Minimal Current Algebra*

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(Received 5 September 1968)

We devise an algebra of currents and their first time derivatives designed to damp at high momentum the asymptotic behavior of lepton-pair scattering amplitudes from hadrons consequent from the local current algebra of Gell-Mann. Given certain criteria, the algebra we find is unique, and the commutators are expressed linearly in terms of the currents themselves. The Jacobi identity, however, is formally violated for this algebra; we argue that this does not invalidate it. A possible realization of this "minimal" algebra is found in terms of the formal limit of a massive Yang-Mills theory as g_0, $m_0 \to 0$; $g_0/m_0^2 \to \text{const} \neq 0$. With this algebra, all electromagnetic masses of hadrons are finite. Experimental consequences, the strongest of which occurs in inelastic lepton-hadron scattering, are outlined.

I. INTRODUCTION

MANY of the predictions of local current algebra,[1] notably the sum rules derived by Adler,[2] Fubini,[3] Dashen and Gell-Mann,[4] and similar asymptotic sum rules valid at high q^2,[5–9] imply that very far off the mass-shell, current-hadron scattering matrix elements are at least as singular as those of free particles. This result, if verified experimentally, would give one great confidence in the general validity of the locality assumptions on the weak currents which underlie the supposed pointlike nature of these amplitudes. On the other hand, the converse is not true. Were all the sum rules to fail experimentally, local current algebra would not necessarily fail. There are many loopholes. One possibility is that the equal-time commutators are ambiguous.[10] Another is that, although the commutators are taken to exist, technical assumptions (interchange of limits in the $P \to \infty$ method, or absence of subtractions in dispersion relations for certain amplitudes) needed in the derivations of the sum rules may be incorrect. Also, alterations of the highly model-dependent space-space commutators and/or of the usually assumed high-q^2 behavior of amplitudes can invalidate many existing sum rules. It is this latter loophole which is explored in this paper.

We formulate criteria designed to minimize the experimental consequences of the local algebra. They are to be applied in the limit of large q^2 (where q is a momentum carried in by a current). Given these criteria, it follows that many of the existing sum rules should be damped at high q^2 by at least an extra power of q^2, and that all electromagnetic self-energies converge. It also turns out that these criteria uniquely determine the commutation relations not only of the currents with each other, but also with their time derivatives. These commutators turn out to be *linear* in the currents and derivatives thereof. We call this algebra the *minimal algebra*. In Sec. II we describe in detail the "unobservability criteria" which are supposed to minimize the observable effects of the local algebra of currents. In Sec. III it is shown how these criteria are sufficient to lead to a unique set of commutators of currents with their time derivatives. In Sec. IV the minimal algebra is shown to result from a limit of a massive Yang-Mills theory as $m_0 \to 0$ and $g_0/m_0^2 \to$ constant. In Sec. V, we discuss the experimental implications of the minimal algebra.

II. CRITERIA FOR THE MINIMAL ALGEBRA

According to the Gell-Mann[11] philosophy of current algebra, matrix elements of time-ordered products of two currents $\langle p | T^*(J_\mu{}^a(x) J_\nu{}^b(0)) | p' \rangle = M_{\mu\nu}{}^{ab}$ are considered as observables, because they can in principle be related to S-matrix elements for scattering of lepton pairs or photons from hadrons[12]:

$$S_{fi} \propto l_\mu l_\nu M_{\mu\nu}(p',q';p,q),$$

with

$$l_\mu = \bar{u}(k+q)\gamma_\mu u(k) \quad \text{or} \quad \bar{u}\gamma_\mu(1-\gamma_5)u$$

the lepton current. In order that $M_{\mu\nu}$ itself be observable, it is necessary that the factors l_μ are allowed to be removed; i.e., that all four components are inde-

* Work supported in part by the U. S. Atomic Energy Commission.

[1] For a summary see S. L. Adler and R. F. Dashen, *Current Algebra* (W. A. Benjamin, Inc., New York, 1968).

[2] S. L. Adler, Phys. Rev. **143**, 1144 (1966).

[3] S. Fubini, Nuovo Cimento **43A**, 475 (1966).

[4] R. F. Dashen and M. Gell-Mann, *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmuller, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966).

[5] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[6] J. D. Bjorken, Phys. Rev. Letters **16**, 408 (1966).

[7] J. D. Bjorken, Phys. Rev. **163**, 1767 (1967).

[8] J. D. Bjorken and M. Bander, Phys. Rev. **174**, 1704 (1968).

[9] C. C. Callan and D. J. Gross, Phys. Rev. Letters **21**, 311 (1968).

[10] R. A. Brandt and J. Sucher, Phys. Rev. Letters **20**, 1131 (1968); and (unpublished).

[11] M. Gell-Mann, Phys. Rev. **125**, 1062 (1962); Physics **1**, 63 (1964).

