Low-Energy Theorem from Pair Suppression and some Applications*

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Assuming separate conservation of quark and antiquark number in the static limit, a general theorem on the low-energy limit of scattering amplitudes of hadron processes is derived. Numerous applications of this theorem are made. A slight modification of the procedure leads also to predictions on pion-pion scattering lengths.

I. GENERAL THEOREM

IN this paper we derive a general theorem on the static limit of scattering amplitudes, starting from a simple and physically plausible assumption. Application of this theorem reproduces (except for the relations concerning the absolute magnitudes) the well-known sum rules[1,2] on scattering lengths, which were previously obtained from current algebra and partially conserved axial-vector current (PCAC). When combined with $SU(3)$ and $SU(6)$ symmetry, our theorem also leads to new sum rules. A slight modification of the procedure permits the derivation of pion-pion scattering length sum rules.

We start with the following postulate:

Postulate. In the static limit, where the momenta of all participating particles are set zero, the quark number N and the antiquark number $\bar{N}$ are separately conserved. In those channels of a vertex where this condition is not met, the amplitude vanishes in the static limit.

We may motivate this postulate as follows: The origin of $SU(6)$ symmetry is best visualized in the framework of the nonrelativistic quark model,[3] and in this model it is natural and plausible to adopt[4] our postulate. The fact that one can get many successful predictions from the realistic quark model, where all hadrons are supposed to be composite systems with *definite numbers* of quarks and antiquarks, seems to lend support to our postulate. However, in this paper we do not resort to the model,[5] except that we assume that baryons have definite quark and definite antiquark numbers [for example, but not necessarily $(N,\bar{N})=(3,0)$] and that mesons have definite nonzero quark and antiquark numbers [for example, but not necessarily $(N,\bar{N})=(1,1)$]. Finally, we note that our above postulate also played an important role in a recent work of ours[6] where a prescription was given for avoiding the defects of the usual static $SU(6)$ theory.

From our postulate we now derive the following theorem:

Theorem. Let A, B, a, b denote some particles and $\bar{a}$, $\bar{b}$ the antiparticles of a, b, with $N_{a,b}\neq 0$ and/or $\bar{N}_{a,b}\neq 0$. Consider the two channels $A+a\rightleftarrows B+b$ and $A+\bar{b}\rightleftarrows B+\bar{a}$ which are both supposed to be such that the quark number and the antiquark number are separately conserved. Then, in the static limit (when the momenta of all participating particles are set zero), the amplitude M satisfies the relation

$$M(A+a\rightleftarrows B+b)=-M(A+\bar{b}\rightleftarrows B+\bar{a})\,, \quad (1)$$

provided the masses μ^a and μ^b are sufficiently small.

Proof. The general form of the S matrix corresponding to the vertex is given by[7]

$$S=-i\delta(P_\nu{}^A+P_\nu{}^B+P_\nu{}^a+P_\nu{}^b)\,|P_0{}^A P_0{}^B P_0{}^a P_0{}^b|^{-1/2}$$
$$\times M(P_\nu{}^A,P_\nu{}^B,P_\nu{}^a,P_\nu{}^b)(b^\dagger a+\bar{b}\bar{a}^\dagger+\bar{b}a+b^\dagger\bar{a}^\dagger)B^\dagger A\,, \quad (2)$$

where a ($a^\dagger$), etc. is the annihilation (creation) operator for the particle a with momentum P^a. The momentum assignments are shown in Fig. 1. In Eq. (2), the substitution law for the particles a and b is explicitly represented, and the amplitude M describes all processes corresponding to the channels $C1$ through $C4$ shown in Fig. 2. We now assume that in the vicinity of the region $P_\nu{}^a=P_\nu{}^b=0$, the amplitude M can be well approximated by a first-order expansion in terms of the four-momenta $P_\nu{}^a$ and $P_\nu{}^b$; i.e., we set

$$M(P^A,P^B,P^a,P^b,)={}_0M+{}_1M_\nu{}^a P_\nu{}^a+{}_1M_\nu{}^b P_\nu{}^b\,, \quad (3)$$

where ${}_0M$, ${}_1M_\nu{}^a$, and ${}_1M_\nu{}^b$ depend only on P^A and P^B. In the static limit, Eq. (3) becomes

$$M(P_0{}^a,P_0{}^b)=C+D^aP_0{}^a+D^bP_0{}^b\,, \quad (4)$$

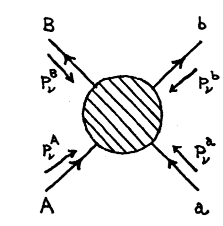

FIG. 1. Momentum assignment.

