Equal-Time Commutator and Superconvergence Sum Rule*

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(Received 30 August 1968)

Applying the infinite-momentum technique to the matrix elements of equal-time commutators, we obtain superconvergence sum rules for the zero-mass-pion–nucleon scattering amplitude, which are identical with those derived from dispersion relations, apart from a possible restriction imposed upon trajectories with integer intercepts. Using this result, we show that a modification of the equal-time commutation relation of pion fields (or of divergence of the axial-vector currents) motivated by a simple quark model is not compatible with the Pomeranchuk theorem.

I. INTRODUCTION

An application of the infinite-momentum technique[1] to the matrix elements of the equal-time commutator of the axial-vector current $A^i{}_\mu(x)$ has led to a successful sum rule (the Adler-Weisberger relation[2]), under the assumptions of partially conserved axial-vector current (PCAC)[3] and current algebra.[4] The sum rule is intimately related to the low-energy theorem on pion scattering, i.e., relations between the s-wave scattering lengths,[5] which is also in good accord with experimental data.[6] As a matter of fact, one of the reasons for the success of the infinite-momentum technique is that, when it is applied to the commutator of the axial-vector currents, one is led to a sum rule which contains a convergent integral.

A similar method has been applied to other equal-time commutators to derive high-energy theorems[7,8] or the ratio of the renormalization constants.[9] In these cases, however, the relevant sum rules contain divergent integrals. In order to avoid this difficulty, various limiting procedures have been used, the validity of which appears to be dubious.

In this article, we discuss similar problems, but take into consideration the high-energy (Regge) behavior of the scattering amplitude that appears in the sum rule. In other words, we subtract the Regge amplitude before taking the infinite-momentum limit. In Sec. II, applying this method to the equal-time commutator of the axial-vector current and its divergence (or the pion field), one obtains a superconvergence sum rule[10–12] for the symmetric part of the zero-mass-pion–nucleon scattering amplitude. The sum rule thus obtained is shown to be identical to that given by Igi's method, i.e., by taking the infinite-*energy* limit in dispersion relations after subtracting the Regge amplitude. A similar calculation is carried out for equal-time commutators of pion fields (or of divergences of the axial-vector currents), leading to a superconvergence sum rule for the antisymmetric amplitude, which is also identical to that obtained from dispersion relations (Sec. III), apart from a possible restriction on trajectories with integer intercepts.

Using the above result, a question of modifying the equal-time commutation relation of pion fields is discussed (Sec. IV).

II. SYMMETRIC AMPLITUDE OF PION-NUCLEON SCATTERING

We start with the relation[13]

$$(2\pi)^4\delta^4(0)\frac{m}{p_0}4\pi f^{(+)}(0,0) = -if_\pi^{-2}2\pi\delta(0)$$
$$\times\langle p|\left[\int A_0^{(+)}(\mathbf{x},x_0)d^3x, \int \partial_\lambda A_\lambda^{(-)}(\mathbf{y},x_0)d^3y\right]|p\rangle$$
$$+if_\pi^{-2}(2\pi)^4\delta^4(0)\langle 0|\left[\int A_0^{(+)}(\mathbf{x},x_0)d^3x,\right.$$
$$\left.\int \partial_\lambda A_\lambda^{(-)}(\mathbf{y},x_0)d^3y\right]|0\rangle, \quad (1)$$

* Work supported in part by the U. S. Atomic Energy Commission.
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where $f^{(+)}(\nu_L,k_0^2)$ $[f^{-}(\nu_L,k_0^2)]$ is the symmetric [antisymmetric] amplitude for mass-k_0-pion–nucleon scattering, the lab energy ν_L is defined by

$$\nu_L=-pk/m=p_0k_0/m\,, \tag{2}$$

and m, p, μ, k, and f_π stand for the nucleon mass, its four-momentum, the pion mass, its four-momentum, and the PCAC constant defined by

$$\partial_\lambda A_\lambda{}^{(\pm)}=f_\pi\mu^2\phi^{(\pm)}\,, \tag{3}$$

$\phi^{(\pm)}$ being the pion field. The quantity $f^{(+)}(0,0)$ is understood as[13]

$$f^{(+)}(0,0)=\lim_{k_0\to 0,\,k_0^2/\nu\to 0} f^{(+)}(\nu,k_0^2)\,. \tag{4}$$

The states $|p\rangle$ and $|0\rangle$ stand for a proton state and the vacuum, respectively, and the second term of the right-hand side of Eq. (1) represents the subtraction of the disconnected diagram.

The method of Refs. 1 and 2 enables us to write Eq. (1) as

$$f^{(+)}(0,0)=\frac{\mu^4|\mathbf{p}|}{2\pi^2}\int_{\mu+\mu^2/2m}^{\infty}\frac{d\nu}{(p_0^2+2m\nu)^{1/2}}\frac{\sigma^{(+)}(\nu_L,k_0^2)}{(\mu^2-k_0^2)^2}\,, \tag{5}$$

where

$$k_0=(\mathbf{p}^2+W^2)^{1/2}-(\mathbf{p}^2+m^2)^{1/2}\,,$$

$$\nu=\frac{W^2-m^2}{2m}=\nu_L+\frac{k_0^2}{2m}\,,$$

$$\sigma^{(\pm)}(\nu,k_0^2)=\tfrac{1}{2}[\sigma_{\pi^-p}(\nu,k_0^2)\pm\sigma_{\pi^+p}(\nu,k_0^2)]\,,$$

