Alternative Constructions of Crossing-Symmetric Amplitudes with Regge Behavior*

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A representation for the scattering amplitude that contains Regge behavior, crossing symmetry, and analyticity is discussed. It is shown that it provides a different ghost-eliminating mechanism (the Mandelstam-Wang one) from that given by Veneziano's proposal. Furthermore, it does not restrict the external masses, but reduces to Veneziano's formula when $\alpha(s)+\alpha(t)+\alpha(u)$ equals a particular integer that depends on the reaction. Several examples are discussed. The behavior of the Regge residue $\beta_{\rho\pi\pi}(t)$ at $\alpha_\rho(t)=0$ is proposed as a test that distinguishes the two representations.

MIXING analyticity, crossing symmetry, and Regge behavior in a suitable approximation scheme has been the aim of all the recent developments in superconvergence and finite-energy sum rules. The idea has been stated that these requirements could produce enough constraints on coupling constants and masses to become the basis of a bootstrap.[1,2] In this work we concluded, in fact, that the program was both feasible and predictive. However, it was not clear whether the whole content of crossing symmetry was indeed taken into account by the sum rules.

The next, important step forward was taken by Veneziano,[3] who recently was able to take advantage of all the hints given by the sum-rule approach to propose a closed, explicit form for the amplitude that embodies the following properties: (a) complete crossing symmetry; (b) that the only singularities present (for linear trajectories, in the narrow-resonance approximation) are the simple poles corresponding to resonances on Regge trajectories; (c) that it satisfies all superconvergence sum rules; and (d) that, when averaged (as in the duality-principle approach,[4]) it displays Regge behavior at asymptotic energies.

For the particularly interesting example $\pi\pi\to\pi\omega$ and the invariant amplitude $A(s,t,u)$ defined by

$$T=\epsilon_{\mu\nu\rho\sigma}e_\mu{}^{(\omega)}p_\nu{}^{(\pi1)}p_\rho{}^{(\pi2)}p_\sigma{}^{(\pi3)}A(s,t,u)\,,$$

his representation becomes

$$A(s,t,u)=\frac{\beta}{\pi}\left(\frac{\Gamma(1-\alpha(t))\Gamma(1-\alpha(s))}{\Gamma(2-\alpha(s)-\alpha(t))}+(s\to u)+(t\to u)\right). \quad (1)$$

The properties (a)–(d) above are essential ingredients of the program stated at the beginning, and Veneziano's formula may be looked at as one complete fulfillment of this program. But in addition, the expression (1) provides a particular ghost-eliminating mechanism and restricts the external masses by the condition $\alpha(s)+\alpha(t)+\alpha(u)=2$.[5] Both these aspects are shared by the sum-rule solution.[2] The ghost-eliminating mechanism provided may be an interesting property for certain reactions, but it is obviously not on an equal footing with (a)–(d).

In this paper we want to show that a different closed solution exists that satisfies the properties (a)–(d) but provides for a different, more general ghost-eliminating mechanism and does not restrict the external masses in the same way. Furthermore, this new solution becomes Veneziano's when $\alpha(s)+\alpha(t)+\alpha(u)=2$ (or a corresponding integer in other reactions). The idea for it stems from the observation that each term of the formula (1) is very similar to a product representation of poles in two different channels.[6] Therefore the obvious generalization is to write the amplitude as a triple product of poles in the three channels. We propose

$$A(s,t,u)=\beta\frac{\Gamma(\frac12-\frac12\alpha(s))\Gamma(\frac12-\frac12\alpha(t))\Gamma(\frac12-\frac12\alpha(u))}{\Gamma(1-\frac12\alpha(t)-\frac12\alpha(u))\Gamma(1-\frac12\alpha(s)-\frac12\alpha(u))\Gamma(1-\frac12\alpha(s)-\frac12\alpha(t))}. \quad (2)$$

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[1] M. Ademollo, H. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters **19**, 402 (1967); Phys. Letters **27B**, 99 (1968); and, independently, S. Mandelstam, Phys. Rev. **166**, 1539 (1968). For a general account of the application of this kind of program to mesons, see Ref. 2.

[2] M. Ademollo, H. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. **176**, 1904 (1968).

[3] G. Veneziano, CERN Report No. Th. 924 (unpublished).

