to the massless case; otherwise the longitudinal amplitudes become infinite as $m_\gamma \to 0$, since the longitudinal polarization vector $\epsilon_\mu^{(0)}(k)=(k,k_0\mathbf{k}/k)/m_\gamma$ blows up in this limit. This fact has led to the conclusion, that it is absolutely necessary to impose gauge invariance when dealing with zero mass. This is not so. We can see this by looking at the unitary matrix, which connects the helicity amplitudes for different Lorentz frames. This matrix depends on the Wigner rotation for helicities $W^H(\Lambda,k)=R^{-1}(\Lambda k)B^{-1}(\Lambda k)\Lambda B(k)R(k)$, where $R(k)$ is a rotation which takes the z axis into the unit vector $\mathbf{k}/k$ and $B(k)=(k\cdot\sigma/m)^{1/2}$ is a boost which takes the vector $(m,0,0,0)$ into $k=(k_0,k_1,k_2,k_3)$. The Wigner rotations $W^H(\Lambda,k)$ go to a finite diagonal matrix as $m\to 0$, even though the individual boosts $B(k)$ and $B^{-1}(\Lambda k)$ go to infinity. Consequently, *the helicity amplitudes can have finite values even though gauge invariance is not imposed.* Our *KF* K_h^0 stay the same, independent of whether the longitudinal amplitudes vanish or not—as long as they are finite. This follows since the components of the crossing matrix which connect the transverse s-channel amplitudes to the longitudinal t-channel amplitudes vanish for $m_\gamma=0$. Therefore, guage invariance is not responsible for the discontinuities of the *KF*. We need gauge invariance for massless particles only if we use the usual invariant-amplitude decomposition; the problem lies with the decomposition itself, since it is not valid for the general case and it is not suitable for describing massless vector particles except for the special case when gauge invariance is assumed.

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Some Considerations on Nonlinear Realizations of Chiral $SU(3)\times SU(3)$

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We consider how nonlinear realizations of the chiral $SU(3)\times SU(3)$ symmetry arise in certain limits of models in which fields transform linearly under the group of transformations which leave the Lagrangian (approximately) invariant. All possible cases of nonlinear realizations of the chiral $SU(3)\times SU(3)$, wherein at least the $U(2)$ group generated by isospin and hypercharge is linearly represented, are considered. The connection between the nonlinear realizations and the spontaneous breakdown of the symmetry is made clear. Our results are clarified from a group-theoretic point of view, thereby establishing the connection between our results and a more general consideration of Coleman *et al.* We discuss theoretical and experimental implications of nonlinar realizations of the chiral $SU(3)\times SU(3)$, with reference to the κ meson.

I. INTRODUCTION

In this paper we shall consider various nonlinear realizations[1] of the chiral $SU(3)\times SU(3)$ symmetry in the context of a dynamical model. The problem of realizing a symmetry group G in the representation spaces of its subgroup has been discussed in generality by Coleman, Wess, and Zumino[2]; the special cases of realizing $SU(2)\times SU(2)$ [or $SU(3)\times SU(3)$] in the representation spaces of its subgroup $SU(2)$ [or $SU(3)$] have attracted special attention and have been discussed by a number of authors.[3]

Our aim in studying the nonlinear realizations of the chiral $SU(3)\times SU(3)$ symmetry is to establish, in the context of a dynamical model, how nonlinear realizations of the symmetry arise as a limit of a model in which fields transform linearly under the group of transformations which leave the Lagrangian (approximately) invariant. To this end, we shall study a generalized version of the $SU(3)\times SU(3)$ σ model[4,5] in which the full symmetry is broken in a way suggested by the validity of the hypothesis of partially conserved axial-vector currents (PCAC) and the Gell-Mann–Okubo mass formula. It will suffice, in establishing the connection between linear and nonlinear models, to consider a system consisting solely of scalar and pseudoscalar mesons. We shall show that a nonlinear model arises when the full symmetry (neglecting the symmetry-breaking term) is

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[1] We shall use the term "nonlinear realizations" in the sense of S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[2] S. Coleman, J. Wess, and B. Zumino, this issue, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).

[3] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967); J. Schwinger, Phys. Letters **24B**, 473(1967); J. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967); W. A. Bardeen and B. W. Lee, in *Nuclear and Particle Physics* edited by B. Margolis and C. S. Lam (W. A. Benjamin, Inc., New York, 1968), p. 273; L. S. Brown, Phys. Rev. **163**, 1802 (1967); P. Chang and F. Gürsey, *ibid.* **164**, 1752 (1964); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968); J. A. Cronin, *ibid.* **161**, 1483 (1967); R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967).

[4] The original σ model was studied by J. Schwinger, Ann. Phys. (N. Y.) **2**, 407 (1958); J. C. Polkinghorne, Nuovo Cimento **8**, 179 (1958); **8**, 781 (1958); M. Gell-Mann and M. Lévy, *ibid.* **16**, 705 (1960).

[5] The $SU(3)$ σ model was first studied by M. Lévy, Nuovo Cimento **52**, 23 (1967). See also S. Gasiorowicz and D. Geffen, Argonne Lecture Notes (unpublished); P. Mitter and S. Swank (to be published); D. J. Majumdar (to be published).

spontaneously broken,[6] and the $SU(3)\times SU(3)$ partners of the Goldstone bosons acquire infinite mass. The "would-be" Goldstone bosons acquire finite masses only because of the *intrinsic* symmetry breaking. Thus in an $SU(3)\times SU(3)$ σ model as considered by Lévy, a nonlinear model which contains only an octet of pseudoscalar mesons arises if the $SU(3)\times SU(3)$ symmetry is broken down spontaneously to the $SU(3)$ and in the limit in which the singlet pseudoscalar meson and the nonet of scalar mesons acquire infinite masses. The Goldstone bosons, then, undergo nonlinear transformations under the full group, and other fields may be made to form the bases of nonlinear realizations of the group. Thus the physical content of these models is precisely that envisaged by Nambu and collaborators[7] some time ago in connection with the PCAC and the spontaneous breakdown of the chiral symmetry.

In our study, we shall consider all possible cases of the spontaneous breakdowns of the chiral $SU(3)\times SU(3)$ symmetry, in which at least the $U(2)$ symmetry spanned by the isospin and hypercharge generators is preserved.

To some extent, the intent of this paper is pedagogical in that the nonlinear model of the chiral symmetry can be studied *per se*, without recourse to linear models. However, it may be that the nonlinear transformation properties of pions, and possibly kaons and κ mesons, in nature, may be merely low-energy manifestations of what is basically a linear realization of the chiral symmetry in the presence of a large symmetry breaking—be it spontaneous or intrinsic. The main purpose of this paper is to demonstrate that the second viewpoint is a tenable one, and to suggest that at the level of phenomenological description, the nonlinear model may be regarded simply as a device to suppress any reference to the degrees of freedom that are hard to excite at low energies.

The plan of this paper is as follows. In Sec. II we discuss the usual $SU(2)$ σ model and its connection with the nonlinear model of chiral symmetry. Nothing really new is presented in this section, but it is believed that our views and methods may best be illustrated in a familiar context. In Sec. III, we extend the same consideration to an $SU(3)$ version of the σ model. It is shown here that various nonlinear realizations of the $SU(3)$ symmetry arise in the limit as the masses of non-Goldstone bosons tend to infinity. This result is further clarified from a group-theoretic point of view in Sec. IV, thereby establishing the connection between our result and the more general consideration of Coleman *et al.*[2] In Sec. V we discuss theoretical and experimental implications of nonlinear realizations of the chiral $SU(3)\times SU(3)$, with reference to the possible existence (or absence) of the κ meson.