and W is the energy in the c.m. system. In deriving Eq. (5), we insert the intermediate-state projections between the two operators of Eq. (1) and sum over them. In contradistinction to the case of the equal-time commutator of axial-vector current, we do not have a contribution from the one-nucleon state, since

$$\langle\mathbf{p}_n|\int\phi^{(-)}(\mathbf{x},x_0)d^3x|\mathbf{p}\rangle=(2\pi)^3\delta(\mathbf{p}-\mathbf{p}_n)\langle\mathbf{p}_n|\phi^{(-)}(0)|\mathbf{p}\rangle$$

$$\propto(2\pi)^3\delta(\mathbf{p}-\mathbf{p}_n)\frac{m}{p_0}\bar{u}(\mathbf{p})\gamma_5u(\mathbf{p})=0\,.$$

As regards multiparticle intermediate states, the contributions from purely disconnected diagrams of the first term are cancelled by the second term, and what remains is brought, by using unitarity, into the form of Eq. (5), where the singularity of the integral has been proved to be the principal value.[2,14]

If one interchanges the integral and the limit $p_0\to\infty$ in Eq. (5), one encounters a divergent integral,

$$\int^{\infty}\sigma^{(+)}(\nu,0)d\nu\,.$$

[14] The proof of Ref. 2 on this point, though plausible, does not seem mathematically rigorous, since it deals with a special limit for an ill-defined quantity such as a product of distributions.

The high-energy behavior of $\sigma^{(+)}(\nu_L,k_0^2)$ for fixed k_0^2 should be $\nu_L{}^{\alpha_P(0)-1}$, where $\alpha_P(0)$ is the intercept of the Pomeranchon, since the Regge trajectory should not depend on the external mass k_0. Therefore, one has to subtract the Regge asymptotic form $\sigma_R{}^{(+)}(\nu_L,k_0^2)$ from $\sigma^{(+)}(\nu_L,k_0^2)$ before taking the infinite-momentum limit. That is, instead of Eq. (5), one should deal with the integral

$$f^{(+)}(0,0)=\frac{\mu^4|\mathbf{p}|}{2\pi^2}\int_{\mu}^{\infty}\frac{d\nu}{(p_0^2+2m\nu)^{1/2}}\frac{1}{(\mu^2-k_0^2)^2}\times\left[\sigma^{(+)}(\nu_L,k_0^2)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sigma_R{}^{(+)}(\nu_L,k_0^2)\right]+\frac{\mu^4|\mathbf{p}|}{2\pi^2}\int_{\mu}^{\infty}\frac{d\nu}{(p_0^2+2m\nu)^{1/2}}\frac{\sigma_R{}^{(+)}(\nu_L,k_0^2)}{(\mu^2-k_0^2)^2}\,. \tag{5'}$$

Here θ stands for the step function, and $\sigma_R{}^{(+)}(\nu_L,k_0^2)$ contains the contribution from the Regge trajectories whose intercepts α are non-negative, so that one could interchange the above-mentioned limiting procedures in the first term of Eq. (5′). For the second term, one should integrate first, then take the limit $p_0\to\infty$. Since the integrand contains a singular function, there is a possibility of obtaining a sensible result by this procedure.[15] If one chooses, however, the expression for the subtraction term as

$$\sigma_R{}^{(+)}(\nu_L,k_0^2)=\sum_{0<\alpha\leq 1}4\pi\beta_\alpha{}^{(+)}P_\alpha(\nu_L/\mu)/\nu_L\,, \tag{6}$$

as is suggested by Regge theory, the second integral of Eq. (5′) leads to a term that behaves like $p_0{}^\alpha$, provided that the residue function is independent of k_0^2. This means that Eq. (6) is not a correct subtraction term to obtain a sensible result through the infinite-momentum limit in Eq. (5′). The simplest choice for $\sigma_R{}^{(+)}(\nu_L,k_0^2)$ to get rid of the $p_0{}^\alpha$ dependence from the

[15] Such an example is a principal-value integral

$$g(y)=P\int_1^{\infty}\frac{y^2f(x)dx}{x^2-y^2}\,,$$

where

$$f(x)\underset{x\to\infty}{\longrightarrow}C+O(1/x^{1+\epsilon})\quad(\epsilon>0).$$

Obviously one cannot perform the limit $y\to\infty$ inside the integral. However, arranging the integral as

$$g(y)=y^2\int_1^{\infty}\frac{f(x)-C}{x^2-y^2}+Cy^2\int_1^{\infty}\frac{1}{x^2-y^2}=y^2\int_1^{\infty}\frac{f(x)-C}{x^2-y^2}+\tfrac{1}{2}Cy\ln\left|\frac{1+y}{1-y}\right|,$$

one obtains the result

$$g(\infty)=-\int_1^{\infty}(f(x)-C)+C.$$

If the kernel $P(1/(x^2-y^2))$ of the integral is replaced by $(1/(x^2+y^2))$, the limit does not exist.

second integral of Eq. (5′) is to set

$$\sigma_R^{(+)}(\nu_L,k_0^2)=\sigma^{(+)}(\infty,0)G_1^{(+)}(k_0^2)+\sum_{0<\alpha<1}4\pi\beta_\alpha^{(+)}G_\alpha^{(+)}(k_0^2)P_\alpha(\nu_L/\mu)/\nu_L\,, \quad (7)$$

where

$$G_\alpha^{(+)}(k_0^2)=1+(2-\alpha)/(\alpha)(k_0^2/\mu^2),\quad 0<\alpha\leq 1. \quad (8)$$

