partial\phi)^2+(\partial\sigma)^2] \\ -\tfrac{1}{2}\mu_0^2\phi^2-\tfrac{1}{2}(\mu_0^2+2\lambda_0/f_0^2)\sigma^2-\lambda_0[(\phi^2+\sigma^2)^2 \\ -2f_0^{-1}\sigma(\phi^2+\sigma^2)]-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-e_0\bar\psi\gamma_\mu\psi A^\mu, \end{aligned} \qquad \text{(A8)}$$

with the coupling constant f_0 given by

$$f_0=g_0/(2m_0). \qquad \text{(A9)}$$

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[14] C. N. Yang, Phys. Rev. **77**, 242 (1950).

[15] Our results do not contradict those of Bell and Jackiw, but rather complement them. The main point of Bell and Jackiw is that the σ model interpreted in the conventional way, does not satisfy the requirements of PCAC. Bell and Jackiw modify the σ model in such a way as to restore PCAC. We, on the other hand, stay within the conventional σ model, and try to systematize and exploit the PCAC breakdown.

The axial-vector current is

$$j_\mu{}^5(x)=\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)+2\left[\sigma(x)\frac{\partial}{\partial x^\mu}\phi(x)-\phi(x)\frac{\partial}{\partial x^\mu}\sigma(x)\right]-f_0{}^{-1}\frac{\partial}{\partial x^\mu}\phi(x),\qquad \text{(A10)}$$

and the divergence of the axial-vector current, as calculated by *formal use of the equations of motion*, is

$$\frac{\partial}{\partial x_\mu}j_\mu{}^5(x)=\frac{\mu_0{}^2}{f_0}\phi(x).\qquad \text{(A11)}$$

This is, of course, the usual operator PCAC equation.

The paradox noted by Bell and Jackiw is obtained by applying Eq. (A11) to the calculation of $\pi^0\rightarrow 2\gamma$ decay. Let us concentrate first on the left-hand side of Eq. (A11). The matrix element $\mathfrak{M}_\mu$ of the axial-vector current between the vacuum and a state with two photons has the following general structure, imposed by the requirements of Lorentz invariance, gauge invariance, and Bose statistics [cf. Eq. (17)]:

$$\begin{aligned}\mathfrak{M}_\mu&=\epsilon_1{}^\sigma\epsilon_2{}^\rho S_{\sigma\rho\mu}(k_1,k_2)\,,\\ S_{\sigma\rho\mu}(k_1,k_2)&=C_1k_1{}^\tau\epsilon_{\tau\sigma\rho\mu}+C_2k_2{}^\tau\epsilon_{\tau\sigma\rho\mu}+C_3k_{1\rho}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\sigma\mu}\\ &\quad+C_4k_{2\rho}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\sigma\mu}+C_5k_{1\sigma}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\rho\mu}\\ &\quad+C_6k_{2\sigma}k_1{}^\xi k_2{}^\tau\epsilon_{\xi\tau\rho\mu}\,,\\ C_1&=k_1\cdot k_2C_3+k_2{}^2C_4\,,\\ C_2&=k_1{}^2C_5+k_1\cdot k_2C_6\,,\\ C_3(k_1,k_2)&=-C_6(k_2,k_1)\,,\\ C_4(k_1,k_2)&=-C_5(k_2,k_1)\,.\end{aligned}\qquad \text{(A12)}$$

As in Eq. (17), k_1 and k_2 denote the photon four-momenta. The matrix element of the divergence of the axial-vector current is proportional to $(k_1+k_2)^\mu\mathfrak{M}_\mu$, and a straightforward algebraic rearrangement[3] using Eq. (A12) shows that

$$\begin{aligned}&(k_1+k_2)^\mu\epsilon_1{}^\sigma\epsilon_2{}^\rho S_{\sigma\rho\mu}(k_1,k_2)\,|_{k_1{}^2=k_2{}^2=0}\\ &\qquad=\tfrac{1}{2}(C_3-C_6)(k_1+k_2)^2k_1{}^\xi k_2{}^\tau\epsilon_1{}^\sigma\epsilon_2{}^\rho\epsilon_{\xi\tau\sigma\rho}\,.\end{aligned}\qquad \text{(A13)}$$