The ensuing discussions in Secs. II and III are to be understood in the context of a field theory in which one retains only tree diagrams.[8]

II. NONLINEAR LIMIT OF THE $SU(2)$ σ MODEL

As a preliminary to the ensuing discussions, we shall elaborate on the connection between the linear $SU(2)$ σ model of Schwinger[3] and of Gell-Mann and Lévy[3] and the nonlinear model.[9] This connection is in fact well known, and forms the basis of Weinberg's derivation of the chiral-dynamics Lagrangian. It is believed, however, that the inclusion of this topic here will serve as an illustration of our viewpoint adopted, and as the basis of certain assertions made in the later discussions. For the sake of lucidity, we shall ignore the coupling of mesons to nucleons, since this is irrelevant to our discussion.

The Lagrangian that we have in mind is

$$L=\tfrac{1}{2}(\partial\sigma)^2+\tfrac{1}{2}(\partial\boldsymbol{\pi})^2-\tfrac{1}{2}\mu_0^2(\sigma^2+\boldsymbol{\pi}^2)+\tfrac{1}{4}\lambda(\sigma^2+\boldsymbol{\pi}^2)^2+f_\pi\mu_\pi^2\sigma\,, \quad (1)$$

whence f_π and μ_π are, respectively, the pion decay constant and the pion mass. Save for the last term, the Lagrangian is invariant under the chiral $SU(2)\times SU(2)$ transformations, the fields

$$M^\alpha{}_{\dot\beta}\equiv\tfrac{1}{2}\sqrt{2}(\sigma 1+i\boldsymbol{\tau}\cdot\boldsymbol{\pi})_{\alpha\beta}\,,\quad M^{\dot\alpha}{}_\beta\equiv\tfrac{1}{2}\sqrt{2}(\sigma 1-i\boldsymbol{\tau}\cdot\boldsymbol{\pi})_{\alpha\beta} \quad (2)$$

transforming, respectively, as the $(2,\bar{2})$ and $(\bar{2},2)$ representations of the of the chiral group. (Of course, these two fields are unitarily equivalent.) In terms of these chiral fields $M^\alpha{}_{\dot\beta}$ and $M^{\dot\alpha}{}_\beta$, the Lagrangian (1) may be rewritten as

$$L=\tfrac{1}{2}\,\mathrm{Tr}\partial_\mu M^\dagger\partial^\mu M-\tfrac{1}{2}\mu_0^2\,\mathrm{Tr}M^\dagger M+\tfrac{1}{4}\lambda\,\mathrm{Tr}(M^\dagger M)^2+(1/2\sqrt{2})f_\pi\mu_\pi^2\,\mathrm{Tr}(M+M^\dagger)\,, \quad (3)$$

where $(M)_{\alpha\beta}=M^\alpha{}_{\dot\beta}$ and $(M^\dagger)_{\alpha\beta}=M^{\dot\alpha}{}_\beta$.

The Lagrangian (1) has a term linear in σ, which allows the transition $\sigma\rightarrow$ vacuum. By translating the σ field by a constant c number, we define a new scalar field s whose vacuum expectation value vanishes[10]:

$$s=\sigma-f_\pi\,. \quad (4)$$

In terms of the s field, the Lagrangian may be written as

$$L=\tfrac{1}{2}(\partial s)^2-\tfrac{1}{2}\mu_\sigma^2 s^2+\tfrac{1}{2}(\partial\boldsymbol{\pi})^2-\tfrac{1}{2}\mu_\pi^2\boldsymbol{\pi}^2-\frac{\mu_\sigma^2-\mu_\pi^2}{2f_\pi}s(s^2+\boldsymbol{\pi}^2)-\frac{\mu_\sigma^2-\mu_\pi^2}{8f_\pi^2}(s^2+\boldsymbol{\pi}^2)^2\,, \quad (5)$$

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[6] For a survey of this subject, see T. W. B. Kibble, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (John Wiley & Sons, Inc., New York, 1967), p. 277.

[7] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962); Y. Nambu and G. Jona-Lasinio, *ibid.* **122**, 345 (1961); **124**, 246 (1961); Y. Nambu and E. Shrauner *ibid.* **128**, 862 (1962); Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **11**, 42 (1963).

[8] B. W. Lee and H. T. Nieh (Ref. 3); S. Coleman *et al.* (Ref. 2); Y. Nambu, Phys. Letters **26B**, 626 (1968); D. G. Boulware and L. S. Brown, Phys. Rev. **172**, 1628 (1968).

[9] See, in particular, F. Gürsey, Nuovo Cimento **16**, 705 (1960); Ann. Phys. (N. Y.) **12**, 91 (1961); M. Gell-Mann and M. Lévy (Ref. 4); S. Weinberg (Ref. 3); W. A. Bardeen and B. W. Lee (Ref. 3).

[10] The constant f_π is the pion decay constant. See, for example, W. A. Bardeen and B. W. Lee (Ref. 3).

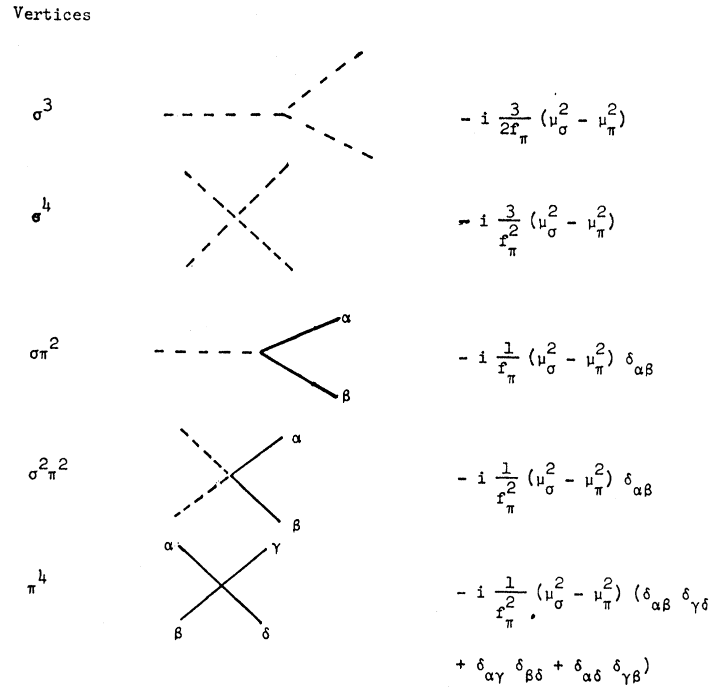

FIG. 1. Feynman rules for Lagrangian (5). Dotted lines are σ lines; undotted ones are π lines. The greek indices α, β, $\cdots$, are isospin labels of pions.

where

$$\mu_\pi^2=\mu_0^2-\lambda f_\pi^2\,,\qquad \mu_\sigma^2=\mu_0^2-3\lambda f_\pi^2\,. \tag{6}$$

The limit in which $\mu_\sigma^2\to\infty$ while μ_π^2 remains finite is to be obtained and is equivalent to $\lambda\to-\infty$ and $\mu_0^2\to-\infty$ in such a way that $\mu_0^2-\lambda f_\pi^2$ is positive and finite. The equation of motion of the s field is

$$(\partial^2+\mu_\sigma^2)s=-\frac{\mu_\sigma^2-\mu_\pi^2}{2f_\pi}(3s^2+\boldsymbol{\pi}^2)-\frac{\mu_\sigma^2-\mu_\pi^2}{2f_\pi^2}s(s^2+\boldsymbol{\pi}^2)\,. \tag{7}$$