In Eq. (7) we have separated the contribution of the Pomeranchon, for convenience. The residue function can still be a function of k_0^2 in such a way that

$$\lim_{k_0^2\to 0}\beta_\alpha^{(+)}(k_0^2)=\beta_\alpha^{(+)}.$$

We assume, however, that such a k_0^2 dependence does not change the essential part of our discussion. This will be justified *a posteriori* by comparing the result with the superconvergence sum rule which will follow from the dispersion relation. The necessity of factor (8) can be seen from the calculation in the Appendix. In fact, it is just an operation to drop the unwanted p_0^α-dependent terms, leaving the constant term unchanged, and thus showing that the procedure is rather stable.

We also assume the absence of the trajectory with $\alpha=0$, since otherwise it would require the existence of a mass-zero particle with spin zero, which has never been observed. This assumption then makes harmless the singularity which appears in the expression of $G_\alpha^{(+)}(k_0^2)$. Needless to say, if there is a Regge cut which runs over $1>\alpha>0$, the sum in α should be replaced by the corresponding integral. Our discussion will not be changed by this modification.

With these preliminaries, and taking the limit $p_0\to\infty$ in Eq. (5′), one obtains

$$f^{(+)}(0,0)=\frac{1}{2\pi^2}\int_\mu^\infty d\nu\left[\sigma^{(+)}(\nu,0)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sigma_R^{(+)}(\nu,0)\right]-\frac{\mu}{2\pi^2}\sigma^{(+)}(\infty,0)-\sum_{0<\alpha<1}\frac{2\beta_\alpha^{(+)}}{\sin\pi\alpha}P_\alpha(0). \quad (9)$$

This is identical to the superconvergence sum rule of Igi type in the case of zero-mass-pion–nucleon scattering amplitude. In order to see this in more detail, and for convenience, we repeat his argument in a slightly different form which is closely related to the procedure described above.

One may write the once-subtracted dispersion relation

$$\mathrm{Re}f^{(+)}(\nu,k_0^2)=\mathrm{Re}f^{(+)}(\mu,k_0^2)-\frac{g^2K^2(k_0^2)}{4\pi m}\left[\frac{\nu^2}{\nu^2-\nu_B^2(k_0^2)}-\frac{\mu^2}{\mu^2-\nu_B^2(k_0^2)}\right]+\frac{2(\nu^2-\mu^2)}{\pi}\int_{\mu+\mu^2/2m-k_0^2/2m}^\infty\frac{\nu'd\nu'\,\mathrm{Im}f^{(+)}(\nu',k_0^2)}{(\nu'^2-\mu^2)(\nu'^2-\nu^2)}\,, \quad (10)$$

where $gK(k_0^2)$ stands for the "pion"-nucleon coupling constant,

$$K(\mu^2)=1,\quad \nu_B(k_0^2)=-k_0^2/2m\,,$$

and

$$\mathrm{Im}f^{(+)}(\nu,k_0^2)=\frac{(\nu^2-k_0^2)^{1/2}}{4\pi}\sigma^{(+)}(\nu,k_0^2). \quad (11)$$

Putting $k_0^2=0$ and subtracting the expression for $\nu=0$, after rearrangement of several terms, one gets the formula

$$f^{(+)}(0,0)+\frac{\nu^2}{2\pi^2}\int_\mu^\infty\frac{d\nu'}{\nu'^2-\nu^2}\left[\sigma^{(+)}(\nu,0)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sigma_R^{(+)}(\nu,0)\right]$$
$$=\mathrm{Re}f^{(+)}(\nu,0)-\frac{\nu^2}{2\pi^2}\int_\mu^\infty\frac{d\nu'}{\nu'^2-\nu^2}[\sigma^{(+)}(\infty,0)+\sum_{0<\alpha<1}4\pi\beta_\alpha^{(+)}P_\alpha(\nu'/\mu)/\nu']$$
$$=\mathrm{Re}f^{(+)}(\nu,0)-\frac{1}{2\pi^2}\left[\sigma^{(+)}(\infty,0)\frac{\nu}{2}\ln\left|\frac{\nu+\mu}{\nu-\mu}\right|+\sum_{0<\alpha<1}4\pi\beta_\alpha^{(+)}\frac{\pi}{\sin\pi\alpha}\{P_\alpha(0)-\tfrac{1}{2}[P_\alpha(\nu/\mu)+\bar{P}_\alpha(-\nu/\mu)]\}\right], \quad (12)$$

where $\bar{P}_\alpha(-\nu/\mu)$ stands for the principal value

$$\bar{P}_\alpha(-x)=\tfrac{1}{2}[P_\alpha(-x+i\epsilon)+P_\alpha(-x-i\epsilon)]=\cos\pi\alpha\,P_\alpha(x)-\frac{2}{\pi}\sin\pi\alpha\,Q_\alpha(x),\quad x>1. \quad (13)$$

One notices that the Born term in the dispersion relation (10) disappears by the required limiting process (4). Since the assumption of the Regge behavior implies that

$$\lim_{\nu\to\infty}\left[\mathrm{Re}f^{(+)}(\nu,0)+\sum_{0<\alpha<1}\beta_\alpha^{(+)}\frac{P_\alpha(\nu/\mu)+\bar{P}_\alpha(-\nu/\mu)}{\sin\pi\alpha}\right]=0, \quad (14)$$

the limit $\nu\to\infty$ in Eq. (12) leads to Eq. (9).