[12] Our metric is $(1, -1, -1, -1)$; $\mu, \nu, \lambda = 0, 1, 2, 3$; $k, l = 1, 2, 3$; a, b, c refer to internal indices; for example, $a = 1-16$ for $SU(3) \otimes SU(3)$. For typographical convenience, we neglect to raise and lower indices when invoking the summation convention.

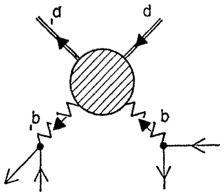

FIG. 1. Scattering of a current from a hadron.

pendent. This is true provided the lepton mass is *not* neglected; otherwise

$$q_\mu l_\mu = 0$$

and $M_{\mu\nu}$ is ambiguous up to terms proportional to q_μ or q_ν'. On the other hand, sum rules which test the local algebra, such as Adler's neutrino sum rules,[2] involve high-energy leptons where the neglect of lepton mass would appear to be justifiable.

As $q_0 \to \infty$, $\mathbf{q}$ fixed, $M_{\mu\nu}$ can be expected (but not proven) to be at least as singular as q_0^{-1} with the coefficients controlled by equal-time commutators.[5] This asymptotic behavior is characteristic of point particles. If no such behavior is manifested experimentally, it may mean the commutation relations are ambiguous.[10] It may also mean that the leading asymptotic behavior of $M_{\mu\nu}$ is contained in pieces proportional to q_μ or q_ν', with the result that observable consequences in the S matrix are limited to terms of the order lepton mass.

We shall adopt this behavior for $M_{\mu\nu}$, and assume that *through order* q_0^{-2}, $M_{\mu\nu}$ *contains only pieces proportional to* q_μ *or* q_ν'. In this way, experimental consequences of the local current algebra would be expected to be minimized. This possibility will be explored in a quantitative way in Sec. III.

III. MINIMAL ALGEBRA

We consider the process shown in Fig. 1 to lowest order in the weak and electromagnetic interaction. The corresponding S-matrix element is proportional to

$$l_\mu^a(q) l_\nu'^b(q') M_{\mu\nu}^{ab}(q,q',p)\,, \tag{1}$$

where l and l' are lowest-order matrix elements of leptonic weak or electromagnetic currents and M is the covariant hadronic current correlation function. We assume that

$$M_{\mu\nu}^{ab} = T_{\mu\nu}^{ab} + S_{\mu\nu}^{ab}\,, \tag{2}$$

where T is the connected time-ordered product

$$T_{\mu\nu}^{ab} = -i \int d^4x\, e^{+iq\cdot x} \langle p | T(J_\mu^a(x) J_\nu^b(0)) | p' \rangle \tag{3}$$

and S is a polynomial in q. The hadronic currents are assumed to be conserved,[13]

$$\partial_\mu J_\mu^a(x) = 0\,, \tag{4}$$

[13] To the extent that the divergences of nonconserved currents are operators whose matrix elements are damped at large q^2 (generalized PCAC hypothesis), our results can be extended to include nonconserved currents.

and M is assumed to satisfy the divergence conditions

$$q_\mu M_{\mu\nu}^{ab}(q,q',p) = i f^{abc} \langle p | J_\nu^c | p' \rangle\,, \tag{5}$$

$$q_\nu' M_{\mu\nu}^{ab}(q,q',p) = i f^{abc} \langle p | J_\mu^c | p' \rangle\,. \tag{6}$$

The leptonic currents satisfy

$$q_\mu l_\mu^a(q) = 0\,, \tag{6'}$$

neglecting the leptonic masses.

Let us first assume that $S=0$ and that the T product in Eq. (3) is well defined and has an expansion in inverse powers of $q_0 \equiv \omega$ up to order ω^{-3}. Then this expansion is given by[5]

$$T_{\mu\nu}^{ab}(q,q',p) = \langle p | \int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}} \left\{ \frac{1}{\omega} [J_\mu^a(0,\mathbf{x}), J_\nu^b(0)] + \frac{i}{\omega^2} [\dot{J}_\mu^a(0,\mathbf{x}), J_\nu^b(0)] \right\} | p' \rangle + O\!\left(\frac{1}{\omega^3}\right). \tag{7}$$

We now ask if the commutators in Eq. (7) can be chosen so that (1) is $O(1/\omega^3)$ (in the limit $q_0, q_0' \to \infty$ with $\mathbf{q}, \mathbf{q}'$, and $\Delta \equiv q - q'$ fixed) for all leptonic currents and hadronic scattering states. Then all observable effects of the theory will be $O(1/\omega^3)$ (neglecting leptonic masses) and the theory will be as smooth as possible in the above framework. We shall therefore refer to the resulting current algebra as the *minimal* one.