Thus, if we write the matrix element for $\pi^0\rightarrow 2\gamma$ in the form

$$\mathfrak{M}(\pi^0\rightarrow 2\gamma)=k_1{}^\xi k_2{}^\tau\epsilon_1{}^\sigma\epsilon_2{}^\rho\epsilon_{\xi\tau\sigma\rho}F\,,\qquad \text{(A14)}$$

then Eqs. (A11) and (A13) tell us that in the σ model (or any other PCAC model), F vanishes when the pion mass $(k_1+k_2)^2$ is extrapolated to zero. This statement, of course, must hold in each order of perturbation theory. So let us check by calculating $\mathfrak{M}(\pi^0\rightarrow 2\gamma)$ directly in the σ model in lowest-order perturbation theory, where the only diagram which contributes is the pseudoscalar coupling triangle diagram (i.e., Fig. 2 with $\gamma_\mu\gamma_5$ replaced by the pion-nucleon coupling $ig_0\gamma_5$). We find, comparing with Eqs. (19) and (20), that

$$\begin{aligned}\mathfrak{M}(\pi^0\rightarrow 2\gamma)_{\text{lowest order}}&=\frac{-ie_0{}^2}{(2\pi)^4}ig_0\epsilon_1{}^\sigma\epsilon_2{}^\rho R_{\sigma\rho}\\ &=k_1{}^\xi k_2{}^\tau\epsilon_1{}^\sigma\epsilon_2{}^\rho\epsilon_{\xi\tau\sigma\rho}(2\alpha_0/\pi)g_0m_0I_{00}(k_1,k_2)\,,\end{aligned}\qquad \text{(A15)}$$

so that

$$F_{\text{lowest order}}=\frac{2\alpha_0}{\pi}g_0m_0I_{00}(k_1,k_2)\,|_{k_1{}^2=k_2{}^2=0}\,.\qquad \text{(A16)}$$

Setting $(k_1+k_2)^2=0$ then gives

$$F_{\text{lowest order}}\,|_{(k_1+k_2)^2=0}=-\frac{\alpha_0}{\pi}\frac{g_0}{m_0}\,,\qquad \text{(A17)}$$

which *does not vanish*, contradicting the conclusion obtained indirectly from PCAC. The nonzero value of Eq. (A17) is the paradox of Bell and Jackiw.

Bell and Jackiw attempt to circumvent this contradiction by introducing a regulator nucleon field ψ_1 which is quantized with commutators rather than anticommutators. The coupling of the regulator field to the mesons is described by the interaction Lagrangian density

$$\bar{\psi}_1g_1(\sigma+i\phi\gamma_5)\psi_1\,;\qquad \text{(A18)}$$

to maintain the PCAC equation the regulator coupling and mass must satisfy the relation

$$g_1/m_1=g_0/m_0\,.\qquad \text{(A19)}$$

Thus, as the regulator mass approaches infinity, the regulator coupling to the mesons becomes infinite as well. As a consequence, even in the limit of infinite regulator mass the regulator field triangle diagram makes a contribution to the amplitude for $\pi^0\rightarrow 2\gamma$ decay,

$$F_{\text{regulator triangle diagram}}\xrightarrow[m_1\rightarrow\infty]{}\frac{\alpha_0}{\pi}\frac{g_1}{m_1}=\frac{\alpha_0}{\pi}\frac{g_0}{m_0}\,.\qquad \text{(A20)}$$

The total amplitude is the sum of Eqs. (A16) and (A20), and *does* vanish at $(k_1+k_2)^2=0$, in accord with the PCAC prediction.

Unfortunately, however, the regulator procedure of Bell and Jackiw leads to grave difficulties when we turn to purely strong interaction phenomena. Let us, in particular, consider the regulator loop contribution to the scattering of $2n$ σ particles. In the limit of large regulator mass, this loop is proportional to

$$g_1{}^{2n}\int d^4r\,\mathrm{Tr}\left\{\left[\frac{1}{r-m_1}\right]^{2n}\right\}\propto m_1{}^4\left(\frac{g_1}{m_1}\right)^{2n},\qquad \text{(A21)}$$

and thus, on account of Eq. (A19), becomes infinite as $m_1\rightarrow\infty$. This means that the regulator procedure of Bell and Jackiw introduces unrenormalizable infinities into the strong interactions in the σ model, and therefore is not satisfactory.