Although one ends up with an identity, one might infer the following two points from the analysis of this section:

(i) The Regge behavior for the asymptotic form of the scattering amplitude is compatible with the infinite-momentum method; and especially

(ii) The k_0^2 dependence of $\sigma_R^{(+)}(\nu_L,k_0^2)$ given by Eqs. (7) and (8) may be justified *a posteriori.*

Finally, one should remark that the conclusion will not be changed if one includes the trajectory with $\alpha(0)=0$ which has a vanishing residue, such as in the case of the conspirator[16,17] of the pion in the zero-mass approximation. The Regge amplitude due to this trajectory is

$$\mathrm{Re}f^{(+)}{}_{\alpha=0}=-\lim_{\alpha\to 0}\left[\frac{\beta_\alpha{}^{(+)}(1+\cos\pi\alpha)P_\alpha(\nu/\mu)}{\sin\pi\alpha}-\frac{2}{\pi}\beta_\alpha{}^{(+)}Q_\alpha(\nu/\mu)\right]$$

$$=-\lim_{\alpha\to 0}\frac{2\beta_\alpha{}^{(+)}}{\pi\alpha}=C^{(+)} \tag{15}$$

and

$$\mathrm{Im}f^{(+)}{}_{\alpha=0}=\lim_{\alpha\to 0}\beta_\alpha{}^{(+)}P_\alpha(\nu/\mu)=0\,, \tag{16}$$

while the contribution to $\sigma_R^{(+)}(\nu_L,k_0^2)$ may be considered as

$$\lim_{\alpha\to 0}4\pi\beta_\alpha{}^{(+)}G_\alpha{}^{(+)}(k_0^2)P_\alpha(\nu_L/\mu)/\nu_L=-\frac{4\pi^2C^{(+)}}{\nu_L}\frac{k_0^2}{\mu^2}, \tag{17}$$

which amounts to adding $C^{(+)}$ to the right-hand side of Eq. (9). Although Eq. (12) is not changed by the addition of this trajectory, one should add $C^{(+)}$ to the right-hand side of Eq. (14), which establishes the equivalence of the two methods in this case. The argument applies equally to the trajectory of the moving pole, as well as the fixed pole, i.e., $\alpha(t)=0$, as far as they are handled with the above-described limiting procedure. Incidentally, in the case of a fixed pole with vanishing residue, one may consider the real constant $C^{(+)}$ as representing a non-Regge behavior of the scattering amplitude. That is, one may subtract $C^{(+)}$ from the amplitude, then Reggeize the remainder as far as the dispersion relation $f^{(+)}(\nu,0)$ is concerned. In the infinite-momentum method, on the other hand, the definition by the limit, as described above, is the only way of getting a consistent result. Therefore, if the fixed pole with vanishing residue could not be defined as the limiting case $\alpha\to 0$, which may likely be the case, then one would be led to conclude that there should not be such a fixed pole or, equivalently, a real constant $C^{(+)}$ of the non-Regge type in the asymptotic amplitude.

III. ANTISYMMETRIC AMPLITUDE

We now turn our attention to the antisymmetric part of the pion-nucleon scattering amplitude. Application of a similar technique to the preceding section to the matrix element

$$\langle p'|\left[\int\phi^{(+)}(\mathbf{x},x_0)d^3x,\ \int\phi^{(-)}(\mathbf{y},x_0)d^3y\right]|p\rangle=0 \tag{18}$$

leads to

$$0=\int_{\mu+\mu^2/2m}^{\infty}\frac{d\nu}{(p_0^2+2m\nu)^{1/2}}\frac{k_0\sigma^{(-)}(\nu_L,k_0^2)}{(\mu^2-k_0^2)^2}$$

$$=\int_{\mu}^{\infty}\frac{d\nu}{(p_0^2+2m\nu)^{1/2}}\frac{k_0}{(\mu^2-k_0^2)^2}\left[\sigma^{(-)}(\nu_L,k_0^2)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sigma_R{}^{(-)}(\nu_L,k_0^2)\right]+\int_{\mu}^{\infty}\frac{d\nu}{(p_0^2+2m\nu)^{1/2}}\frac{k_0\sigma_R{}^{(-)}(\nu_L,k_0^2)}{(\mu^2-k_0^2)^2}. \tag{19}$$

Here $\sigma_R^{(-)}(\nu_L,k_0^2)$ may be chosen either as[18]

$$\sigma_R{}^{(-)}(\nu_L,k_0^2)=\sum_{-1\le\alpha<1}\frac{4\pi\beta_\alpha{}^{(-)}G_\alpha{}^{(-)}(k_0^2)N_\alpha(\nu_L/\mu)^{\alpha-1}}{\mu} \tag{20a}$$

or as

$$\sigma_R{}^{(-)\prime}(\nu_L,k_0^2)=\sum_{0\le\alpha<1}4\pi\beta_\alpha{}^{(-)\prime}G_\alpha{}^{(-)}(k_0^2)P_\alpha(\nu_L/\mu)/\nu_L+\sum_{-1\le\alpha<0}\frac{4\pi\beta_\alpha{}^{(-)\prime}G_\alpha{}^{(-)}(k_0^2)N_\alpha(\nu_L/\mu)^{\alpha-1}}{\mu}, \tag{20b}$$

where

$$G_\alpha{}^{(-)}(k_0^2)=1+(1-\alpha)/(1+\alpha)k_0^2/\mu^2 \tag{21}$$

and

$$P_\alpha(x)=N_\alpha x^\alpha+N_{-\alpha-1}x^{-\alpha-1}+O(x^{\alpha-2},x^{-\alpha-3})\,, \tag{22}$$

with

$$N_\alpha=2^\alpha\Gamma(\tfrac{1}{2}+\alpha)/\{\sqrt{\pi}\Gamma(1+\alpha)\}\,. \tag{23}$$