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[1] Y. Tomozawa, Nuovo Cimento **46**, 707 (1966).
[2] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[3] For a recent review, see, for example, R. H. Dalitz, in *Proceedings of the Thirteenth International Conference on High-Energy Physics* (University of California Press, Berkeley, 1967), p. 125.
[4] K. Kikkawa [Phys. Rev. 1753 (1968)] used the same postulate in a different context.
[5] Our central theorem can be also obtained in a model where the meson is a composite system of a baryon and an antibaryon and where separate baryon and antibaryon number conservation is assumed in the static limit.
[6] S. Ishida and P. Roman, Phys. Rev. **172**, 1684 (1968).
[7] We indicate only the momenta because other variables (spin, etc.) are irrelevant.

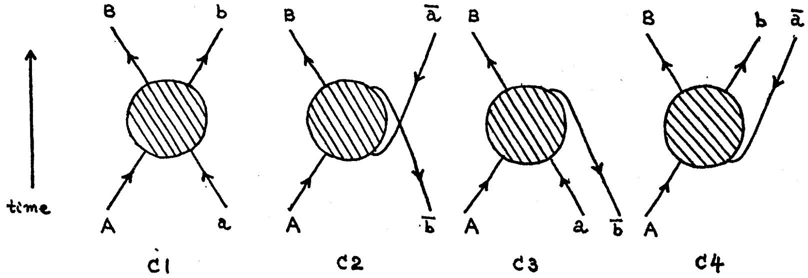

FIG. 2. Possible channels for the vertex (2).

where C, D^a, and D^b are constants. In the channels $C3$ and $C4$, the quark and antiquark number are not separately conserved so that, because of our postulate, the corresponding amplitudes must vanish in the static limit. Since in these channels, $(P_0{}^a, P_0{}^b)$ equals (μ^a,μ^b) and $(-\mu^a,-\mu^b)$, respectively, Eq. (4) leads to

$$C+(D^a\mu^a+D^b\mu^b)=0\,,\quad C-(D^a\mu^a+D^b\mu^b)=0\,.$$

Thus

$$C=0\,. \tag{5}$$

Since in the channels $C1$ and $C2$ (where separate quark and antiquark number conservation holds by assumption) we have the respective values $(\mu^a, -\mu^b)$ and $(-\mu^a, \mu^b)$ for $(P_0{}^a,P_0{}^b)$, Eqs. (5) and (4) give

$$M(P_0{}^a,P_0{}^b)=D^a\mu^a-D^b\mu^b \text{ in } C1\,, \tag{6a}$$

$$M(P_0{}^a,P_0{}^b)=-(D^a\mu^a-D^b\mu^b) \text{ in } C2\,. \tag{6b}$$

This proves our theorem.

Even though our theorem can be valid for only sufficiently small values of μ^a and μ^b, we shall now apply it for physical scattering amplitudes at threshold, hoping that these quantities vary only slightly with the masses. Clearly, the most appropriate problem to consider is meson-baryon scattering, especially pion-baryon scattering.

TABLE I. Scattering lengths for pseudoscalar meson-nucleon reactions.

Reaction amplitude	(A) $SU(3)$+theorem g_1	g_2	g_3	g_4	(B) $SU(6)$ +theorem g
$(\pi^+p\|\pi^+p)$	−1	0	0	0	−1
$(\pi^-p\|\pi^-p)$	1	0	0	0	1
$(K^+n\|K^+n)$	0	1	0	0	−1
$(K^-n\|K^-n)$	0	−1	0	0	1
$(K^+p\|K^+p)$	−1	1	−1	0	−2
$(K^-p\|K^-p)$	1	−1	1	0	2
$(K^-p\|\pi^-\Sigma^+)$	0	−1	1	0	1
$(K^-p\|\pi^+\Sigma^-)$	0	0	1	−1	0
$(K^-p\|K^0\Xi^0)$	0	0	1	1	0
$(K^-p\|K^+\Xi^-)$	0	0	0	0	0
$2\sqrt{3}(K^-p\|\pi^0\Lambda)$	2	−1	0	−1	3
$2(K^-p\|\eta\Lambda)$	2	−1	2	−1	3
$2\sqrt{3}(K^-p\|\eta\Sigma^0)$	0	−3	0	−1	3
$(\pi^+p\|K^+\Sigma^+)$	0	1	−1	0	−1
$(\pi^-p\|K^+\Sigma^-)$	0	0	−1	1	0
$(\sqrt{6})(\pi^-p\|K^0\Lambda^0)$	−2	1	0	1	−3
$(\sqrt{6})(\pi^-p\|\eta n)$	0	0	0	−2	0