We now suggest a different resolution of the paradox, utilizing the ideas developed in the text.[15] As we saw, when triangle graphs are present we cannot naively use the equations of motion to calculate the divergence of the axial-vector current. Rather, we must infer the correct divergence equation from perturbation theory, which tells us that the extra term of Eq. (30) is present. In the σ model, the effect of this extra term is to replace Eq. (A11) by

$$\frac{\partial}{\partial x_\mu} j_\mu{}^5(x) = \frac{\mu_0{}^2}{f_0}\phi(x) + \frac{\alpha_0}{4\pi} F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}. \tag{A22}$$

In other words, *the PCAC equation must be modified in the presence of electromagnetic interactions.* As a result, the argument leading to the conclusion that F vanishes at $(k_1+k_2)^2=0$ must be modified. As before, we conclude that the matrix element of the left-hand side of Eq. (A22) between vacuum and two photons vanishes at $(k_1+k_2)^2=0$. But instead of implying that $\mathfrak{M}(\pi^0 \to 2\gamma)$ vanishes, this now tells us that

$$\begin{aligned}\mathfrak{M}(\pi^0 \to 2\gamma) &= Z_3{}^{-1/2} \times \text{matrix element of } (\mu^2\phi)\\ &= -\mu^2(f_0/\mu_0{}^2)Z_3{}^{-1/2} \times \text{matrix element}\\ &\qquad\qquad \text{of } [(\alpha_0/4\pi)F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}]\\ &= \frac{\mu^2}{\mu_0{}^2} Z_3{}^{-1/2}\left(-\frac{\alpha}{\pi}\frac{g_0}{m_0}\right) k_1{}^\xi k_2{}^\tau \epsilon_1{}^\sigma \epsilon_2{}^\rho \epsilon_{\xi\tau\sigma\rho}; \end{aligned} \tag{A23}$$

in other words,

$$F|_{(k_1+k_2)^2=0} = \frac{\mu^2}{\mu_0{}^2} Z_3{}^{-1/2}\left(-\frac{\alpha}{\pi}\frac{g_0}{m_0}\right). \tag{A24}$$

[In Eqs. (A23) and (A24), Z_3 is the π^0 wave-function renormalization constant.] To lowest order in perturbation theory, Eq. (A24) agrees with Eq. (A17), so our modified PCAC equation leads to no paradox. In addition, Eq. (A22) yields a bonus: From the derivation of Eq. (A24) it is clear that Eq. (A24) is not just a lowest-order perturbation theory result, but in fact is an *exact* statement in the σ model. We can reexpress Eq. (A24) in terms of physical quantities using the equation[16]

$$\frac{g_0}{m_0}\frac{\mu^2}{\mu_0{}^2} Z_3{}^{-1/2} = \frac{g_r(0)}{m_N}\frac{1}{g_A}, \tag{A25}$$

where m_N, $g_r(0)$, g_A are, respectively, the renormalized nucleon mass, the renormalized pion-nucleon coupling constant (evaluated at pion mass zero), and the nucleon axial-vector coupling constant in the σ model. Thus Eq. (A24) becomes

$$F|_{(k_1+k_2)^2=0} = -\frac{\alpha}{\pi}\frac{g_r(0)}{m_N g_A}. \tag{A26}$$

Let us now make the standard PCAC assumption that F is slowly varying as the pion mass $(k_1+k_2)^2$ is varied from μ^2 to 0, so that we can use Eq. (A26) for the physical π^0-decay matrix element. We also replace $g_r(0)$ by the on-shell coupling constant g_r. Using the physical values for μ, m_N, g_r, g_A,[17] we find for the pion lifetime

$$\tau^{-1} = (\mu^3/64\pi)F^2 = 9.7\ \text{eV}, \tag{A27}$$

in good agreement with the experimental value[18]

$$\begin{aligned}\tau_{\text{expt}}{}^{-1} &= (1.12 \pm 0.22) \times 10^{16}\ \text{sec}^{-1}\\ &= (7.37 \pm 1.5)\ \text{eV}.\end{aligned} \tag{A28}$$

So we see that the σ model, as interpreted with Eq. (A22), gives a reasonable account of $\pi^0 \to 2\gamma$ decay.[19] This also makes it clear that the use of regulators to cancel away the triangle graph contribution to F up to terms of order $\mu^2/m_N{}^2$ will tend to give much too small a value for the $\pi^0 \to 2\gamma$ matrix element.