The choice of expression (21) for $G_\alpha^{(-)}(k_0^2)$ is again dictated to guarantee the existence of the proper limit $p_0\to\infty$ of Eq. (19). The equality signs at $\alpha=0$ or -1 of the summation in Eqs. (20) stand for the inclusion of terms which are considered as the limiting case taken

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from the inside of the boundary, i.e., $\alpha\to 0+0$ or $\alpha\to -1+0$. The other terms at $\alpha=-1$, those not considered by the above-mentioned limit, are denoted by $\alpha=-1-0$, and are excluded from the summation of Eqs. (20). Having the vanishing residue factor N_α, all trajectories with intercept -1 give no contribution to $\sigma_R^{(-)}(\nu,0)$, but do contribute to $\mathrm{Re}f_R^{(-)}(\nu,0)$. In other words, we have

$$\mathrm{Re}f_R^{(-)}(\nu,0)=\sum_{-1\leq\alpha<1}\frac{\beta_\alpha^{(-)}N_\alpha(1-\cos\pi\alpha)(\nu/\mu)^\alpha}{\sin\pi\alpha}+\frac{2\beta_{-1-0}^{(-)}}{\pi}\frac{\mu}{\nu} \quad (24a)$$

and

$$\mathrm{Re}f_R^{(-)\prime}(\nu,0)=\sum_{-1\leq\alpha<1}\frac{\beta_\alpha^{(-)\prime}N_\alpha(1-\cos\pi\alpha)(\nu/\mu)^\alpha}{\sin\pi\alpha}+\frac{2\beta_{-1-0}^{(-)\prime}}{\pi}\frac{\mu}{\nu}+\frac{\beta_0^{(-)\prime}}{\pi}\ln\left|\frac{\nu+\mu}{\nu-\mu}\right|+O\left(\frac{1}{\nu}\right). \quad (24b)$$

The last term of Eq. (24b) derives from the fact that $\lim_{\alpha\to 0}\bar{P}_\alpha(-x)$ contains the term

$$Q_0(x)=\tfrac{1}{2}\ln[(x+1)/(x-1)]$$

[see Eq. (13)]. Since the trajectory with $\alpha=-1+0$ contributes to $\sigma_R^{(-)}(\nu,k_0^2)$ as

$$\lim_{\alpha\to -1}4\pi\beta_\alpha^{(-)}\frac{1-\alpha}{1+\alpha}\frac{k_0^2}{\mu^3}N_\alpha\left(\frac{\nu}{\mu}\right)^{\alpha-1}=-\frac{8\pi\beta_{-1+0}^{(-)}k_0^2}{\nu^2\mu}, \quad (25)$$

comparison of Eqs. (20) and (24) leads to the relations

$$\beta_{-1-0}^{(-)}=\beta_0^{(-)\prime}+\beta_{-1-0}^{(-)\prime}$$

and

$$\beta_\alpha^{(-)}=\beta_\alpha^{(-)\prime},\quad -1\leq\alpha<1. \quad (26)$$

Choosing Eq. (20a) for the subtraction term, we perform the limit $p_0\to\infty$ in Eq. (19) to obtain

$$\int_\mu^\infty d\nu\left[\nu\sigma^{(-)}(\nu,0)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sum_{-1<\alpha<1}4\pi\beta_\alpha^{(-)}N_\alpha\left(\frac{\nu}{\mu}\right)^\alpha\right]=\sum_{-1\leq\alpha<1}\frac{4\pi\beta_\alpha^{(-)}\mu N_\alpha}{1+\alpha}, \quad (27)$$

where use has been made of the formulas of the Appendix. Note that only the term $\beta_{-1+0}^{(-)}$ appears in the sum rule (27), but not the term $\beta_{-1-0}^{(-)}$. This difference stems from the singularity of $G_\alpha^{(-)}(k_0^2)$ at $\alpha=-1$, as stressed earlier.