The problem can be most succinctly expressed in terms of the operators $T_{\mu\nu}^{ab}(q,q')$ and $J_\mu^c(\Delta)$ defined by

$$\langle p | T_{\mu\nu}^{ab}(q,q') | p' \rangle = \delta(p + q - p' - q') T_{\mu\nu}^{ab}(q,q',p)\,, \tag{8}$$

$$J_\mu^c(\Delta) = \int d^4x\, e^{i\Delta\cdot x} J_\mu^c(x)\,. \tag{9}$$

Thus we want to find the commutators in Eq. (7) such that

$$l_\mu^a(q) l_\nu'^b(q') T_{\mu\nu}^{ab}(q,q') = O(1/\omega^3)\,, \tag{10}$$

$$q_\mu T_{\mu\nu}^{ab}(q,q') = i f^{abc} J_\nu^c(\Delta)\,, \tag{11}$$

$$q_\nu' T_{\mu\nu}^{ab}(q,q') = i f^{abc} J_\mu^c(\Delta)\,. \tag{12}$$

We mean by Eqs. (10)–(12) that the identities are valid when the equations are sandwiched between arbitrary hadronic *scattering* states.

In view of Eq. (6'), the conditions (10) require that T has the form

$$T_{\mu\nu}^{ab}(q,q') = q_\mu F_\nu^{ab}(q,q') + q_\nu' F_\mu'^{ab}(q,q') + O(\omega^{-3}) \tag{13}$$

for some operators F and F'. The divergence conditions (11) and (12) impose the further restrictions

$$\tilde{J}_\nu^{ab}(\Delta) = q^2 F_\nu^{ab}(q,q') + q_\nu' q\cdot F'^{ab}(q,q') + O(\omega^{-2}) \tag{14}$$

and

$$\tilde{J}_\mu^{ab}(\Delta) = q_\mu q'\cdot F^{ab}(q,q') + q'^2 F_\mu'^{ab}(q,q') + O(\omega^{-2})\,, \tag{15}$$

where we have defined

$$\tilde{J}_\mu^{ab} = i f^{abc} J_\mu^c\,.$$

Finally, the behavior (7) requires that F and F' have the forms

$$F_\nu^{ab}=\omega^{-2}A_\nu^{ab}+\omega^{-3}B_\nu^{ab}+O(\omega^{-3-\epsilon}), \quad (16)$$

$$F_\mu'^{ab}=\omega^{-2}A_\mu'^{ab}+\omega^{-3}B_\mu'^{ab}+O(\omega^{-3-\epsilon}). \quad (17)$$

We shall now show that (13)–(17) uniquely determine the minimal algebra occurring in (7). Substituting (16) and (17) in (14) and equating coefficients of ω^0 and ω^{-1} gives the relations (suppressing the internal indices)

$$\tilde{J}_0=A_0+A_0', \quad (18a)$$

$$0=B_0-\mathbf{q}\cdot\mathbf{A}'+B_0'-\Delta_0A_0', \quad (18b)$$

$$\tilde{J}_k=A_k, \quad (18c)$$

$$0=B_k+q_k'A_0'. \quad (18d)$$

Similarly, Eqs. (16), (17), and (15) give

$$\tilde{J}_0=A_0+A_0', \quad (19a)$$

$$0=B_0-\mathbf{q}'\cdot\mathbf{A}+B_0'-\Delta_0A_0-2\Delta_0A_0', \quad (19b)$$

$$\tilde{J}_k=A_k', \quad (19c)$$

$$0=B_k'+q_kA_0-2\Delta_0A_k'. \quad (19d)$$

Let us exhibit the independent information in (18) and (19). Equations (18a) and (19a) are the same equation:

$$\tilde{J}_0=A_0+A_0', \quad (20)$$

and (18c) and (19c) give

$$\tilde{J}_k=A_k=A_k'. \quad (21)$$

The difference of (18b) and (19b), using (20) and (21), simply gives current conservation

$$0=-(\mathbf{q}'-\mathbf{q})\cdot\tilde{\mathbf{J}}-\Delta_0\tilde{J}_0, \quad (22)$$

whereas the sum gives

$$0=2B_0+2B_0'-(\mathbf{q}'+\mathbf{q})\cdot\tilde{\mathbf{J}}-\Delta_0(A_0+3A_0'). \quad (23)$$

Finally, the difference of (18d) and (19d) gives

$$0=B_k'+q_kA_0-2\Delta_0A_k', \quad (24)$$

and the sum gives

$$0=B_k+q_k'A_0. \quad (25)$$

Equations (20)–(25) are the consequences of (14)–(17). Note that they do *not* uniquely determine the sixteen quantities A_μ, A_μ', B_μ, B_μ'.