II. BASIC APPLICATIONS

We first discuss the scattering of a pion on an arbitrary target. We denote the scattering length by $a(i,A\,;j,B)$, where i, j and A, B refer to the isospin indices of the initial (final) pion and of the target (residual) particle, respectively. Because of charge independence, the general expression is

$$a(i,A\,;j,B)=\alpha\delta_{ij}\delta_{AB}+\beta(\hat{T}_\pi)_{ij}(\hat{T}_t)_{AB}\,. \tag{7}$$

Here $(T_{\pi,k})_{ij}\equiv i\epsilon_{kij}$ and $\hat{T}_t$ are the isospin matrices for the pion and the target particle. Our theorem (1) requires

$$a(i,A\,;i,B)=-a(i,A\,;i,B) \tag{8}$$

so that we get the restriction $\alpha=0$. Thus, the scattering length $a_T{}^\pi$ for a state with isospin T is given by

$$a_T{}^\pi=a_0[T(T+1)-T_t(T_t+1)-2]\,, \tag{9}$$

where a_0 is a constant depending on the target particle and T_t is the isospin of the target.[8] Formula (9) is identical with Weinberg's result[2] which he obtained from current algebra and PCAC, except that our a_0 is an undetermined constant. The comparison of (9) with experiments has been already discussed by Tomozawa.[1] We only call attention to the remarkably well-satisfied[9] sum rule

$$2a_{3/2}{}^{\pi N}+a_{1/2}{}^{\pi N}=0\,, \tag{10}$$

which follows from (9) when specializing to pion-nucleon scattering. It is indeed this particular case when our approximation can be expected to be most adequate.

Application of the same technique to kaon scattering on an arbitrary target leads to the formulas

$$a_T{}^K=\alpha+\beta[T(T+1)-T_t(T_t+1)-\tfrac{3}{4}]\,, \tag{11a}$$

$$a_T{}^{\bar{K}}=-\alpha+\beta[T(T+1)-T_t(T_t+1)-\tfrac{3}{4}]\,, \tag{11b}$$

[8] Equation (9) is not applicable for pion-pion scattering because it does not satisfy the requirement of Bose statistics. However, by a simple modification we can derive a corresponding formula for the π-π scattering length. It does not agree with Weinberg's well-known result of Ref. 2. We shall discuss this problem in Sec. III of this paper.

[9] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963) quote the values $a_{3/2}{}^{\pi N}=-0.088\pm0.004$ and $a_{1/2}{}^{\pi N}=0.171\pm0.005$, in units of $\mu_\pi{}^{-1}$.

where a_T^K ($a_T^{\bar{K}}$) is the scattering length of kaon (antikaon) for the state with isospin T and α and β are constants depending on the target. Specialization of Eq. (11) to kaon-nucleon scattering gives the sum rules

$$-2a_1^{KN}=a_1^{\bar{K}N}+a_0^{\bar{K}N}, \tag{12a}$$

$$-2a_1^{\bar{K}N}=a_1^{KN}+a_0^{KN}. \tag{12b}$$

This is a less restrictive result than the one which has been derived by Tomozawa[1] from current algebra and generalized PCAC, and our result is satisfied by his formula. Unfortunately, (12a), and (12b) are inconsistent with experiments.[10] However, as has been pointed out,[1] the discrepancy may be ascribed to the circumstance that for the $\bar{K}N$ system, the $\pi\Lambda$ and $\pi\Sigma$ channels are open below threshold and that the existence of the $Y_0^*(1405)$ and $Y_1^*(1388)$ resonances make our approximation unreliable.