On the other hand, from the unsubtracted dispersion relation

$$\mathrm{Re}f^{(-)}(\nu,k_0^2)=\frac{2\nu f^2K^2(k_0^2)(k_0/\mu)^2}{\nu^2-\nu_B^2(k_0^2)}+\frac{2\nu}{\pi}\int_{\mu+\mu^2/2m-k_0^2/2m}^\infty\frac{d\nu'\,\mathrm{Im}f^{(-)}(\nu',k_0^2)}{\nu'^2-\nu^2}, \quad (28)$$

where

$$f^2=g^2/4\pi(\mu/2m)^2=0.081, \quad (29)$$

one gets

$$\frac{\nu^2}{2\pi^2}\int_\mu^\infty\frac{\nu'd\nu'}{\nu'^2-\nu^2}\left[\sigma^{(-)}(\nu',0)\theta\left(\nu'-\mu-\frac{\mu^2}{2m}\right)-\sigma_R^{(-)}(\nu',0)\right]=\nu\left[\mathrm{Re}f^{(-)}(\nu,0)-\sum_{-1\leq\alpha<1}\frac{\beta_\alpha^{(-)}N_\alpha(1-\cos\pi\alpha)(\nu/\mu)^\alpha}{\sin\pi\alpha}-\sum_{-1<\alpha<1}\frac{2\beta_\alpha^{(-)}N_\alpha\mu}{\pi(1+\alpha)\nu}\right]+O(1/\nu^2). \quad (30)$$

Equations (24a) and (30) then lead to the sum rule

$$\int_\mu^\infty d\nu\left[\nu\sigma^{(-)}(\nu,0)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sum_{-1<\alpha<1}4\pi\beta_\alpha^{(-)}N_\alpha\left(\frac{\nu}{\mu}\right)^\alpha\right]=\sum_{-1\leq\alpha<1}\frac{4\pi\beta_\alpha^{(-)}\mu N_\alpha}{1+\alpha}-4\pi\beta_{-1-0}^{(-)}\mu, \quad (31)$$

which is compared to Eq. (27) to yield

$$\beta_{-1-0}^{(-)}\mu=\beta_0^{(-)\prime}\mu+\beta_{-1-0}^{(-)\prime}\mu=0. \quad (32)$$

If one had used Eq. (20b) for the subtraction term, one would have obtained

$$\int_\mu^\infty d\nu\left[\nu\sigma^{(-)}(\nu,0)\theta\left(\nu-\mu-\frac{\mu^2}{2m}\right)-\sum_{0\leq\alpha<1}4\pi\beta_\alpha^{(-)\prime}P_\alpha(\nu/\mu)-\sum_{-1<\alpha<0}4\pi\beta_\alpha^{(-)\prime}N_\alpha(\nu/\mu)^\alpha\right]$$
$$=\sum_{-1\leq\alpha<0}\frac{4\pi\beta_\alpha^{(-)\prime}\mu N_\alpha}{1+\alpha}+4\pi\beta_0^{(-)\prime}\mu \quad (27')$$
$$=\sum_{-1\leq\alpha<0}\frac{4\pi\beta_\alpha^{(-)\prime}\mu N_\alpha}{1+\alpha}-4\pi\beta_{-1-0}^{(-)\prime}\mu, \quad (31')$$

which lead again to Eq. (32). Equations (27′) and (31′) can be derived also from Eqs. (27) and (31) by using the relation

$$\int_1^\infty[P_\alpha(x)-N_\alpha x^\alpha]dx=N_\alpha/(1+\alpha),\quad 1>\alpha>0$$
$$=0,\qquad \alpha=0. \quad (33)$$

Relation (32) would imply that the real part of the antisymmetric amplitude should not have a component $1/\nu$, unless it comes from the trajectory with $\alpha\to -1+0$. This may exclude the existence of a fixed pole at $\alpha(t)=-1$. Of course, it is not clear whether this statement is valid also for the physical pion-nucleon scattering amplitude. The residue function $\beta_{-1-0}^{(-)}$ may well vanish only for zero-mass pions. Barring this ambiguity, an interesting case would be $\beta_{-1\pm 0}^{(-)}=0$, so that one has an accessible prediction

$$\beta_0^{(-)\prime}\mu=0. \tag{34}$$

Recently, several authors have noticed the necessity of a singularity other than the ρ trajectory from the numerical analysis of superconvergence sum rules[10,12] or from the observations of the polarization of the πp charge-exchange scattering[17,19,20] or from a remarkable dipole fit of the electromagnetic form factor of the nucleon.[21] The second ρ trajectory, the so-called ρ', has a small intercept, but its precise value is not known. The magnitude and the sign of the residue function is also not well established[20]; they depend very much on the other input information or assumption. If the intercept of ρ' is small, its residue may well be restricted by the condition (34), i.e., the residue function must be small from the continuity reason. Analysis of Ref. 10 shows that the residue function of ρ' at $t=0$ is small indeed. Incidentally, we should note that our conclusion is independent of the existence of any other trajectory or Regge cuts, although we deal with the zero-mass-pion–nucleon scattering. The mathematical analysis of the validity of our subtraction method is, however, left to be investigated in more detail.[22]

IV. MODIFICATION OF EQUAL-TIME COMMUTATION RELATIONS OF PION FIELDS

Finally we remark that a modification of the equal-time commutator of pion fields, such as

$$[\phi^{(+)}(\mathbf{x},x_0),\phi^{(-)}(\mathbf{y},x_0)]=CV_0{}^3(\mathbf{x},x_0)\delta(\mathbf{x}-\mathbf{y}), \tag{35}$$

where

$$I_3=\tfrac{1}{2}\int V_0{}^3(x)d^3x$$

is the third component of the isospin operator, is not compatible with the Pomeranchuk theorem. Equation (35) may be motivated from a simple quark model in which the pion is identified with the pseudoscalar bilinear form of quark fields.[23] This is considered as the equal-time commutation relation in a simplest example of nonelementary pions. The matrix element of Eq. (35) between one-proton states gives

$$C=\frac{2m}{\pi}\frac{|\mathbf{p}|}{p_0}\int_{\mu+\mu^2/2m}^{\infty}\frac{d\nu}{(p_0{}^2+2m\nu)^{1/2}}\frac{k_0\sigma^{(-)}(\nu_L,k_0{}^2)}{(\mu^2-k_0{}^2)^2} \tag{36}$$

instead of Eq. (19). Using the calculation of the Appendix, and taking the limit $p_0\to\infty$ in Eq. (36) associated with the subtraction method described in this article, one obtains

$$C=-\sigma^{(-)}(\infty,0)/\pi\mu^2, \tag{37}$$

which proves our statement.