[4] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1772 (1968). For applications of this approach, see C. Schmid, Phys. Rev. Letters **20**, 689 (1968); G. F. Chew and A. Pignotti, *ibid.* **20**, 1078 (1968); and Ref. 2.

[5] It is perhaps important to stress in what sense this restriction is needed. First, if it is not satisfied, and the amplitude is still given by (1), then there are poles at even values of $\alpha(s)$. This in turn requires the existence of a trajectory spaced by one unit of angular momentum but with the same signature as the leading one. Its existence does not contradict any fundamental principle, but is neither predicted nor required by any other theoretical scheme [analyticity at $t=0$ or $O(4)$ symmetry]. In fact, it should belong to an independent Toller family. On the other hand, if we want to eliminate these *ad hoc* poles, we can always add nonleading terms to (1) so as to push them to higher energies. Then the asymptotic region where Regge behavior holds will be reached at higher energies too. In principle, if we add an infinite number of nonleading terms (and the coefficients do not decrease fast enough), then not even the whole sum will display Regge behavior.

[6] In fact, one can obtain each term of (1) by writing an infinite product of polesin s and t multiplied by zeros to prevent double poles. In this way the relevance of linearities of the trajectories becomes clear. I thank C. Goebel for these remarks.

To prove that this expression reduces to (1) when $\alpha(s)+\alpha(t)+\alpha(u)=2$, one uses the duplication formula for the Γ function. It is noteworthy that Eq. (2) contains poles only at odd values of s, independently of any restriction on the external masses. Furthermore, the limit $s\to\infty$, t fixed, is

$$\lim_{s\to\infty} A(s,t,u)=\beta[\tfrac{1}{2}\alpha(s)]^{\alpha(t)-1}[\tan\tfrac{1}{2}\pi\alpha(s)+\tan\tfrac{1}{2}\pi\alpha(t)]\frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha(t))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))}.$$

At $\alpha(t)=0$, ImA is, in general, different from zero. This means that the Regge residue that has the factorization, β_{sn} is proportional to $\alpha^{-1/2}$, i.e., it corresponds to the Mandelstam-Wang mechanism.[7] If we try to impose $\beta_{sn}\propto\alpha^{1/2}$ (choosing sense or nonsense mechanism), then we find once more that $\alpha(s)+\alpha(t)+\alpha(u)=2n$, as in Veneziano's solution. In this way, the restriction appears here as in the sum-rules approach.[8] In the more general case the triple-product representation will have a zero at $\alpha(s)+\alpha(u)=1$, but this zero depends on the trajectories in the crossed channels. If there is more than one trajectory in the s or u channel that contributes to the t-channel Regge trajectory, then the residue function that has the factorization need not have a zero at any particular value.

For other cases, a modification of (2) may be used. For instance, for $\pi\eta\to\pi\rho$ (also discussed in Refs. 2 and 3) one finds that

$$A(s,t,u)=\frac{\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t))\Gamma(1-\frac{1}{2}\alpha(u))}{\Gamma(\frac{3}{2}-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(\frac{3}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))} \tag{3}$$

is a suitable representation. Once more, for $\alpha(s)+\alpha(t)+\alpha(u)=2$, it reduces to Veneziano's proposal. One can easily imagine reactions for which the restriction is not satisfied: for instance, $\pi\eta'\to\pi\rho$. In these cases, obviously the triple-product representation is more flexible. On the other hand, for those cases where in one of the channels there are no resonances, one is forced to use a double-product formula.

An interesting final example which may be a more definite test to decide between expressions (1) and (2) is $\pi\pi$ scattering. In fact, for this case the double-product representation predicts also a zero of the $\beta_{\rho\pi\pi}$ at $\alpha_\rho(t)=0$ *independently of any nonleading term that one may add and/or any other trajectory in the s, u, or t channel.* There is definite evidence against such a zero.[9] Furthermore, the condition $\alpha(s)+\alpha(t)+\alpha(u)=1$ is badly violated (using the parameters of the ρ trajectory obtained from the same condition applied to $\pi\pi\to\pi\omega$ gives 1.46 for this sum, while the experimental $\alpha_0=0.57$ gives 1.79, for a sum that may vary only between 0 and 3). So it will be interesting if a solution of the triple-product type exists. In fact, the expressions