In the following we consider the general problem of the 0^- meson octet and $\frac{1}{2}^+$ baryon octet scattering. As is well known, the $SU(3)$ invariant amplitude has eight independent terms. Application of our theorem reduces this number to four, and the amplitude at threshold is given by

$$\begin{aligned}M=&g_1\bar{B}_\gamma{}^\alpha B_\delta{}^\gamma[P_\alpha{}^\delta(f)P_\delta{}^\beta(i)-P_\delta{}^\beta(f)P_\alpha{}^\delta(i)]\\&+g_2\bar{B}_\delta{}^\gamma B_\gamma{}^\alpha[P_\alpha{}^\delta(f)P_\delta{}^\beta(i)-P_\delta{}^\beta(f)P_\alpha{}^\delta(i)]\\&+g_3\bar{B}_\beta{}^\alpha B_\delta{}^\gamma[P_\alpha{}^\beta(f)P_\gamma{}^\delta(i)-P_\gamma{}^\delta(f)P_\alpha{}^\beta(i)]\\&+g_4\bar{B}_\beta{}^\alpha B_\delta{}^\gamma[P_\alpha{}^\delta(f)P_\gamma{}^\beta(i)-P_\gamma{}^\beta(f)P_\alpha{}^\delta(i)].\end{aligned} \tag{13}$$

Here $P_\beta{}^\alpha(i)$ ($P_\beta{}^\alpha(f)$) represents the initial (final) meson octet and $B_\beta{}^\alpha$ ($\bar{B}_\beta{}^\alpha$) represents the initial (final) baryon octet. In column (A) of Table I we tabulated the contributions to the terms with the coefficients g_1 through g_4 for various processes with incident charged kaons or pions, which are not related by charge independence. We then obtain ten new sum rules which are independent of (10) and (12). However, there is only one relation which is not related to $\bar{K}N$ states and which can be presently compared with experiment, viz.,

$$a(K^+p\to K^+p)+a(\pi^-p\to\pi^-p)=a(\pi^+p\to\Sigma^+K^+). \tag{14}$$

There is some ambiguity if one wishes to compare symmetry-consideration predictions with experimental data in case of a broken symmetry. In the present problem, it appears plausible to assume that our theoretical predictions refer to the generalized scattering length which coincides with the standard scattering length for elastic processes and which is related in the general case to the total cross sections σ near threshold by the equation

$$\sigma=(P_f/P_i)4\pi a^2. \tag{15}$$

Here P_i (P_f) is the c.m. system momentum in the initial (final) state. Using this definition, the experimental data[11] when substituted into both sides of (14) give

$$(-0.13\pm0.015)\mu_\pi{}^{-1}=(-0.031_{-0.04}{}^{+0.02})\mu_\pi{}^{-1}. \tag{16}$$

Thus, qualitative agreement is obtained.[12]

Finally, we combine our theorem with $SU(6)$ symmetry. The general $SU(6)$-invariant amplitude for the 35-tuplet meson and 56-tuplet baryon scattering has four independent terms.[13] If we apply our theorem (1), this number is reduced to 1, and the amplitude at threshold is given by

$$\begin{aligned}M=g\bar{B}^{ABC}B_{ABD}[&M_C{}^E(f)M_E{}^D(i)\\&-M_E{}^D(f)M_C{}^E(i)].\end{aligned} \tag{17}$$

Here $M_B{}^A(i)$ ($M_B{}^A(f)$) and B_{ABC} ($\bar{B}^{ABC}$) represent the initial (final) 35-tuplet and 56-tuplet, respectively. In column (B) of Table I, we tabulated the contributions for the same processes as above. We see that the present $SU(6)$ predictions were obtained from the corresponding $SU(3)$ results by setting $g_1=-g_2=g$ and $g_3=g_4=0$. From Table I, we can get several predictions which are independent of our previous results (10), (12), and (14) and which can be confronted with experiment. First of all, we find

$$a_1^{KN}=2a_{3/2}{}^{\pi N}=-0.176\mu_\pi{}^{-1},\quad(-0.205\pm0.005)\mu_\pi{}^{-1}, \tag{18a}$$

$$a_0^{KN}=0,\ (0.03\pm0.03)\mu_\pi{}^{-1}. \tag{18b}$$

The experimental values,[10] which are given in parentheses, show good agreement. In addition, we find for the generalized scattering length of the process $\pi^-+p\to K^0+\Lambda$ the result

$$a(\pi^+p\to K^0\Lambda)=(\sqrt{\tfrac{3}{2}})a_{3/2}{}^{\pi N}=-0.108\mu_\pi{}^{-1}. \tag{19}$$