The replacement of the fourth component of a vector $CV_0{}^3(\mathbf{x},x_0)$ in Eq. (35) by a scalar $C_S\varphi(\mathbf{x},x_0)$ or a tensor $C_T\varphi_{00}(\mathbf{x},x_0)$, etc., leads to an equation similar to Eq. (36), in which C is replaced by the term $(C_Sm)/p_0$ or $(C_Tm)/p_0$. Since the right-hand side of Eq. (36) starts with $1/p_0{}^2$ when $\sigma^{(-)}(\infty,0)=0$, we conclude[24] that $C_S=C_T=0$. Therefore it is not possible to modify Eq. (18) in a simple manner within the scheme of the local quantum field theory.

ACKNOWLEDGMENTS

This work has been initiated while the author was a summer visitor (1967) at Stanford Linear Accelerator Center (SLAC) and at the Argonne National Laboratory, and was completed at CERN. The author wishes to thank Professor S. D. Drell, Professor K. C. Wali, Professor L. Van Hove, Professor W. Thirring, and Professor J. Prentki for the hospitality extended to him at their respective institutions. Thanks are also due to J. D. Bjorken, Y. Frishman, H. Harari, J. Sullivan, J. Weyers, K. Kawarabayashi, N. N. Khuri, C. B. Chiu, J. Finkelstein, M. Jacob, E. Leader, M. Toller, and W. Weisberger, as well as to his colleagues at the University of Michigan for useful discussions and comments. He is indebted to the Faculty Research Fund of the Horace H. Rackham School of Graduate Studies of the University of Michigan for support which enabled him to travel to Europe.

APPENDIX: RELEVANT INTEGRALS

We compute the following principal-value integrals which are necessary for the discussion of the text:

$$I_\alpha{}^{(+)}=J_\alpha{}^{(+)}+\frac{2-\alpha}{\alpha}K_\alpha{}^{(+)},\quad 0<\alpha\leqslant 1, \tag{A1}$$

where

$$J_\alpha{}^{(+)}=\mu^4p_0\int_\mu^\infty\frac{d\nu}{(p_0{}^2+2m\nu)^{1/2}}\frac{1}{(\mu^2-k_0{}^2)^2}\frac{P_\alpha(\nu_L/\mu)}{\nu_L} \tag{A2}$$

[19] P. Bonamy *et al.*, Phys. Letters **23**, 501 (1966).

[20] H. Högaasen and A. Frisk, Phys. Letters **22**, 90 (1966); R. K. Logan, J. Beaupré, and L. Sertorio, Phys. Rev. Letters **18**, 259 (1967); T. J. Gajdicar, R. K. Logan, and J. W. Moffat, Phys. Rev. **170**, 1599 (1968); W. Rarita and B. M. Schwarzschild, *ibid.* **162**, 1378 (1967).

[21] W. Panofsky, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience Publishers, Inc., New York, 1968), p. 371.

[22] Similar problems as in this article have been discussed recently in a different way by G. Furlan and C. Rossetti (unpublished); S. Fubini and G. Furlan (unpublished). The present author would like to thank these authors for having sent him these articles prior to publication.

[23] The author is indebted to Professors J. D. Bjorken and W. Weisberger for useful suggestions on this question.

[24] A weaker statement would be that the relevant form factor should vanish for the zero momentum transfer. This, however, seems an unlikely case since there is no dynamical reason for that to happen.

and

$$K_\alpha{}^{(+)}=\mu^2p_0\int_\mu^\infty \frac{d\nu}{(p_0{}^2+2m\nu)^{1/2}}\frac{k_0{}^2}{(\mu^2-k_0{}^2)^2}\frac{P_\alpha(\nu_L/\mu)}{\nu_L}. \quad \text{(A3)}$$

Changing the variables

$$\nu_L=\mu x,\quad dx=\frac{d\nu_L}{\mu}=\frac{p_0}{m\mu}\frac{WdW}{(W^2+\mathbf{p}^2)^{1/2}}=\frac{p_0}{\mu}\frac{d\nu}{(p_0{}^2+2m\nu)^{1/2}},$$

$$p_0/m=y,\quad k_0=m\nu_L/p_0=\mu x/y,$$

and defining

$$x_1=1-1/y^2\,\mu/2m+O(1/y^4),$$

one obtains

$$\begin{aligned}J_\alpha{}^{(+)}&=\int_{x_1}^\infty\frac{dx\,P_\alpha(x)}{[1-(x/y)^2]^2x}\\&=\int_1^\infty dx\bigg[\frac{1}{x}-\frac{1}{2}\frac{1}{x-y}-\frac{1}{2}\frac{1}{x+y}\\&\qquad+\frac{1}{4}\frac{y}{(x-y)^2}-\frac{1}{4}\frac{y}{(x+y)^2}\bigg]P_\alpha(x)+O\Big(\frac{1}{y^2}\Big)\\&=\frac{\pi}{\sin\pi\alpha}[-P_\alpha(0)+\tfrac{1}{4}(1+\cos\pi\alpha)(2-\alpha)N_\alpha y^\alpha]\\&\qquad+O(y^{-\alpha-1},y^{\alpha-2})\end{aligned} \quad \text{(A4)}$$