In the limit $\mu_\sigma^2\to\infty$, the "kinetic energy" term $\partial^2 s$ may be neglected in Eq. (7) (see below). Thus, in this limit, we have

$$-2f_\pi^2 s=f_\pi(3s^2+\boldsymbol{\pi}^2)+s(s^2+\boldsymbol{\pi}^2)$$

or

$$(s+f_\pi)[(s+f_\pi)^2+\boldsymbol{\pi}^2-f_\pi^2]=0\,. \tag{8}$$

The solution $\sigma=s+f_\pi=0$ of Eq. (8) corresponds to the (negative) infinite pion mass [see Eq. (1)] and is physically unacceptable. The other solution,

$$\sigma^2=f_\pi^2-\boldsymbol{\pi}^2\,, \tag{9}$$

when substituted in Eq. (1), leads to the standard form of the nonlinear σ model. We recall that this is the unique nonlinear Lagrangian,[11] which is at most quadratic in the derivatives of pion fields and in which the divergence of the axial-vector current transforms like $(2,\bar{2})+(\bar{2},2)$ under the chiral algebra.

[11] The uniqueness of the $SU(2)$ chiral-invariant Lagrangian is shown in L. S. Brown (Ref. 3); W. A. Bardeen and B. W. Lee (Ref. 3); S. Weinberg, Phys. Rev. **166**, 1568 (1968).

The amplitudes that one computes from the nonlinear Lagrangian

$$L_{\mathrm{NL}}=\tfrac{1}{2}[(\partial\sigma)^2+(\partial\boldsymbol{\pi})^2]+f_\pi\mu_\pi^2\sigma\,,$$

where

$$\sigma=(f_\pi^2-\boldsymbol{\pi}^2)^{1/2}\,, \tag{10}$$

correspond to the $\mu_\sigma^2\to\infty$ limit of the amplitudes that one constructs from the Lagrangian (5). The Feynman rules for the Lagrangian (5) are summarized in Fig. 1. As $\mu_\sigma\to\infty$, we need only consider pion external lines; the σ lines appear only as internal lines. A consistent perturbation procedure is obtained if one expands the T matrix in powers of f_π^{-1} and expands the σ propagator $i(k^2-\mu_\sigma^2)^{-1}$ in powers of k^2:

$$\frac{-i}{\mu_\sigma^2-k^2}=-i\left(\frac{1}{\mu_\sigma^2}+\frac{k^2}{\mu_\sigma^4}+\cdots\right). \tag{11}$$

[Here we see that we must assume that $|k^2|\ll\mu_\sigma^2$. A similar assumption is made in deriving Eq. (8).] It is not very difficult to see that in any tree diagram[8] the first term in Eq. (11) gives rise to a term in the T matrix diverging as μ_σ^2, but this divergent term is cancelled by the similar, diverging contact term. The second term gives rise to a finite term proportional to k^2. [Similarly, the Lagrangian (10) contains terms quadratic in derivatives of field, and no higher.] The rest of the terms on the right-hand side of Eq. (11) do not contribute in the limit $\mu_\sigma^2\to\infty$. We see also why the Lagrangian (10) is not renormalizable (in the usual sense), while the Lagrangian (5) is; the Lagrangian (10) corresponds to the limit of (5) in which the σ propagator is replaced with $-i(\mu_\sigma^{-2}+k^2\mu_\sigma^{-4})$, and μ_σ^2 is let go to infinity, and becomes infinite; clearly loop diagrams involving σ lines do not converge in this limit.

As an illustration, we consider the $\pi\pi$ scattering: $\pi(k_1,\alpha)+\pi(k_2,\beta)\rightarrow\pi(k_3,\gamma)+\pi(k_4,\delta)$. The σ pole terms give

$$(-i)^3\left[\delta_{\alpha\beta}\delta_{\gamma\delta}\left(\frac{\mu_\sigma^2-\mu_\pi^2}{f_\pi}\right)^2\times\left(\frac{1}{\mu_\sigma^2}+\frac{(k_1+k_2)^2}{\mu_\sigma^2}+\cdots\right)+\text{permutations}\right],$$

while the contact term gives

$$(-i)\left(\delta_{\alpha\beta}\delta_{\gamma\delta}\frac{\mu_\sigma^2-\mu_\pi^2}{f_\pi^2}+\text{permutations}\right).$$

The sum of these two terms is

$$\delta_{\alpha\beta}\delta_{\gamma\delta}\left(i\frac{\mu_\sigma^2}{f_\pi^2}+i\frac{(k_1+k_2)^2}{f_\pi^2}-i\frac{\mu_\sigma^2-\mu_\pi^2}{f_\pi^2}\right)+\text{permutations}$$
$$=(i/f_\pi^2)[\delta_{\alpha\beta}\delta_{\gamma\delta}(s-\mu_\pi^2)+\delta_{\alpha\gamma}\delta_{\beta\delta}(t-\mu_\pi^2)+\delta_{\alpha\delta}\delta_{\beta\gamma}(u-\mu_\pi^2)]$$

in the $\mu_\sigma^2\rightarrow\infty$ limit and agrees with the T matrix obtained from Eq. (10).

The nonlinear Lagrangian (10) may also be obtained by the following device. We perform a nonlinear canonical transformation on the fields M in Eq. (3). [We call a nonlinear transformation of fields $x\rightarrow y=f(x)$ canonical[12] if $f(x)$ is formally expandable in powers of x, with $f(0)=0$, $f'(0)=df/dx|_{x=0}=1$, and the inverse transformation exists. In the present context, the fields x and y are understood to have zero vacuum expectation values.] We consider the canonical transformation $(s,\boldsymbol{\pi})\leftrightarrow(\phi,\mathbf{p})$ given by

$$\begin{aligned}M&=\tfrac{1}{2}\sqrt{2}(f_\pi+s+i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\\&=\tfrac{1}{2}\sqrt{2}(f_\pi+\phi)e^{i\mathbf{p}\cdot\boldsymbol{\tau}/f_\pi},\\M^\dagger&=\tfrac{1}{2}\sqrt{2}(f_\pi+\phi)e^{-i\mathbf{p}\cdot\boldsymbol{\tau}/f_\pi}.\end{aligned}\tag{12}$$

It is easy to see that for $f_\pi\neq0$ this transformation is canonical in the sense described above. On substituting Eq. (12) in Eq. (3) we obtain a new form of the Lagrangian in terms of ϕ and $\mathbf{p}$:

$$L=\tfrac{1}{2}\operatorname{Tr}(\partial_\mu M^\dagger)(\partial^\mu M)-\tfrac{1}{2}\mu_0^2(f_\pi+\phi)^2+\tfrac{1}{4}\lambda(f_\pi+\phi)^4+\tfrac{1}{2}f_\pi\mu_\pi^2(f_\pi+\phi)\operatorname{Tr}\cos(\boldsymbol{\varrho}\cdot\boldsymbol{\tau}/f_\pi).\tag{13}$$

In this form of the Lagrangian, the terms proportional to μ_0^2 and λ are now functions only of ϕ. The constant f_π is determined from the condition that there be no term linear in ϕ in Eq. (13). This gives

$$-\mu_0^2f_\pi+\lambda f_\pi^3+f_\pi\mu_\pi^2=0.\tag{14}$$

[12] This definition is tacitly used in W. A. Bardeen and B. W. Lee (Ref. 3).