From experiment[14] we know only the cross section near threshold, from which we estimate the absolute magnitude of the generalized scattering length as

$$|a(\pi^+p\to K^0\Lambda)|=(0.048\pm0.007)\mu_\pi{}^{-1}.$$

It appears that our theorem is reasonably favored by experiments, even though the analysis was based only on a simple approximation. Thus we feel that our basic postulate may give an important clue for a future

[10] S. Goldhaber *et al.*, Phys. Rev. Letters **9**, 135 (1962) and V. J. Stenger *et al.*, Phys. Rev. **134**, B1111 (1964) give the values $a_1^{KN}=-0.205\pm0.005$ and $a_0^{KN}=0.03\pm0.03$ (in units of $\mu_\pi{}^{-1}$). The $\bar{K}N$ scattering lengths are complex, and J. K. Kim, Phys. Rev. Letters **14**, 29 (1965), obtained the values $\mathrm{Re}\,a_1^{\bar{K}N}=-0.002\pm0.041$ and $\mathrm{Re}\,a_0^{\bar{K}N}=-1.172\pm0.027$ (same units).

[11] For K^+p and π^+p, we used the data of Refs. 9 and 10. For $\pi^+p\to\Sigma^+K^+$, we quoted the absolute value given by F. Grard and G. A. Smith, Phys. Rev. **127**, 607 (1962), and we assumed that there is no change in sign from somewhat higher-energy data, of F. S. Crawford *et al.*, Phys. Rev. Letters **3**, 394 (1959).

[12] Using the more recent total cross-section data at a somewhat higher energy [N. L. Carayannopoulos *et al.*, Phys. Rev. **138B**, 433 (1965)] and assuming that the sign of the amplitude is the same as found at higher energies by F. S. Crawford *et al.*, Phys. Rev. Letters **3**, 394 (1959), the right-hand side of Eq. (16) becomes $(-0.044\pm0.003)\mu_\pi{}^{-1}$.

[13] V. Barger and M. Rubin, Phys. Rev. Letters **14**, 713 (1965).

[14] L. Bertanza *et al.*, Phys. Rev. Letters **8**, 332 (1962) report $\sigma=0.056\pm0.015$ mb at a lab-system pion kinetic energy 775 MeV (which corresponds to $P_i=525$ MeV/c and $P_f=50$ MeV/c).

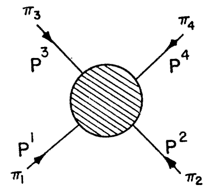

FIG. 3. Four-pion vertex.

dynamical theory. Further detailed experimental tests of the predictions given in Table I appear desirable.

III. PION-PION SCATTERING

We define the invariant pion-pion scattering amplitude M from the S-matrix relation

$$\langle 3,4|S|1,2\rangle = -i\delta(P_\nu^1+P_\nu^2+P_\nu^3+P_\nu^4) \times |P_0^1P_0^2P_0^3P_0^4|^{-1/2}M(1,2,3,4). \quad (20)$$

Here P_ν^n denotes the four-momentum of the nth pion, and the signs are chosen, as indicated in Fig. 3.

A crucial point in the subsequent analysis is that, because of the substitution law (crossing), the amplitude M describes not only the pion-pion scattering process, $C1$ of Fig. 4, but also the other two[15] processes, $C2$ and $C3$.

We now assume that, near the static point $P_i^n=0$ ($n=1$, 2, 3, 4 and $i=1$, 2, 3), the amplitude M can be well approximated by a second-order expansion[16] in terms of the four-momenta P_ν^n, which reads as follows:

$$\begin{aligned} M(1,2,3,4) &= (\varphi(1)\varphi(2))(\varphi(3)\varphi(4))\{A+B[(P^1P^2)+(P^3P^4)] \\ &\quad +C[(P^1P^3)+(P^2P^3)+(P^1P^4)+(P^2P^4)]\} \\ &+(\varphi(1)\varphi(3))(\varphi(2)\varphi(4))\{A+B[(P^1P^3)+(P^2P^4)] \\ &\quad +C[(P^1P^2)+(P^2P^3)+(P^1P^4)+(P^3P^4)]\} \\ &+(\varphi(1)\varphi(4))(\varphi(2)\varphi(3))\{A+B[(P^1P^4)+(P^2P^3)] \\ &\quad +C[(P^1P^2)+(P^2P^4)+(P^1P^3)+(P^3P^4)]\}. \quad (21) \end{aligned}$$

Here $\varphi(n)$ is the isospin wave function of the nth pion, $(\varphi(n)\varphi(m))$ and (P^nP^m) stand for the inner products of $\varphi(n)$, $\varphi(m)$ and of P_ν^n, P_ν^m, respectively, and A, B, C are constants. In deriving (21), we assumed the substitution law, isospin conservation, and Bose statistics.