The above ideas are readily extended to other field theoretical models, and hopefully, to the physical axial-vector current as well. Let $\mathfrak{F}_3{}^{5\lambda}$ be the third component of the axial-vector octet. (It corresponds to $\frac{1}{2}j^{5\lambda}$ in the model discussed above.) Let us suppose that the world is really described by a field theory, and that there are only spin-0 or spin-$\frac{1}{2}$ elementary fields.[20] We then make the following two assumptions:

(i) The usual PCAC equation,

$$\frac{\partial}{\partial x^\lambda}\mathfrak{F}_3{}^{5\lambda} = C_\pi \mu^2 \phi_{\pi^0}, \quad C_\pi = m_N g_A / g_r(0), \tag{A29}$$

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[16] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[17] We take $g_r \approx 13.4$, $g_A \approx 1.18$. If we used $g_A \approx 1.24$, then we would get $\tau^{-1} = 8.9$ eV. We can also evaluate Eq. (A26) by using the relation $g_r(0)/(m_N g_A) = \sqrt{2}\mu_+{}^2/f_\pi$, with f_π the charged-pion decay amplitude and μ_+ the charged-pion mass. (See S. L. Adler and R. F. Dashen, Ref. 11, pp. 41–45.) This gives $F|_{(k_1+k_2)^2=0} = -(\alpha/\pi)\sqrt{2}\mu_+{}^2/f_\pi$. Using the experimental value $f_\pi \approx 0.96\ \mu_+{}^3$, we find from Eq. (A27) that $\tau^{-1} = 7.4$ eV.

[18] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[19] Comparing Eqs. (A26) and (A17), we see that apart from a factor of $g_A{}^{-2}$, our PCAC expression for the π^0 lifetime is the same as the expression obtained from the pseudoscalar coupling triangle graph if one uses the physical nucleon mass and pion-nucleon coupling rather than the bare mass and coupling appearing in Eq. (A17). That the triangle graph, evaluated using physical quantities, gives a good value for $\pi^0 \to 2\gamma$ decay has been noted by J. Steinberger, Phys. Rev. **76**, 1180 (1949); and J. Steinberger (private communication).

[20] This assumption is not strictly necessary for the calculation of the $\pi^0 \to 2\gamma$ rate. If there is also a single elementary neutral vector-meson field B^λ, then there will be an additional term in Eq. (A30) proportional to $F^{\xi\sigma}\partial B^\tau/\partial x_\rho \epsilon_{\xi\sigma\tau\rho}$. However, because the gauge-invariant coupling of a massive vector boson to a physical photon vanishes [G. T. Feldman and P. T. Matthews, Phys. Rev. **132**, 823 (1963)], this term makes no contribution to the physical $\pi^0 \to 2\gamma$ decay. In general, there will be no change in the $\pi^0 \to 2\gamma$ prediction if *only* isoscalar vector mesons or *only* isovector vector mesons are present. If *both* isoscalar and isovector vector meson$_\sigma$ are present, there will be additional terms like $\partial B^\xi(I=1)/\partial x s$ $\partial B^\tau(I=0)/\partial x_\rho \epsilon_{\xi\sigma\tau\rho}$, which do affect the $\pi^0 \to 2\gamma$ prediction.

should, on account of triangle graphs, be replaced by

$$\frac{\partial}{\partial x^{\lambda}}\mathfrak{F}_3{}^{5\lambda}=C_{\pi}\mu^2\phi_{\pi^0}+S\frac{\alpha_0}{4\pi}F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}, \tag{A30}$$

with S a constant.[21]