The mass of the scalar particle is given by $\mu_0^2-3\lambda f_\pi^2$, as before. The solution $f_\pi=0$ of Eq. (14) is to be rejected, since the transformation in Eq. (12) would no longer be canonical. The other two solutions are physically equivalent, being

$$f_\pi^2=(\mu_0^2-\mu_\pi^2)/\lambda.$$

In the limit as μ_0^2, $\lambda\rightarrow-\infty$ and μ_0^2/λ fixed, we obtain

$$f_\pi^2=\mu_0^2/\lambda.\tag{15}$$

Since the mass of the scalar excitation goes to infinity, reference to the scalar field ϕ may be dropped in Eqs. (12) and (13). We thereby obtain

$$M=f_\pi e^{i\mathbf{p}\cdot\boldsymbol{\tau}/f_\pi}/\sqrt{2}.\tag{16}$$

Equation (16) is equivalent to the one that follows from Eq. (9):

$$M=[(f_\pi^2-\boldsymbol{\pi}^2)^{1/2}+i\boldsymbol{\pi}\cdot\boldsymbol{\tau}]/\sqrt{2},\tag{17}$$

in that the fields $\boldsymbol{\pi}$ and $\mathbf{p}$ are related by a canonical transformation. Note also that, in either case,

$$MM^\dagger=M^\dagger M=\tfrac{1}{2}f_\pi^2.$$

As shown elsewhere,[2,8] all these equivalent forms of M give rise to the same T matrix, when all external lines are put on the mass shell.

Note that if μ_0^2 and λ were finite, then Eq. (14) would be precisely the condition for the spontaneous breakdown of the chiral symmetry.[6,7] In this case, σ can have a nonvanishing vacuum expectation value even in the absence of the symmetry-breaking term $f_\pi\mu_\pi^2\sigma$ of Eq. (1). Equation (15) would imply $\mu_\pi^2=0$ [see Eq. (6)], and the pion would be the Goldstone boson of zero mass. In our case, Eq. (15) is true only asymptotically as $\mu_\sigma^2\rightarrow0$, so that pions can have a finite mass. Nonetheless, pions do behave much like Goldstone bosons in this theory, as suggested first by Nambu.[7,13]

III. $SU(3)$ σ MODEL

We consider the $SU(3)$ version of the σ model first studied by Lévy.[5] In this model, nonets of the scalar and pseudoscalar particles are assigned to the $(3,\bar{3})+(\bar{3},3)$ representation of the chiral $SU(3)\times SU(3)$. It is convenient to denote the meson fields by

$$M^{\alpha}{}_{\dot\beta}=(\Sigma+i\Pi)_{\alpha\beta}\equiv(M)_{\alpha\beta},\tag{18}$$

which transform like $(3,\bar{3})$, and

$$M^{\dot\alpha}{}_{\beta}=(\Sigma-i\Pi)_{\alpha\beta}\equiv(M^\dagger)_{\alpha\beta},\tag{18'}$$

which transform like $(\bar{3},3)$. In Eqs. (18) and (18′), Σ and Π are the usual 3×3 Hermitian matrix representations of the scalar and pseudoscalar nonets. Let Q_i, $i=1,2,\cdots,8$, be the $SU(3)$ charge operators and Q_i^5 be the corresponding axial charge operators. The chiral $SU(3)\times SU(3)$ is formed by the operators $2Q_i^\pm=Q_i$

[13] Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

$\pm Q_i^5$. The fields M and $M^\dagger$ transform under chiral transformations as

$$\begin{aligned}
[Q_i^+, M^\alpha{}_{\dot\beta}] &= (-\tfrac{1}{2}\lambda)^\alpha{}_\gamma M^\gamma{}_{\dot\beta}\,,\\
[Q_i^-, M^\alpha{}_{\dot\beta}] &= M^\alpha{}_{\dot\gamma}(\tfrac{1}{2}\lambda)^{\dot\gamma}{}_{\dot\beta}\,,\\
[Q_i^+, M^{\dot\alpha}{}_\beta] &= M^{\dot\alpha}{}_\gamma(\tfrac{1}{2}\lambda)^\gamma{}_\beta\,,\\
[Q_i^-, M^{\dot\alpha}{}_\beta] &= (-\tfrac{1}{2}\lambda)^{\dot\alpha}{}_{\dot\gamma} M^{\dot\gamma}{}_\beta\,.
\end{aligned}\tag{19}$$

The Lagrangian that we wish to consider is[14]

$$\begin{aligned}
L = \tfrac{1}{2}\,&\mathrm{Tr}(\partial^\mu M)(\partial_\mu M^\dagger)\\
&-\mu_0^2[\tfrac{1}{2}\,\mathrm{Tr}MM^\dagger + H(W^{(2)},W^{(3)},W^{(4)})]\\
&+\mathrm{Tr}A(M+M^\dagger)\,,
\end{aligned}\tag{20}$$

where H is an arbitrary polynomial in chiral $SU(3)\times SU(3)$ invariants:

$$W^{(2)}=\tfrac{1}{2}\,\mathrm{Tr}MM^\dagger,\quad W^{(3)}=\det M+\det M^\dagger,$$
$$W^{(4)}=\tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger)^2.$$

Without loss of generality, we may assume that H does not contain a term linear in $W^{(2)}$. Except for the last term, the Lagrangian (20) is fully symmetric under the chiral $SU(3)\times SU(3)$. The Hermitian matrix A can always be diagonalized. We assume that it is a linear combination of λ_0 and λ_8. Again, by translating the Σ fields by a c-number matrix, we define new scalar fields $\tilde\Sigma$ which have vanishing vacuum expectation values:

$$\tilde\Sigma=\Sigma-F,\quad \langle\Sigma\rangle_0=F.\tag{21a}$$

The c-number matrix F then must satisfy the condition

$$F+\alpha F^3+\beta F\,\mathrm{Tr}F^2+2\gamma F^{-1}\det F=2A/\mu_0^2\,,\tag{21b}$$

where $\alpha=(\partial H/\partial W^{(4)})_{M=M^\dagger=F}$, $\beta=(\partial H/\partial W^{(2)}{}_{M=M^\dagger=F}$, and $\gamma=(\partial H/\partial W^{(3)})_{M=M^\dagger=F}$.

There are, in general, many solutions to Eq. (21b) for given A. All the solutions, however, can be diagonalized simultaneously with A: Since F is assumed to be Hermitian, F may be diagonalized; Eq. (21b) then tells us that A is diagonal in that representation.

As $\mu_0^2\to\infty$ in Eq. (20), certain of the particles will acquire infinite masses. In this limit Eq. (21b) reduces to

$$F+\alpha F^3+\beta F\,\mathrm{Tr}F^2+2\gamma F^{-1}\det F=0.\tag{21c}$$

The trivial solution of Eq. (21c), $F=0$, corresponds to all 18 particles acquiring an infinite mass and is of no interest to us. In general, F has the structure

$$F=\begin{pmatrix} f_1 & & \\ & f_2 & \\ & & f_3\end{pmatrix}.$$

The cases $f_1=f_2$ are of special interest to us, since in these cases *at least* isospin and hypercharge invariances are maintained. If $f_1=f_2=f_3$ in the limit $\mu_0^2\to\infty$, we say that the $SU(3)$ breaking is *induced*, since the $SU(3)$-symmetry breaking is due entirely to the input term $\mathrm{Tr}A(M+M^\dagger)$ in Eq. (20). Otherwise, the symmetry is broken *spontaneously* [in this case, even in the absence of the last term of Eq. (20), the symmetry is broken].