$$\begin{aligned}
A^{I=0}={}&10\frac{\Gamma(1-\frac{1}{2}\alpha(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))}\\
&+5\frac{\Gamma(1-\frac{1}{2}\alpha(t))\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(1-\frac{1}{2}\alpha(u))}{\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))},\\
A^{I=1}={}&5\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t))\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))}\\
&-5\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))\Gamma(1-\frac{1}{2}\alpha(u))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))},\\
A^{I=2}={}&2\frac{\Gamma(1-\frac{1}{2}\alpha(t))\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(1-\frac{1}{2}\alpha(u))}{\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(s))}\\
&-2\frac{\Gamma(1-\frac{1}{2}\alpha(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))}\\
&+3\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))\Gamma(1-\frac{1}{2}\alpha(u))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))}\\
&+3\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t))\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(1-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))}
\end{aligned} \tag{4}$$

[7] S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967).

[8] The solution obtained from the sum rules is even closer to the form (1). In fact, in Ref. 2 we found it necessary to change the expression for ImA from the usual one

$$\frac{\beta}{\Gamma(\alpha(t))}\left(\frac{\nu}{\nu_0}\right)^{\alpha(t)-1} \quad\text{to}\quad \frac{\Gamma(\alpha(s)+\alpha(t)-1)}{\Gamma(\alpha(s))\Gamma(\alpha(t))}$$

in order to get an exact algebraic solution to the sum-rules equations. This latter expression is exactly equal to the averaged imaginary part of (1).

[9] See L. Bertocchi, in *Proceedings of the Heidelberg International Conference on Elementary Particles,* edited by A. Filthuth (Interscience Publishers, Inc., New York, 1968).

are completely crossing-symmetric. The ρ and f trajectories appear degenerate. These expressions show resonances in the $I=2$ channel, but they are not on the leading trajectories and can be pushed to higher energies by the usual trick of adding nonleading terms. So if the present evidence against the ρ choosing nonsense is confirmed, then the triple-product representation must be chosen to describe $\pi\pi$ scattering.

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K^+-Nucleon Scattering in an Effective Chiral Lagrangian Model

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The S-wave K^+-nucleon scattering lengths are calculated in a phenomenological chiral model containing baryons and pseudoscalar mesons. Good agreement with experiment is found, in contrast to the usual current-algebra results which correspond to unphysical limits of our expressions. The sensitivity of the calculation to various modifications of the Langrangian is also discussed. In particular, the scattering lengths are observed to be rather dependent on the way $SU(3)$ symmetry breaking is introduced into the model.

INTRODUCTION

Many authors[1] have recently discussed applications of chiral $SU(2)\times SU(2)$ and $SU(3)\times SU(3)$ effective Lagrangians wherein the pseudoscalar mesons are taken to transform nonlinearly under the group. In this way many of the so-called current-algebra (CA) results can be obtained when the appropriate extrapolations to zero meson momenta are made. In this note, we compute the K^+-nucleon S-wave scattering lengths with a chiral $SU(3)\times SU(3)$ Lagrangian consisting of baryons and pseudoscalar mesons. Our results reduce to the CA results[2] in the appropriate unphysical limit but are rather different for the physical processes. In this way, good agreement with experiment[3] is found and the reason that the CA results are so poor becomes understandable. We must point out, however, that the results depend on the way in which $SU(3)$ symmetry breaking is introduced. We do not consider here the case of $\bar{K}^+$-nucleon scattering which is complicated because inelastic channels are already open at threshold.

Our choice of effective Lagrangian is made so that the (partially) conserved axial-vector current (PCAC) computed in the canonical way has the experimental D/F ratio. However, the results for the scattering lengths turn out to be extremely insensitive to this ratio. The pion-nucleon scattering lengths are also computed and the same insensitivity is found.

The choice of chiral representation for the octet baryons is, of course, not unique. The $[(8,1),(1,8)]$ representation which does not contain any additional particles is used in the first part of our paper. We may also group a $\frac{1}{2}^-$ singlet together with the $\frac{1}{2}^+$ octet to form the $[(3,3^*),(3^*,3)]$ representation. A discussion of this case is also given and the results are also found to be in good agreement with experiment. Our Lagrangian gives the correct width for the decay $Y_0\rightarrow\Sigma\pi$.

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