(ii) If $\mathfrak{F}_3{}^{5\lambda}$ is expressed in terms of the elementary fields by

$$\mathfrak{F}_3{}^{5\lambda}=\sum_j g_j\bar{\psi}_j\gamma^{\lambda}\gamma^5\psi_j+\text{meson terms}, \tag{A31}$$

then S is given by

$$S=\sum_j g_jQ_j{}^2, \tag{A32}$$

where the charge of the jth fermion is Q_je_0. Equation (A32) means that we count only triangle graphs of the elementary fermions, but do not include triangles involving nonelementary bound states. It may be possible to decide in model calculations whether this rule, which we conjecture, is really correct.

Using Eq. (A30) to calculate the $\pi^0\rightarrow 2\gamma$ matrix element then gives

$$F\approx-(\alpha/\pi)2S(g_r/m_Ng_A). \tag{A33}$$

The experimentally measured π^0 lifetime corresponds[22] to $|S|=0.44$; for comparison, S in the σ model is $\frac{1}{2}1^2-\frac{1}{2}0^2=\frac{1}{2}$, while S in the quark model is $\frac{1}{2}(\frac{2}{3})^2-\frac{1}{2}(-\frac{1}{3})^2=\frac{1}{6}$. More generally, in any triplet model in which the electromagnetic current is a U-spin singlet, the triplet charges will be $(Q_p,Q_n,Q_{\lambda})=(Q,Q-1,Q-1)$ and we have $S=\frac{1}{2}Q^2-\frac{1}{2}(Q-1)^2=Q-\frac{1}{2}$. That is, in triplet models we have $S=\langle Q\rangle_{\text{av}}$, where $\langle Q\rangle_{\text{av}}$ is the average charge of the triplet particles taking part in both the $\Delta S=0$ weak $V-A$ current and the $|\Delta S|=1$ weak $V-A$ current. This means that the condition $\langle Q\rangle_{\text{av}}=-\frac{1}{2}$, necessary[23] for the radiative corrections to the $\Delta S=0$ and $|\Delta S|=1$ weak currents to be finite, also predicts a $\pi^0\rightarrow 2\gamma$ rate in good accord with experiment.[24]

The two-photon decay $\eta\rightarrow 2\gamma$ can be treated in a similar manner. The analog of Eq. (A30) for $\mathfrak{F}_8{}^{5\lambda}$ is

$$\frac{\partial}{\partial x^{\lambda}}\mathfrak{F}_8{}^{5\lambda}=C_{\eta}\mu_{\eta}{}^2\phi_{\eta}+\frac{1}{\sqrt{3}}S\frac{\alpha_0}{4\pi}F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}, \tag{A34}$$

where S is the same constant as in Eq. (A30) and where the factor $3^{-1/2}$ appears because the electromagnetic current is a U-spin singlet.[25] If there were no $\eta - X^0$ mixing, then ϕ_{η} would be the η field; in the presence of mixing, ϕ_{η} would be a mixture of the η and X^0 fields. In the SU_3 limit, one has, of course, $C_{\eta}=C_{\pi}$. To get a prediction for the $\eta\rightarrow 2\gamma$ rate from Eq. (A34), we sandwich Eq. (A34) between the η state and a two-photon state and make the following three approximations: (i) We neglect $\eta-X^0$ mixing; (ii) we take $C_{\eta}=C_{\pi}$; (iii) we ngelect the left-hand side of Eq. (A34), which makes a contribution of order $\mu_{\eta}{}^2$ [equivalently, we assume that the *exact* prediction $F_{\eta}(\mu_{\eta}{}^2=0)=-(\alpha/\pi)\times(2S/\sqrt{3})(1/C_{\eta})$ can be smoothly extrapolated from $\mu_{\eta}{}^2=0$ to the physical η mass]. These approximations give the standard SU_3 prediction[26]

$$\Gamma(\eta\rightarrow 2\gamma)=\tfrac{1}{3}(\mu_{\eta}/\mu)^3\Gamma(\pi^0\rightarrow 2\gamma)=(165\pm 34)\text{ eV}, \tag{A35}$$

about a factor of 8 smaller than the experimental value of

$$\Gamma(\eta\rightarrow 2\gamma)=(1210\pm 260)\text{ eV}. \tag{A36}$$

In view of the approximations made, the discrepancy is not too disturbing; in particular, the terms of order $\mu_{\eta}{}^2$ are by no means negligible, and could easily make a contribution to the $\eta\rightarrow 2\gamma$ matrix element as important as the $S/\sqrt{3}$ term which we have retained.[27]