If F is not equal to zero identically, it is possible to write M in the form

$$M=F+\tilde\Sigma+i\Pi=e^{iP}e^{iS}(F+X+iY)e^{-iS}e^{iP}\,,\tag{22}$$

where S, P, X, and Y are Hermitian 3×3 matrices, and S and P are traceless. If we choose S, P, X, and Y such that

$$\begin{aligned}
&\text{if}\quad f_\alpha-f_\beta=0,\quad S_{\alpha\beta}=0\quad\text{and}\quad X_{\alpha\beta}\neq0,\\
&\text{if}\quad f_\alpha-f_\beta\neq0,\quad S_{\alpha\beta}\neq0\quad\text{and}\quad X_{\alpha\beta}=0,
\end{aligned}\tag{23a}$$

$$\begin{aligned}
&\text{if}\quad f_\alpha+f_\beta=0\quad(\alpha\neq\beta),\quad P_{\alpha\beta}=0\ \text{and}\ Y_{\alpha\beta}\neq0,\\
&\text{if}\quad f_\alpha+f_\beta\neq0\quad(\alpha\neq\beta),\quad P_{\alpha\beta}\neq0\ \text{and}\ Y_{\alpha\beta}=0,
\end{aligned}\tag{23b}$$

$$\mathrm{Tr}P=0,\quad Y_{11}=Y_{22}=Y_{33},\tag{23b$'$}$$

the transformation from $\tilde\Sigma$ and Π to nonvanishing elements of S, P, X, and Y is canonical, except for trivial renormalization of S and P that may be involved. The proof of this statement and the explicit construction of S, P, X, and Y in terms of $\tilde\Sigma$ and Π are given in the Appendix. Since under parity conjugation P, $M(\mathbf{x}=0)\to M^\dagger(\mathbf{x}=0)$, it is clear that S and X are scalar and P and Y are pseudoscalar fields.

When the expression (22) is substituted in Eq. (20), the term proportional to μ_0^2 becomes

$$\begin{aligned}
L'(X,Y)=-\mu_0^2[\tfrac{1}{2}\,&\mathrm{Tr}ZZ^\dagger\\
&+H(\mathrm{Tr}ZZ^\dagger,\ \det Z+\det Z^\dagger,\ \mathrm{Tr}(ZZ^\dagger)^2)]\,,
\end{aligned}\tag{24}$$

where $Z=F+X+iY$.

A remarkable fact about Eq. (24) is that it depends only on X and Y. Consequently *the particles represented by S and P would have zero mass but for the symmetry-breaking input, i.e., the last term of Eq. (20)*. Equations (23) guarantee that there are no terms linear in X or Y in Eq. (24). Provided that

$$-\left.\frac{\delta^2L'}{\delta X_i\delta X_j}\right|_{X=Y=0}\quad\text{and}\quad-\left.\frac{\delta^2L'}{\delta Y_i\delta Y_j}\right|_{X=Y=0}$$

are positive definite (considered as real symmetric matrices with indices i and j), the particles corresponding to X and Y will acquire positive infinite masses as $\mu_0^2\to\infty$. (Whether the positive-definiteness condition is satisfied depends on the form of H.[15])

Since, as $\mu_0^2\to\infty$, the interactions among the fields X and Y alone become infinitely strong [while the inter-

[14] Lévy's Lagrangian [M. Lévy (Ref. 5)] is a special case of Eq. (20), in which $H=\alpha W^{(4)}+\beta[W^{(2)}]^2+\gamma W^{(3)}$. This choice has the merit of being renormalizable. The last term of Eq. (20) implies that the intrinsic symmetry-breaking term transforms like $(3,\bar3)\oplus(\bar3,3)$. This is an assumption which we adopt for convenience. Insofar as the nonlinear realizations of the $SU(3)\times SU(3)$ are concerned, the precise nature of the intrinsic symmetry breaking is irrelevant. Note, however, that the relations such as Eqs. (31′) and (34b′) below depend on the particular representation to which the intrinsic symmetry-breaking term is assigned.

[15] The simplest form of H, $H=\alpha W^{(4)}+\beta[W^{(2)}]^2+\gamma W^{(3)}$, does not satisfy the positive-definiteness condition if $f_1=f_2\neq f_3$ for any choice of α, β, and γ. It does, however, if $f_1=f_2=f_3$ for some ranges of α, β, and γ.

actions of the fields X and Y with S and P, through both the term $\frac{1}{2}\,\mathrm{Tr}(\partial^\mu M)(\partial_\mu M^\dagger)$ and the term $\mathrm{Tr}A \times(M+M^\dagger)$, remain finite] and the masses of the fields X and Y become infinite, these fields may be dropped from the Lagrangian.

In this limit, the meson matrix M of Eq. (22) reduces to

$$M=e^{iP}e^{iS}Fe^{-iS}e^{iP} \tag{25}$$

and the term proportional to μ_0^2 in the Lagrangian (20) [or Eq. (24)] becomes a c number. The nonlinear Lagrangian that one obtains in this manner is, *irrespective of the form of H*,

$$L=\frac{1}{2}\,\mathrm{Tr}(\partial^\mu M)(\partial_\mu M)+\mathrm{Tr}A(M+M^\dagger), \tag{26}$$

with M given by Eq. (25). Note that the process of letting $\mu_0^2\rightarrow\infty$ (and thereby eliminating the fields X and Y) does not destroy the chiral symmetry of currents, since Eq. (20) is chiral-symmetric (but for the last term) for all values of μ_0^2.

The number and nature of the boson excitations present in the nonlinear Lagrangian depend on the matrix F, according to the prescription (23). We shall enumerate all possible cases with $f_1=f_2$.

(i) $f_1=f_2=0$, $f_3\neq 0$; Scalar κ and pseudoscalar K mesons and pseudoscalar η meson are present. The symmetry of the system not broken spontaneously is the direct product of the chiral $SU(2)\times SU(2)$ and the hypercharge group $U(1)$. This case is of little interest to us because the pion, the hadron of the least mass, is absent. In the remainder, we parametrize F as

$$F=\frac{f}{\sqrt{2}}\begin{pmatrix}1&&\\&1&\\&&w\end{pmatrix},\quad f\neq 0. \tag{27}$$

(ii) $w=1$: The octet of pseudoscalar mesons is present. The symmetry is the usual $SU(3)$. This is the case considered by Cronin[3] and by Lee,[16] among others.

(iii) $w=0$: Pions, kaons, an isosinglet pseudoscalar meson, and κ mesons are present. The symmetry is the direct product of the isospin $SU(2)$ and two one-parameter Abelian groups. [In this case, the $SU(2)\otimes U(1)\otimes U(1)$ is a subgroup of $U(3)\otimes U(3)/U(1)$ and not of $SU(3)\otimes SU(3)$.]

(iv) $w=-1$: Pions, κ mesons, and η mesons are present. The symmetry is an unusual $SU(3)$, spanned by Q_1, Q_2, Q_3, Q_4^5, Q_5^5, Q_6^5, Q_7^5, and Q_8.

(v) $w\neq 1$, 0, -1: The octet of pseudoscalar meson and the κ mesons are present. The symmetry is that of the isospin-hypercharge group $U(2)$.

The number and nature of the boson excitations are exactly those of the Goldstone bosons. As shown by Goldstone, Salam, and Weinberg[17] and by Bludman and Klein,[18] the Goldstone bosons manifest themselves with those quantum numbers which correspond to the generators of the original group which are not contained in the conserved subgroup.