If we now substitute (16) and (17) into (13) and use (20)–(25), we find

$$T_{00}=\omega^{-1}\tilde{J}_0+\omega^{-2}\mathbf{q}\cdot\tilde{\mathbf{J}}+O(\omega^{-3}), \quad (26a)$$

$$T_{0k}=\omega^{-1}\tilde{J}_k+O(\omega^{-3}), \quad (26b)$$

$$T_{k0}=\omega^{-1}\tilde{J}_k+\omega^{-2}\Delta_0\tilde{J}_k+O(\omega^{-3}), \quad (26c)$$

$$T_{kl}=\omega^{-2}(q_k\tilde{J}_l+q_l'\tilde{J}_k)+O(\omega^{-3}). \quad (26d)$$

Thus, as claimed, the conditions (10)–(12) uniquely determine the minimal algebra (to within c numbers). Comparison with (7) gives it to be ($x_0=y_0$)

$$[J_0^a(x),J_\mu^b(y)]_T=if^{abc}J_\mu^c(x)\delta(\mathbf{x}-\mathbf{y}), \quad (27a)$$

$$[J_k^a(x),J_l^b(y)]_T=0, \quad (27b)$$

$$[\dot{J}_0^a(x),J_\mu^b(y)]_T=ig_{\mu0}f^{abc}J_i^c(y)\frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{y}), \quad (27c)$$

$$[\dot{J}_k^a(x),J_0^b(y)]_T=if^{abc}\dot{J}_k^c(x)\delta(\mathbf{x}-\mathbf{y}), \quad (27d)$$

$$[\dot{J}_k^a(x),J_l^b(y)]_T=if^{abc}\left(J_k^c(x)\frac{\partial}{\partial x_l}-J_l^c(y)\frac{\partial}{\partial y_k}\right)\delta(\mathbf{x}-\mathbf{y}). \quad (27e)$$

Here the subscript "T" refers to "truncated" commutators

$$[A,B]_T\equiv[A,B]-\langle 0|A,B]|0\rangle.$$

Strictly speaking, we have not yet established the existence of the minimal algebra but have only shown that, if it does exist, then it is given by (27). Thus we must show that the commutators (27) constitute (part of) a consistent algebra and that there exists a solution to Eqs. (20)–(25) such that the resulting $T_{\mu\nu}$ given by (13) is an acceptable amplitude.

We first show that the latter requirement is satisfied. We shall exhibit a $T_{\mu\nu}$ consistent with (10)–(12) and with the usual conditions—particularly Lorentz covariance. To this end, we let $\bar{T}_{\mu\nu}$ be an arbitrary acceptable amplitude [satisfying (11) and (12)] and consider[14]

$$T_{\mu\nu}(q,q')\equiv\bar{T}_{\mu\nu}(q,q')-(g_{\mu\alpha}-q_\mu q_\alpha/q^2)\times(g_{\nu\beta}-q_\nu'q_\beta'/q'^2)\bar{T}_{\alpha\beta}(q,q'). \quad (28)$$

This amplitude is Lorentz-covariant, satisfies (11) and and (12), and, in view of (6′), also satisfies (10). Thus a consistent solution of (20)–(25) is guaranteed to exist. To see what this solution is, we use the divergence conditions (11) and (12) to write (28) as

$$T_{\mu\nu}(q,q')=(q_\mu/q^2)\tilde{J}_\nu(\Delta)+(q_\nu'/q'^2)\tilde{J}_\mu(\Delta)-q_\mu q_\nu'/q^2q'^2(\tfrac{1}{2}q+q')\cdot\tilde{J}(\Delta). \quad (29)$$

Comparison with (13), (16), and (17) now gives, for example,

$$\begin{aligned} &A_k=A_k'=\tilde{J}_k, && A_0=A_0'=\tfrac{1}{2}\tilde{J}_0,\\ &B_k=-\tfrac{1}{2}q_k'\tilde{J}_0, && B_0=\tfrac{1}{4}(\mathbf{q}+\mathbf{q}')\cdot\tilde{\mathbf{J}}-\tfrac{1}{4}\Delta_0J_0,\\ &B_k'=-\tfrac{1}{2}q_k\tilde{J}_0+2\Delta_0\tilde{J}_k, && B_0'=\tfrac{1}{4}(\mathbf{q}+\mathbf{q}')\cdot\tilde{\mathbf{J}}+5/4\Delta_0\tilde{J}_0. \end{aligned} \quad (30)$$

This solution, of course, satisfies (20)–(25).

Next we consider the minimal algebra (27) itself. Equation (27a) holds in all of the usual models. Equation (27b) holds in models, such as the σ model[15] and the algebra of fields,[16] in which the currents are con-

[14] To the order ω^{-2} of interest, the q^2 denominators in (28) can be replaced by $q^2-m^2+i\epsilon$. The $+i\epsilon$ should, in any case, be present in order to maintain the causal structure of T.

[15] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[16] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

structed from Bose operators. Equations (27c) and (27d) follow from (27a) and (27b) together with current conservation (4). Equation (27b) implies that (27e) must be symmetric under ($k\leftrightarrow l$, $a\leftrightarrow b$, $x\leftrightarrow y$), as it is. Finally, all of the Jacobi identities are formally satisfied except the one involving $\dot{J}_k$, J_l, and J_0, which cannot be satisfied unless the currents vanish. The fact that this double commutator does not formally satisfy the Jacobi identity is not necessarily an inconsistency. Our derivation only implies and requires that (27) is valid when sandwiched between physical scattering states $|p\rangle$, whereas the formulation of the Jacobi identity would require them to be valid between, say, the physical state $|p'\rangle$ and the state $J(x_0,\mathbf{z})|p\rangle$. Thus no difficulties can arise if we only use the relations (27) between scattering states. Put differently, field theoretic equal-time commutators must be defined as equal-time limits[17] and the Jacobi identity will not hold when certain of these limits can not be interchanged. We shall illustrate and discuss this further in Sec. IV within the context of a specific model.