According to our postulate, in the static limit of the channels $C2$ and $C3$ the amplitude M must vanish. In channel $C2$, we have in the static limit $(P^3P^n)=-\mu^2$ for $n\neq 3$ and $(P^nP^m)=\mu^2$ for n, $m\neq 3$. Thus, Eq. (2) and our postulate give the condition

$$A=0. \quad (22a)$$

In channel $C3$, we have in the static limit $(P^nP^m)=\mu^2$ for all n, m. Hence, we are led to the additional condition

$$A+(2B+4C)\mu^2=0. \quad (22b)$$

Because of these conditions [(22a) and (22b)], the amplitude M can be represented in terms of a single parameter B and is given explicitly by

$$\begin{aligned} M(1,2,3,4) = B\{&(\varphi(1)\varphi(2))(\varphi(3)\varphi(4))(s-\tfrac{1}{2}t-\tfrac{1}{2}u) \\ &+(\varphi(1)\varphi(3))(\varphi(2)\varphi(4))(t-\tfrac{1}{2}u-\tfrac{1}{2}s) \\ &+(\varphi(1)\varphi(4))(\varphi(2)\varphi(3))(u-\tfrac{1}{2}s-\tfrac{1}{2}t)\}. \quad (23) \end{aligned}$$

Here, we introduced in place of (P^nP^m) the usual Mandelstam variables

$$\begin{aligned} s &\equiv (P^1+P^2)^2=(P^3+P^4)^2, \\ t &\equiv (P^1+P^3)^2=(P^2+P^4)^2, \quad (24) \\ u &\equiv (P^1+P^4)^2=(P^2+P^3)^2. \end{aligned}$$

Next we define the scattering amplitude a_{lT} with specific orbital momentum l and isospin T by setting, as usual,

$$M(1,2,3,4)=\sum_{l,T}(2l+1)P_T(1,2,3,4)a_{lT}P_l(\cos\theta), \quad (25)$$

where P_T denotes the set of isospin projection matrices. We then easily obtain from Eq. (23) the relations

$$a_{00}=2B(4\mu^2+6k^2), \quad (26a)$$

$$a_{02}=-B(4\mu^2+6k^2), \quad (26b)$$

$$a_{11}=2Bk^2, \quad (26c)$$

where $k^2=\frac{1}{4}(s-4\mu^2)$. The remaining amplitudes a_{01}, a_{10}, and a_{12} are zero, as required by statistics. It is remarkable that the three nonvanishing amplitudes are represented in terms of one parameter. In particular, we get from (26) the S-wave amplitude relation

$$a_{00}/a_{02}=-2. \quad (27)$$

Our results are different from those which were obtained by Weinberg[17] with the use of current algebra and PCAC.[18] The accuracy of presently available experimental data does not permit a decisive comparison of the two sets of results. In any case, detailed experimental tests of our predictions (26) are most desirable.

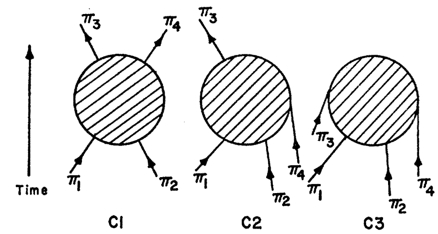

FIG. 4. Possible channels of the four-pion vertex.

[15] We indicate only one of a pair of channels which arise from each other by time reversal.

[16] In order to apply our Postulate, we consider all momenta P^n to be independent variables, in contrast to the usual procedure. All pions are always on the mass shell, $(P^n)^2=\mu^2$. Concerning the adequacy of the expansion (21), we refer to a discussion by Weinberg (Ref. 17) which essentially applies also to the present case.

[17] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[18] We find this discrepancy interesting, because in Sec. II of this work we obtained from our postulate similar results for the simpler problem of meson-baryon scattering to those obtained from current algebra.