Of the five cases considered above, cases (ii) and (v) appear to be physically relevant. Since case (ii) has been considered extensively elsewhere,[19] we shall discuss some of the physical consequences of case (v). Case (ii) corresponds to an induced breaking of the $SU(3)$, while case (v) corresponds to a spontaneous breaking.

In case (v), the matrix S and P may be taken as[20]

$$S=\frac{1}{2f}\begin{pmatrix}0&\dfrac{2i}{w-1}\kappa\\[2mm]\dfrac{-2i}{w-1}\kappa^\dagger&0\end{pmatrix},$$

$$P=\frac{1}{2f}\begin{pmatrix}\dfrac{1}{\sqrt{2}}\left(\boldsymbol{\pi}\cdot\boldsymbol{\tau}+\dfrac{\eta}{(1+2w^2)^{1/2}}\right)&\dfrac{2}{w+1}K\\[2mm]\dfrac{2}{w+1}K^\dagger&-\dfrac{2}{(1+2w^2)^{1/2}}\eta\end{pmatrix}, \tag{28}$$

where the fields π, K, η, and κ are properly normalized in the sense that

$$\frac{1}{2}\mathrm{Tr}(\partial_\mu M)(\partial^\mu M^\dagger)=\frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2+\frac{1}{2}(\partial_\mu\eta)^2+(\partial_\mu K^\dagger)(\partial^\mu K)+(\partial_\mu\kappa^\dagger)(\partial^\mu\kappa)+\cdots. \tag{29}$$

The terms higher than quadratic in fields have been omitted. To understand the physical meaning of the parameters f and w, we compute the vector and axial-vector currents V_μ and A_μ of this theory:

$$V_\mu=-\tfrac{1}{2}i[\Sigma,\partial_\mu\Sigma]-\tfrac{1}{2}i[\Pi,\partial_\mu\Pi]\simeq-\tfrac{1}{2}i[F,\partial_\mu\Sigma]+\cdots, \tag{30}$$
$$A_\mu=-\tfrac{1}{2}\{\Sigma,\partial_\mu\Pi\}+\tfrac{1}{2}\{\pi,\partial_\mu\Sigma\}\simeq-\tfrac{1}{2}\{F,\partial_\mu\Pi\}+\cdots.$$

Hence we see that

$$f_\pi=f,\quad f_K=\tfrac{1}{2}f(1+w),\quad w=2f_K/f_\pi-1,\quad f_\kappa=\tfrac{1}{2}f(1-w), \tag{31}$$

where f_π, f_K, and f_κ are, respectively, the pion, kaon, and κ-meson decay constants. The mass spectrum is

[16] B. W. Lee, Phys. Rev. Letters **20**, 617 (1968); Phys. Rev. **170**, 1359 (1968).

[17] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[18] S. Bludman and A. Klein, Phys. Rev. **131**, 2363 (1963).

[19] For example, J. A. Cronin (Ref. 3); B. W. Lee (Ref. 16).

[20] In general, one can write $M=UWFW^\dagger U$, where U and W are unitary, unimodular, and canonically equivalent to e^{iP} and e^{iS}. A convenient form for W is

$$W=\frac{1}{2f}\begin{pmatrix}\left[1-\left(\dfrac{1}{w-1}\right)^2\kappa\kappa^\dagger\right]^{1/2}&\dfrac{2}{w-1}\kappa\\[2mm]\dfrac{2}{w-1}\kappa^\dagger&\left[1-\left(\dfrac{2}{w-1}\right)^2\kappa^\dagger\kappa\right]^{1/2}\end{pmatrix}.$$

given by the term

$$\mathrm{Tr}A(M+M^\dagger)=\text{const}-\frac{a}{2f}\left[\pi^2+2\frac{1+\xi}{1+w}K^\dagger K+\frac{1+2\xi w}{1+2w^2}\eta^2+2\frac{\xi-1}{w-1}\kappa^\dagger\kappa\right]+\text{higher-order terms},\quad(32)$$

where we have written

$$A=a\begin{pmatrix}1&&\\&1&\\&&\xi\end{pmatrix}.$$

We find that, by fitting the pion and kaon masses,

$$a=f_\pi\mu_\pi^2,\qquad \xi=2\left(\frac{\mu_K}{\mu_\pi}\right)^2\frac{f_K}{f_\pi}-1.\quad(33)$$

The mass of the η and κ mesons are predicted to be

$$\mu_\eta^2=\frac{1-2w}{1+2w^2}\mu_\pi^2+\frac{2w(1+w)}{1+2w^2}\mu_K^2,\quad(34\text{a})$$

$$\mu_\kappa^2=\frac{1}{f_K-f_\pi}(f_K\mu_K^2-f_\pi\mu_\pi^2).\quad(34\text{b})$$

Equations (34) give, for $f_K/f_\pi=1.2$ (1.3),

$$\mu_\eta=574\ (578)\ \text{MeV},\qquad \mu_\kappa=1170\ (1000)\ \text{MeV}.$$

We do not suggest taking the predictions (31) and (34) seriously. The η-X mixing has not been taken into account, nor the gauge-vector particles. When the gauge-vector (and axial) particles are introduced in the theory, there will be a mixing of the longitudinal components of the vector (axial-vector) fields and the scalar (pseudoscalar) fields.[21] The latter effect is expressed by renormalizations of the π, K, and κ fields:

$$(\pi)_{\text{unrenorm}}=Z_\pi^{1/2}(\pi)_{\text{renorm}},\quad (f_\pi)_{\text{unrenorm}}=Z_\pi^{1/2}(f_\pi)_{\text{renorm}},\quad (\mu_\pi^2)_{\text{unrenorm}}=Z_\pi^{-1}(\mu_\pi^2)_{\text{renorm}},\quad\text{etc.}$$

The content of Eqs. (31) is summarized by a sum rule:

$$Z_K^{1/2}f_K+Z_\kappa^{1/2}f_\kappa=Z_\pi^{1/2}f_\pi,\quad(31')$$

and Eq. (34b) is modified to read

$$\mu_\pi^2f_\pi Z_\pi^{-1/2}=\mu_K^2f_KZ_K^{-1/2}+\mu_\kappa^2f_\kappa Z_\kappa^{-1/2}.\quad(34\text{b}')$$

Equations (31′) and (34b′) have been previously derived by Glashow and Weinberg[22]; they also discuss their various physical consequences.

In summary, we note that, depending on the structure of the matrix F of Eq. (27), a certain subgroup of $U(3)\times U(3)$ is linearly realized. Members of the multiplet which form a linear representation of the subgroup will acquire infinite masses as we let $\mu_0^2\to\infty$, and the remaining fields transform nonlinearly under $U(3)\times U(3)$, as we shall see in Sec. IV. While the form of the matrix M of Eq. (25) is deduced on the basis of a model in which the breaking of the chiral $SU(3)\times SU(3)$ transforms like a member of $(3,\bar{3})+(\bar{3},3)$, we infer from the work of Coleman, Zumino, and Wess[2] that this form is essentially unique.

IV. GROUP-THEORETICAL CONSIDERATIONS

The purpose of this section is to establish the claim in Sec. III that the limit of the $SU(3)$ σ model given by Eq. (25) corresponds to the nonlinear realization of the chiral $SU(3)\times SU(3)$ in the representation spaces of its subgroup.[2] We shall consider case (v) of Sec. III in detail; for the other cases, essentially the same arguments apply.