The relations (27) are, furthermore, consistent with our initial assumption that $T_{\mu\nu}$ has an expansion in powers of ω^{-1} to the order of ω^{-3}. The nonsingular nature of the right-hand sides of (27) (e.g., the absence of local operator products) suggests that the terms in (7) are well defined so that the expansion should be valid.

We finally note that dropping the assumption $S=0$ would not change any of our results. If we add to $T_{\mu\nu}$ any polynomial $S_{\mu\nu}$ in q, then the conditions (1)=0, (5), and (6) require that $S_{\mu\nu}=0$.

IV. LIMIT OF MASSIVE YANG-MILLS THEORY

In this section we show that the minimal current algebra of Sec. III is the algebra corresponding to a particular formal limit of the massive Yang-Mills theory. This will shed light on both the singular aspects of the model (such as the failure of the formal Jacobi relation) and the smooth aspects (such as the good high-q_0 behavior). It will also enable the incorporation of electromagnetism, PCAC (partially conserved axial-vector current), and $SU(3)$-breaking into the model. The approach is along the lines given by Bardakci, Frishman, and Halpern.[18]

The massive Yang-Mills[19] theory is defined by the Lagrangian density,

$$L(x)=-\tfrac{1}{4}F_{\mu\nu}{}^a(x)F_{\mu\nu}{}^a(x)+\tfrac{1}{2}m_0{}^2\phi_\mu{}^a(x)\phi_\mu{}^a(x)\,,\quad(31)$$

where

$$F_{\mu\nu}{}^a=\partial_\mu\phi_\nu{}^a-\partial_\nu\phi_\mu{}^a-\tfrac{1}{2}g_0f^{abc}(\phi_\mu{}^b\phi_\nu{}^c+\phi_\nu{}^c\phi_\mu{}^b)\,.\quad(32)$$

The equations of motion are

$$\partial_\mu F_{\mu\nu}{}^a+m_0{}^2\phi_\nu{}^a=\tfrac{1}{2}g_0f^{abc}(F_{\nu\mu}{}^b\phi_\mu{}^c+\phi_\mu{}^cF_{\nu\mu}{}^b)\,,\quad(33)$$

the stress-energy tensor is

$$\theta_{\mu\nu}=\tfrac{1}{2}(F_{\mu\lambda}{}^aF_{\lambda\nu}{}^a+F_{\nu\lambda}{}^aF_{\lambda\mu}{}^a)+\tfrac{1}{2}m_0{}^2(\phi_\mu{}^a\phi_\nu{}^a+\phi_\nu{}^a\phi_\mu{}^a)-g_{\mu\nu}L\,,\quad(34)$$

and the canonical commutation rules imply ($x_0=y_0$)

$$[\phi_0{}^a(x),\phi_0{}^b(y)]=i\lambda^{-1}f^{abc}\phi_0{}^c(x)\delta(\mathbf{x}-\mathbf{y})\,,\quad(35a)$$

$$[\phi_0{}^a(x),\phi_k{}^b(y)]=i\lambda^{-1}f^{abc}\phi_k{}^c\delta(\mathbf{x}-\mathbf{y})+im_0{}^{-2}\delta^{ab}(\partial/\partial x_k)\delta(\mathbf{x}-\mathbf{y})\,,\quad(35b)$$

$$[\phi_k{}^a(x),\phi_l{}^b(y)]=0\,,\quad(35c)$$

$$[\partial_0\phi_k{}^a(x)-\partial_k\phi_0{}^a(x),\,\phi_l{}^b(y)]=i\delta^{ab}g_{kl}\delta(\mathbf{x}-\mathbf{y})+i\lambda^{-1}f^{abc}\phi_k{}^c(x)(\partial/\partial x_l)\delta(\mathbf{x}-\mathbf{y})-iC^{-1}f^{ace}f^{bde}\phi_l{}^e(x)\phi_k{}^d(x)\delta(\mathbf{x}-\mathbf{y})\,,\quad(35d)$$

where we have defined

$$\lambda=m_0{}^2/g_0,\quad C=m_0{}^2/g_0{}^2.\quad(36)$$

The assumption of field-current identity[20] and field algebra[16] is that the hadronic currents are given by

$$J_\mu{}^a=\lambda\phi_\mu{}^a.\quad(37)$$

One then has current conservation

$$\partial_\mu J_\mu{}^a=0.\quad(38)$$

Bardakci, Frishman, and Halpern[18] have shown that in the limit

$$m_0\to0,\quad g_0\to0,\quad C=\text{const},\quad(39)$$

the above model becomes the Sugawara[21] model ($x_0=y_0$):