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[21] In Eq. (A30), $\phi_{\pi0}$ does not necessarily mean a canonical pion field, but only a suitable interpolating field for the pion. For example, in the quark model, $\phi_{\pi0}$ would be proportional to $\bar{\psi}\gamma_5\tau_3\psi$. The separation of $\partial_{\lambda}\mathfrak{F}_3{}^{5\lambda}$ into two terms in Eq. (A30) is made unique by the requirement that $\phi_{\pi0}$ and the photon field be *dynamically independent*, in the sense that $[\phi_{\pi0},A_{\lambda}]=[\phi_{\pi0},\dot{A}_{\lambda}]=0$ at equal times.

[22] If we use instead of Eq. (A33) the formula $F\approx-(\alpha/\pi)(2S)\times(\sqrt{2}\mu_+{}^2/f_{\pi})$, as in Ref. 17, then the experimentally measured π^0 lifetime gives $|S|=0.50$.

[23] N. Cabibbo, L. Maiani, and G. Preparata, Phys. Letters **25B**, 132 (1967); K. Johnson, F. Low, and H. Suura, Phys. Rev. Letters **18**, 224 (1967).

[24] This result was noted previously, in the context of the vector dominance model, by N. Cabibbo, L. Maiani, and G. Preparata, Phys. Letters **25B**, 31 (1967).

[25] The correctness of the factor $1/\sqrt{3}$ is easily verified in the triplet model.

[26] The factor $(\mu_{\eta}/\mu)^3$ comes from phase space.

[27] We discuss briefly two other electromagnetic decays to which current algebra methods have been applied: $\omega\rightarrow\pi^0\gamma$ and $\eta\rightarrow 3\pi$. In the case of $\omega\rightarrow\pi^0\gamma$ it has been argued by D. G. Sutherland [Nucl. Phys. **B2**, 433 (1967)] that the usual PCAC equation [Eq. (A11)] implies vanishing of the decay amplitude at zero π^0 four-momentum. This conclusion, however, is erroneous, and results from the use by Sutherland of an insufficiently general form for the axial-vector-current–vector-meson–photon vertex. The most general such vertex is given by Eq. (A12); an examination of the reasoning leading to Eq. (A13) shows that Eq. (A13) is valid *only* when $k_1{}^2=k_2{}^2=0$. When one of the vectors is massive, as in the case of ω decay, we find instead that

$$(k_1+k_2)^{\mu}\epsilon_1{}^{\sigma}\epsilon_2{}^{\rho}S_{\sigma\rho\mu}(k_1,k_2)\big|_{(k_1+k_2)^2=k_1{}^2=0} =[C_4+C_6-\tfrac{1}{2}(C_3+C_6)]k_2{}^2k_1{}^{\xi}k_2{}^{\tau}\epsilon_1{}^{\sigma}\epsilon_2{}^{\rho}\epsilon_{\xi\tau\sigma\rho}\neq 0,$$

contradicting Sutherland's conclusion. This equation also means that our modified PCAC prediction for $\pi^0\rightarrow 2\gamma$ will be altered when one of the photons is virtual, as is the case in the Primakoff effect.

In the decay $\eta\rightarrow 3\pi$, the only point which we wish to make is that the triangle graphs which we have considered (involving either photons or strongly interacting vector mesons) cannot alter the usual PCAC predictions. The reason is the presence in all matrix elements coming from our extra term of the factor $k_1{}^{\xi}k_2{}^{\tau}\times\epsilon_1{}^{\sigma}\epsilon_2{}^{\rho}\epsilon_{\xi\tau\sigma\rho}$, which vanishes at zero four-momentum for the axial-vector vertex. (In the $\pi^0\rightarrow 2\gamma$ case we were always talking about the matrix element left after removal of this factor.)