We begin by making some preliminary remarks. It is possible to write an arbitrary element $G(\xi.\eta;\zeta)$ of the group $SU(3)\times SU(3)$ as

$$G(\xi,\eta;\zeta)=e^{i\boldsymbol{\xi}\cdot\mathbf{I}}e^{i\boldsymbol{\eta}\cdot\mathbf{J}}e^{i\boldsymbol{\zeta}\cdot\mathbf{A}},\quad(35)$$

where

$$\boldsymbol{\xi}\cdot\mathbf{I}=\sum_{i=1,2,3,8}\xi_iQ_i,\qquad \boldsymbol{\eta}\cdot\mathbf{J}=\sum_{i=4,5,6,7}\eta_iQ_i,\qquad \boldsymbol{\zeta}\cdot\mathbf{A}=\sum_{i=1}^{8}\zeta_iQ_i^5.$$

There exists an involutive automorphism P of $SU(3)\times SU(3)$ which is essentially the parity operation[23]:

$$P:\ Q_i^+\leftrightarrow Q_i^-,$$

or

$$P:\ Q_i\to Q_i,\quad Q_i^5\to -Q_i^5.\quad(36)$$

The spinor ψ^α transforms like (3,0); that is, under the operation (35), ψ^α transforms as

$$\psi^\alpha\to G\psi^\alpha G^{-1}=[g^{-1}(\xi,\eta;\zeta)]^\alpha{}_\beta\psi^\beta,\quad(37)$$

where

$$g(\xi,\eta;\zeta)=\exp(\tfrac{1}{2}i\sum_{i=1,2,3,8}\xi_i\lambda_i)\exp(\tfrac{1}{2}i\sum_{i=4,5,6,7}\eta_i\lambda_i)\times\exp(\tfrac{1}{2}i\sum_{i=1}^{8}\zeta_i\lambda_i).\quad(38)$$

Under the parity operation, ψ^α transforms into $\psi^{\dot\alpha}$, and consequently, under the action of Eq. (35), $\psi^{\dot\alpha}$ transforms as

$$\psi^{\dot\alpha}\to[g^{-1}(\xi,\eta;-\zeta)]^{\dot\alpha}{}_{\dot\beta}\psi^{\dot\beta}.\quad(39)$$

[21] See, for example, the discussion in B. W. Lee and H. T. Nieh (Ref. 3).

[22] S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968). We wish to thank S. Weinberg for illuminating discussions on this work.

[23] The importance of the involutive automorphism (while inessential) in this context is emphasized by L. Biedenharn, in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy*, edited by A. Perlmutter, C. A. Hurst, and B. Kursunoglu (W. A. Benjamin, Inc., New York, 1968).

The transformation law for the tensor $M^{\dot\alpha}{}_\beta$ is therefore given by

$$M^{\dot\alpha}{}_\beta \to [g^{-1}(\xi,\eta;-\zeta)]^{\dot\alpha}{}_{\dot\lambda} M^{\dot\lambda}{}_\mu [g(\xi,\eta;\zeta)]^\mu{}_\beta. \quad (40)$$

Now let s_i (i=4, 5, 6, and 7) and p_i (i=1, 2, $\cdots$. 8) be the fields corresponding to the κ mesons and octet pseudoscalar mesons. Let

$$S=\tfrac{1}{2}\sum_{i=4,5,6,7}\lambda_i s_i,\quad P=\tfrac{1}{2}\sum_{i=1}^{8}\lambda_i p_i.$$

Then, according to Eq. (35), it is possible to write

$$e^{-i\mathbf{s}\cdot\mathbf{J}}e^{i\mathbf{p}\cdot\mathbf{A}}G(\xi,\eta;\zeta)=e^{i\xi'\cdot\mathbf{I}}e^{-i\mathbf{s}'\cdot\mathbf{J}'}e^{i\mathbf{p}'\cdot\mathbf{A}}, \quad (41)$$

where ξ', s', and p' depend, in general, on s and p as well as on ξ, η, and ζ. If we substitute for the generators in Eq. (41) the matrices in the (3,0) representation, we obtain

$$e^{-iS}e^{-iP}g(\xi,\eta;\zeta)=e^{i\xi'\cdot\tau/2}e^{-iS'}e^{-iP'} \quad (42)$$

or

$$g^{-1}(\xi,\eta;\zeta)e^{iP}e^{iS}=e^{iP'}e^{iS'}e^{-i\xi'\cdot\tau/2},\quad \tau_{1,2,3,0}=\lambda_{1,2,3,8}. \quad (42')$$

In the (0,3) representation, Eq. (41) takes the form

$$e^{-iS}e^{+iP}g(\xi,\eta;-\zeta)=e^{i\xi'\cdot\tau/2}e^{-iS'}e^{+iP'} \quad (43)$$

or

$$g^{-1}(\xi,\eta;-\zeta)e^{-iP}e^{iS}=e^{-iP'}e^{iS'}e^{-i\xi'\cdot\tau/2}. \quad (43')$$

If we let F be any matrix that commutes with $\boldsymbol{\tau}$,

$$[F,\tau_i]=0,\quad i=0,1,2,3$$

then it follows from Eqs. (42) and (43′) that

$$g^{-1}(\xi,\eta;-\zeta)[e^{-iP}e^{iS}Fe^{-iS}e^{-iP}]g(\xi,\eta;\zeta) =e^{-iP'}e^{iS'}Fe^{-iS'}e^{-iP'}. \quad (44)$$

Thus, if we identify $(M^\dagger)_{\alpha\beta}=M^{\dot\alpha}{}_\beta$ with

$$M^\dagger=e^{-iP}e^{iS}Fe^{-iS}e^{-iP}, \quad (25')$$

and assign the nonlinear transformation properties to S and P, given by

$$GSG^{-1}=S',\quad GPG^{-1}=P', \quad (45)$$

then we have

$$GM^\dagger(S,P)G^{-1}=M^\dagger(S',P') =g^{-1}(\xi,\eta;-\zeta)M^\dagger(S,P)g(\xi,\eta;\zeta) \quad (46)$$

as required by Eq. (40). Hence, defined as nonlinear functions of S and P, according to Eqs. (25) and (25′), M and $M^\dagger$ transform like $(3,\bar{3})$ representations of the chiral $SU(3)\times SU(3)$ once the nonlinear transformation laws, Eq. (45), are assigned to S and P. Thus the first term of the nonlinear Lagrangian (26) is invariant under the $SU(3)\times SU(3)$ transformations.

If we now define a new field ϕ^a as a canonical transform of ψ^α by

$$\phi^a\equiv(e^{-iS}e^{-iP})^a{}_\alpha\psi^\alpha, \quad (47)$$

then ϕ^a transforms under the action of $G(\xi,\eta;\zeta)$ of Eq. (35) as[24]

$$G\phi^aG^{-1}=[e^{-iS'}e^{-iP'}g^{-1}(\xi,\eta;\zeta)e^{iP}e^{iS}]^a{}_b\phi^b =[e^{-i\xi'\cdot\tau/2}]^a{}_b\phi^b, \quad (48)$$

where use is made of Eq. (42). It shows that, since the matrix $e^{-i\xi'\cdot\tau/2}$ is reducible (block-diagonal in the isospin-hypercharge representation), each isospin-hypercharge multiplet of ϕ^a transforms irreducibly.