$$[J_0{}^a(x),J_0{}^b(y)]=if^{abc}J_0{}^c(x)\delta(\mathbf{x}-\mathbf{y})\,,\quad(40a)$$

$$[J_0{}^a(x),J_k{}^b(y)]=if^{abc}J_k{}^c(x)\delta(\mathbf{x}-\mathbf{y})+iC\delta^{ab}(\partial/\partial x_k)\delta(\mathbf{x}-\mathbf{y})\,,\quad(40b)$$

$$[J_k{}^a(x),J_l{}^b(y)]=0\,,\quad(40c)$$

$$[\partial_0J_k{}^a(x)-\partial_kJ_0{}^a(x),J_l{}^b(y)]=if^{abc}J_k{}^c(x)(\partial/\partial x_l)\delta(\mathbf{x}-\mathbf{y})-iC^{-1}f^{ace}f^{bde}J_l{}^e(x)J_k{}^d(x)\delta(\mathbf{x}-\mathbf{y})\,,\quad(40d)$$

$$\partial_\mu J_\nu{}^a-\partial_\nu J_\mu{}^a=\tfrac{1}{2}C^{-1}f^{abc}(J_\mu{}^bJ_\nu{}^c+J_\nu{}^cJ_\mu{}^b)\,,\quad(41)$$

$$\theta_{\mu\nu}=\tfrac{1}{2}C^{-1}(J_\mu{}^aJ_\nu{}^a+J_\nu{}^aJ_\mu{}^a-g_{\mu\nu}J_\lambda{}^aJ_\lambda{}^a)\,.\quad(42)$$

Bardakci *et al.* used the limiting procedure (39) to incorporate electromagnetism, PCAC, and $SU(3)$ breaking into the model.

We shall show that in the different limit

$$m_0\to0,\quad g_0\to0,\quad\lambda=\text{const},\quad C\to\infty\,,\quad(43)$$

the massive Yang-Mills theory yields the commutation

[17] R. A. Brandt, Phys. Rev. **166**, 1795 (1968).

[18] K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968).

[19] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954); see also Ref. 18.

[20] N. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[21] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

relations of the minimal algebra. We shall (and, in fact, must) simultaneously take the divergence of local field products into account. We assume that the divergence of $\{\phi_l^c(x),\phi_k^d(x)\}$ is mild enough so that

$$C^{-1}\{\phi_l^c(x),\phi_k^d(x)\} \to c \text{ number} \tag{44}$$

in the limit (43). This can be thought of as a boundary condition to be used in solving the theory. Under (43) and (44), (35)–(37) give exactly the commutation rules (27) of the minimal algebra. In addition, the c-number parts of the commutators are given and, for (35b) and possibly (35d), are infinite. This, of course, is acceptable and is exactly what happens in the free-field quark model. What we attempt to do is remove the q-number divergence from (35d) and have it become a c-number divergence in (35b). As we have seen in Sec. III, this makes the physical properties of the theory less singular.

Let us now give a more careful discussion of our limiting procedures. We assume that, in analogy with soluble models and perturbation theory, local field products are to be defined as suitable limits of nonlocal products.[22] Thus the mass term in (31) becomes

$$\lim_{\xi\to 0} \tfrac{1}{2}[m_0(\xi)]^2\phi_\mu^a(x+\xi)\phi_\mu^a(x)\,, \tag{45}$$

where the limit is to be taken in a spacelike direction, say $\xi=(0,\boldsymbol{\xi})$. The vanishing of $m_0(0)$ is supposed to cancel singularities of the local product $\phi(x)\phi(x)$. The equation of motion (33) becomes

$$\partial_\mu F_{\mu\nu}^a(x)=\lim_{\xi\to 0}\{\tfrac{1}{2}g_0(\xi)f^{bac} \times[F_{\nu\mu}^b(x+\xi)\phi_\mu^c(x)+\phi_\mu^c(x)F_{\nu\mu}^b(x+\xi)] -[m_0(\xi)]^2\phi_\nu^a(x)\}\,. \tag{46}$$

We only assume this relation is valid between physical scattering states. Let $J_\mu^a(x;\,\xi)=\lambda(\xi)\phi_\mu^a(x;\,\xi)$ be the nonlocal solution of the nonlocal theory with $\xi\neq 0$. We assume, again in analogy with soluble models and perturbation theory, that the equal-time local current commutators can be calculated as limits of commutators of the corresponding nonlocal currents.[17] Thus we assume, for example, that (35d) becomes ($x_0=y_0$)

$$\begin{aligned}&[\partial_0 J_k^a(x)-\partial_k J_0^a(x),\,J_l^b(y)]_T\\ &=\lim_{\xi\to 0,\xi'\to 0}[\partial_0 J_k^a(x;\xi)-\partial_k J_0^a(x;\xi),\,J_l^b(y;\xi')]_T\\ &=if^{abc}J_k^c(x)\frac{\partial}{\partial x_l}\delta(\mathbf{x}-\mathbf{y})-if^{ace}f^{bde}\\ &\quad\times\lim_{\xi\to 0}[C^{-1}(\xi)J_l^c(x+\xi)J_k^d(x)]\delta(\mathbf{x}-\mathbf{y})\,,\end{aligned} \tag{47}$$

valid between physical states. All the equal-time commutators in the theory are to be defined in this way.

We now specialize to the case (43) and (44). We put

$$m_0(\xi)=\lambda r(\xi),\quad g_0(\xi)=[r(\xi)]^2\,, \tag{48}$$

with[23]

$$r(\xi)\to 0,\quad [r(\xi)]^2\phi_\mu^a(x+\xi)\phi_\nu^b(x)\to c\text{ number}, \tag{49}$$

so that the commutation rules (35), defined in analogy with (47), become those of the minimal algebra. The algebra is, furthermore, now guaranteed to be completely consistent, provided the limits $\xi\to 0$ are taken after all commutation. In particular, the Jacobi identity will now be satisfied. For example, whereas one has

$$\lim_{\xi\to 0} C^{-1}(\xi)J_k^a(y+\xi)J_l^b(y)=C_{kl}^{ab}(y)=c\text{ number}, \tag{50}$$

one nevertheless has ($x_0=y_0$)

$$\begin{aligned}&[J_0^c(x),C_{kl}^{ab}(y)]\\ &=\lim_{\xi\to 0}[J_0^c(x),C_{kl}^{ab}(y;\,\xi)]\\ &=i\lim_{\xi\to 0}C^{-1}(\xi)\{[f^{cad}J_k^d(x)+\delta^{ca}C(\xi)(\partial/\partial x_k)]\\ &\quad\times J_l^b(y)\delta(\mathbf{x}-\mathbf{y}-\boldsymbol{\xi})+J_k^a(y+\xi)[f^{cbd}J_l^d(x)\\ &\quad+\delta^{cb}C(\xi)(\partial/\partial x_l)]\delta(\mathbf{x}-\mathbf{y})\}\\ &=i[\delta^{ca}J_l^b(y)+\delta^{cb}J_k^a(y)]\delta(\mathbf{x}-\mathbf{y})\neq 0\,.\end{aligned} \tag{51}$$

One can now use the method of Bardakci *et al.*[18] to introduce electromagnetism, PCAC, and $SU(3)$ breaking into the theory.

In perturbation theory[22] the divergences in the local products $\phi(x)\phi(x)\phi(x)$ and $\phi(x)\partial_\mu\phi(x)$ will be worse than that of $\phi(x)\,\phi(x)$ and, in order to obtain a nontrivial theory, we assume that this is the case here. We put (as boundary condition on the solution of the theory)[24]

$$[r(\xi)]^2\phi_\mu^a(x+\xi)\phi_\nu^b(x)\phi_\lambda^c(x-\xi)\to\Phi_{\mu\nu\lambda}^{abc}(x)\,, \tag{52}$$

$$[r(\xi)]^2\phi_\mu^a(x+\xi)\partial_\nu\phi_\lambda^b(x)\to\chi_{\mu\nu\lambda}^{ab}(x)\,, \tag{53}$$

for some local operators Φ, χ; and denote by $K_\nu^a(x)$ the particular combination occurring in (33). Thus, in our limit, (31)–(34) become

$$L\to -\tfrac{1}{4}F_{\mu\nu}^aF_{\mu\nu}^a\,, \tag{54}$$

$$F_{\mu\nu}^a\to\partial_\mu\phi_\nu^a-\partial_\nu\phi_\mu^a\,, \tag{55}$$

$$\partial_\mu F_{\mu\nu}^a\to K_\nu^a\,, \tag{56}$$

$$\theta_{\mu\nu}\to\tfrac{1}{2}(F_{\mu\lambda}^aF_{\lambda\nu}^a+F_{\nu\lambda}^aF_{\lambda\mu}^a)-g_{\mu\nu}L\,, \tag{57}$$

between physical scattering states. It is important to note that, for example, (55) can not be substituted into

[22] See, for example, R. A. Brandt, Ann. Phys. (N. Y.) **44**, 221 (1967); and W. Zimmermann, Commun. Math. Phys. **6**, 161 (1967).

[23] An example is the free scalar field $\phi(x)$ which satisfies $\xi^2\phi(x+\xi)\phi(x)\to -1/4\pi^2$.

[24] For the example of Ref. 23, one has

$$\xi^2\phi(x+\xi)\phi(x)\phi(x-\xi)\to -(3/4\pi^2)\phi(x).$$

(54). One must first substitute (32) into (31) and then take the limit $\xi \to 0$ using (52) and (53).