In general, we may assign to any field χ of isospin T and hypercharge Y the nonlinear transformation law

$$G\chi G^{-1}=e^{-i\xi'\cdot\mathbf{t}}\chi, \quad (49)$$

where $\mathbf{t}$ is the representation of I_i, i=0, 1, 2, 3 ($I_0=Q_8$), in the space of isospin T and hypercharge Y. For the construction of the nonlinear Lagrangian which is invariant under the nonlinear transformations of Eqs. (46) and (49), we refer the reader to the extant literature.[25]

V. CONCLUDING REMARKS

It is worthwhile to discuss which one of the possibilities (i)–(v), if any, discussed in Sec. III is realized in, or is the best approximation to, nature. On the ground that $f_K\gtrsim f_\pi$ we have identified cases (ii) and (v) to be of physical relevance.

It is well to review, first, why the nonlinear realization of the chiral $SU(2)\times SU(2)$ symmetry in terms of the pion fields has gained the respectability that it enjoys today. No doubt the success of current algebra is responsible for this: Most results of current algebra are readily understood as theorems on soft-pion emissions based on the approximate dynamical symmetry,[26] the symmetry operations acting nonlinearly on the fields. However, this is not the only way in which to understand the chiral symmetry of low-energy pion processes. As we have shown in Sec. II, it may be that the chiral symmetry acts linearly on fields (such as pions and σ), but, because of the large σ mass, the law of nature looks as if pions transform nonlinearly under chiral $SU(2)$ transformations.

Now let us turn to $SU(3)\times SU(3)$. Here again the nonlinear realization of this symmetry may be of a fundamental nature, or it may not. The advantage of exploring the nonlinear model is that, even when the nonlinear transformation law is merely a low-energy manifestation of a fundamentally algebraic symmetry,[26]

[24] We leave it as an exercise for the reader to show that Eq. (48) defines a representation of the group, i.e., if

$$G'\phi G'^{-1}=\exp(-i\xi''\cdot\tau/2)\phi$$

and $G''=G'G$, then $G''\phi G''^{-1}=\exp(-i\xi''\cdot\tau/2)\exp(-i\xi'\cdot\tau/2)\phi$.

[25] S. Weinberg (Ref. 11); C. G. Callan *et al.* (Ref. 2); W. A. Bardeen (unpublished).

[26] We use the terms "dynamical symmetry" and "algebraic symmetry" in the sense used by S. Weinberg, Phys. Rev. **177** (1968).

it makes no reference to the degrees of freedom which are hard to excite at low energies and describes the interactions among low-lying excitations in terms of low-energy phenomenological parameters (such as masses, f_π, f_K, and f_κ).

In case (ii), since the "unrenormalized" f_K and f_π are identical, the observed difference of f_K and f_π must be attributed to the renormalization effects as discussed at the end of Sec. III. This may be due to the presence of gauge bosons (which are not completely degenerate) or other $SU(3)$-breaking interactions. In case (v), the inequality of f_K and f_π is due, at least partly, to the spontaneous breakdown of $SU(3)$. This case calls for the existence of the κ mesons, whose existence is at present not clear. It is of some interest to note that, according to the work of Glashow and Weinberg,[22] (a) the first and second sum rules of Weinberg applied to strangeness-changing currents, (b) saturation of these sum rules by single-particle states, and (c) the observed mass ratio of K^* and KA demand the existence of the κ mesons (i.e., $f_\kappa \neq 0$). In the work of Lee on K_{l3} decays,[16] on the other hand, case (ii) was assumed, and the observed difference of f_K and f_π was attributed to wave-function renormalization effects. In this case, the second sum rule of Weinberg is not satisfied.

The existence of the κ meson of the type needed in case (v) is an experimental question and cannot be decided on the basis of a theoretical speculation. As long as f_K/f_π is close to unity, the κ mass could be relatively large, and case (v) may prove to be a convenient way to incorporate the $SU(3)$-symmetry-breaking effect on f_K and f_π, as far as low-energy meson phenomena are concerned.

APPENDIX: NONLINEAR CANONICAL TRANSFORMATIONS OF EQ. (22)

It is not difficult to see that, with S, P, X, and Y satisfying Eqs. (23), Eq. (22) defines a unique mapping of S, P, X, and Y into $\tilde{\Sigma}$ and Π. We shall consider here the inverse problem, that of expressing nonvanishing elements of S, P, X, and Y in terms of elements of $\tilde{\Sigma}$ and Π.

It is convenient to write Eq. (22) with a dummy parameter α:

$$F/\alpha+\tilde{\Sigma}+i\Pi=e^{iP}e^{iS}(F/\alpha+X+iY)e^{-iS}e^{iP},$$
$$S=\alpha\sum_{K=0}^{\infty}\alpha^k S_k,\quad P=\alpha\sum_{K=0}^{\infty}\alpha^k P_k,\qquad \text{(A1)}$$
$$X=\sum_{K=0}^{\infty}\alpha^k X_k,\quad Y=\sum_{K=0}^{\infty}\alpha^k Y_k,$$

and try to satisfy Eqs. (23) and (A1) in *each order of* α. To order α^{-1}, we obtain just the identity. Equating the coefficient of α^0 in Eq. (A1), we obtain

$$\tilde{\Sigma}=X_0+i[S_0,F],\qquad \Pi=Y_0+\{P_0,F\},\qquad \text{(A2)}$$

where we have invoked Hermiticity of Σ, Π, S, P, X, and Y to identify the Hermitian and anti-Hermitian parts of Eq. (A1). Equation (A2) can be solved by setting:

(a) if $f_\alpha-f_\beta\neq 0$,
$$(X_0)_{\alpha\beta}=0,\quad (S_0)_{\alpha\beta}=[i/(f_\alpha-f_\beta)](\tilde{\Sigma})_{\alpha\beta};$$
if $f_\alpha-f_\beta=0$,
$$(X_0)_{\alpha\beta}=(\tilde{\Sigma})_{\alpha\beta},\quad (S_0)_{\alpha\beta}=0;$$
(b) if $f_\alpha+f_\beta\neq 0$ $(\alpha\neq\beta)$,
$$(Y_0)_{\alpha\beta}=0,\quad (P_0)_{\alpha\beta}=(f_\alpha+f_\beta)^{-1}(\Pi)_{\alpha\beta};$$
if $f_\alpha+f_\beta=0$ $(\alpha\neq\beta)$,
$$(Y_0)_{\alpha\beta}=(\Pi)_{\alpha\beta},\quad (P_0)_{\alpha\beta}=0.\qquad \text{(A3)}$$
(c) We want P to be traceless. Therefore we set
$$Y_0=1\tfrac{1}{3}\,\mathrm{Tr}\Pi,$$
$$(P_0)_{\alpha\alpha}=(1/2f_\alpha)[\Pi-\tfrac{1}{3}1\,\mathrm{Tr}\Pi]_{\alpha\alpha}.$$

In general, terms of α^n $(n\neq -1, 0)$ are given by

$$X_n+i[S_n,F]=f_n,\qquad Y_n+i\{P_nF\}=g_n,\qquad \text{(A4)}$$

where f_n and g_n are nonlinear functions of S_k, P_k, X_k, and Y_k, with $k=0, 1, \cdots. n-1$. Thus Eq. (A4) may be solved for S_n, P_n, X_n, and Y_n uniquely, just as Eq. (A2) was solved for S_0, P_0, X_0, and Y_0, with the conditions of Eqs. (23) applied to them. In this way ,we can solve Eq. (22) for S, P, X, and Y in terms of $\tilde{\Sigma}$ and Π, and the preceding argument shows that this transformation is canonical, since to first order in $\tilde{\Sigma}$ and Π the nonvanishing elements of S, P, X, and Y are given by Eq. (A3).