It is interesting to note that our expressions (54)–(57) are exactly orthogonal to the Sugawara expressions obtained from (31)–(34) in the limit of Bardakci *et al.* In our limit (43) only the kinetic terms survive whereas in the limit (39) only the mass terms survive.

Although the minimal algebra appears to be more singular than the Yang-Mills or Sugawara theories, many aspects of it are, in fact, less singular. We must use complicated commutator definitions such as (47) and impose boundary conditions such as (44), (52), and (53). We have, however, eliminated the singular local field products from (31)–(34) and (35c). As we have seen in Sec. II, this has a smoothing effect on the high-q_0 properties of the theory. In effect, we have made the mathematical formalism of the theory more complicated in order than its physical consequences become less complicated.

V. EXPERIMENTAL CONSEQUENCES

Because the minimal algebra yields smoother asymptotic behavior in $q_0=\omega$, except in the almost unobservable pieces proportional to q_μ or q_ν', the high-q^2 behavior of various sum rules is weakened. Among the results are the following:

(i) All electromagnetic mass-differences are finite to order α. This follows from the vanishing of the q-number part of $[\dot{J}_\mu^{\text{em}},J_\nu^{\text{em}}]$.[25]

(ii) Asymptotic sum rules for neutrino[2] (and in all likelihood inelastic electron or muon scattering) in the backward direction[7] have a vanishing right-hand side because $[j_i(x),j_j(0)]=0$ for the minimal algebra. In the case of inelastic scattering, where only inequalities exist, one cannot make a rigorous argument, because the inequality goes the wrong way. It is *consistent* with the minimal algebra to have a vanishing right-hand side.

(iii) The sum rule of Callan and Gross[9]

$$\lim_{q^2\to\infty} q^2 \int_0^\infty \frac{d\nu}{\nu} W_2(q^2,\nu)=\text{const}=K\,, \tag{58}$$

where

$$\lim_{E,E'\to\infty} \frac{d\sigma(ep\to\text{hadrons})}{dq^2dE'}=\frac{4\pi\alpha^2}{q^4}W_2(q^2,E-E') \tag{59}$$

(generalized to cases where $[\partial_0 j_i(x),j_j(0)]$ has finite matrix elements between nucleon states), in the minimal algebra has a vanishing right-hand side $K=0$, because to this order (ω^{-2}, $m_l=0$), all observable consequences have been obliterated.

(iv) Similar statements hold for neutrino and antineutrino processes, e.g.,

$$\lim_{q^2\to\infty}\lim_{E\to\infty} q^2 \int_0^\infty \frac{d\nu}{\nu}\left[\frac{d\sigma(\bar{\nu}p\to\text{hadrons})}{dq^2d\nu}\right]=0\,. \tag{60}$$

(v) No statement can be made on the validity of the Fubini–Dashen–Gell-Mann sum rule with the minimal algebra alone. Writing, for the special case of spin-zero matrix elements,

$$M_{\mu\nu}=P_\mu P_\nu A_1(\nu,t,q^2,q'^2)+\cdots\,, \tag{61}$$

the Fubini-Dashen-Gell-Mann sum rule is

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu\; \text{Im}A_1(\nu,t,q^2,q'^2)=F(t)\,, \tag{62}$$

while from the minimal algebra

$$\lim_{q^2,q'^2\to\infty} q^2 \int \frac{d\nu}{\nu}\,\text{Im}A_1(\nu,t,q^2,q'^2)=0\,. \tag{63}$$

What can be said is that the value of ν needed to saturate this sum rule grows more rapidly than linearly with q^2 in the "minimal algebra," contrary to what has been sometimes assumed in the literature.

(iv) The minimal algebra implies that the Weinberg[26] sum rules are valid. In fact, the absence of J^2 terms in the $[\dot{J},J]$ commutator allows the second sum rule to be derived without invoking special limiting processes.

(vii) The failure of the minimal algebra to satisfy the Jacobi identity suggests that some of the vacuum expectation values of the commutators are divergent. This was the case, for example, when the algebra was obtained as a limit of the massive Yang-Mills theory. These divergences themselves have experimental implications. The Dooher[27] relation,

$$0=\lim_{E\to\infty} E^4 \ln E\, \sigma_{\text{tot}}(E)_{e^+e^-\to\text{hadrons}}\,, \tag{64}$$

for example, should no longer hold.

In conclusion, we wish to emphasize that experiments can test the speculations in this paper. Perhaps the most conclusive test is the behavior of the Callen-Gross integral Eq. (58). For it to vanish in the limit is not a consequence of field algebra or most conventional models.

ACKNOWLEDGMENTS

It is a pleasure to thank Y. Frishman for discussions and criticism; some of the initial work was done in collaboration with him. One of us (RAB) thanks the SLAC theory group for the kind hospitality extended to him and for providing a very stimulating atmosphere.

[25] See Ref. 5 for the method. Most other models give divergent mass shifts. See, however, Ref. 10.

[26] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[27] J. Dooher, Phys. Rev. Letters 19, 600 (1967).