and

$$\begin{aligned}K_\alpha{}^{(+)}&=\int_{x_1}^\infty\frac{dx(x/y^2)P_\alpha(x)}{[1-(x/y)^2]^2}\\&=\frac{y}{4}\int_{x_1}^\infty\bigg[\frac{1}{(x-y)^2}-\frac{1}{(x+y)^2}\bigg]P_\alpha(x)dx\\&=-\frac{\pi\alpha(1+\cos\pi\alpha)N_\alpha y^\alpha}{4\sin\pi\alpha}+O(y^{-\alpha-1}),\end{aligned} \quad \text{(A5)}$$

and therefore

$$I_\alpha{}^{(+)}=-\pi/\sin\pi\alpha P_\alpha(0)+O(y^{-\alpha-1},y^{\alpha-2}). \quad \text{(A6)}$$

Here use has been made of formulas

$$\begin{aligned}&\int_1^\infty\frac{P_\alpha(x)}{x+y}dx=-\frac{\pi}{\sin\pi\alpha}P_\alpha(y),\quad -1<\mathrm{Re}\alpha<0,\\&\int_1^\infty\frac{P_\alpha(x)}{(x+y)^2}dx=\frac{\pi}{\sin\pi\alpha}P_\alpha{}'(y),\quad -1<\mathrm{Re}\alpha<1,\end{aligned} \quad \text{(A7)}$$

and analytic continuation in α to the required value has been carried out. For integral values of α, we have

$$I_1{}^{(+)}=J_1{}^{(+)}=-1+O(1/y^2) \quad \text{(A8)}$$

and

$$\lim_{\alpha\to 0}\alpha I_\alpha{}^{(+)}=2K_0{}^{(+)}=-1+O(1/y^2). \quad \text{(A9)}$$

Similar integrals for the antisymmetric part amplitude are

$$I_\alpha{}^{(-)}=J_\alpha{}^{(-)}+(1-\alpha)/(1+\alpha)K_\alpha{}^{(-)},\quad 0<\alpha<1, \quad \text{(A10)}$$

where

$$\begin{aligned}J_\alpha{}^{(-)}&=\frac{\mu^4p_0{}^2}{m}\int_\mu^\infty\frac{d\nu}{(p_0{}^2+2m\nu)^{1/2}}\frac{k_0}{(\mu^2-k_0{}^2)^2}\frac{P_\alpha(\nu_L/\mu)}{\nu_L}\\&=\tfrac{1}{4}\mu y\int_{x_1}^\infty\bigg[\frac{y}{(x-y)^2}+\frac{y}{(x+y)^2}+\frac{1}{x+y}-\frac{1}{x-y}\bigg]P_\alpha(x)dx\\&=-\pi\mu/4\sin\pi\alpha(1-\cos\pi\alpha)\\&\qquad\times[(1-\alpha)N_\alpha y^{1+\alpha}+(2+\alpha)N_{-\alpha-1}y^{-\alpha}]\\&\qquad-\tfrac{1}{2}\mu M_\alpha y^{-\alpha}(2+\alpha)+O(y^{\alpha-1})\end{aligned} \quad \text{(A11)}$$

and

$$\begin{aligned}K_\alpha{}^{(-)}&=\frac{\mu^2p_0{}^2}{m}\int_\mu^\infty\frac{d\nu}{(p_0{}^2+2m\nu)^{1/2}}\frac{k_0{}^3}{(\mu^2-k_0{}^2)^2}\frac{P_\alpha(\nu_L/\mu)}{\nu_L}\\&=\tfrac{1}{4}\mu y\int_{x_1}^\infty\bigg[\frac{y}{(x-y)^2}+\frac{y}{(x+y)^2}\\&\qquad-\frac{1}{x+y}+\frac{1}{x-y}\bigg]P_\alpha(x)dx\\&=\pi\mu/4\sin\pi\alpha(1-\cos\pi\alpha)\\&\qquad\times[(1+\alpha)N_\alpha y^{\alpha+1}-\alpha N_{-\alpha-1}y^{-\alpha}]\\&\qquad-\tfrac{1}{2}\mu M_\alpha\alpha y^{-\alpha}+O(y^{\alpha-1}).\end{aligned} \quad \text{(A12)}$$

Here we have used the asymptotic form of $Q_\alpha(y)$:

$$Q_\alpha(y)=M_\alpha y^{-\alpha-1}+O(y^{-\alpha-3}), \quad \text{(A13)}$$

with

$$M_\alpha=(\sqrt{\pi})\Gamma(1+\alpha)/2^{\alpha+1}\Gamma(\tfrac{3}{2}+\alpha)=1/\{(1+2\alpha)N_\alpha\}.$$

From Eqs. (A10)–(A12), one obtains

$$\begin{aligned}I_\alpha{}^{(-)}&=-\pi\mu/2\sin\pi\alpha(1-\cos\pi\alpha)(1+2\alpha)/(1+\alpha)N_{-\alpha-1}y^{-\alpha}\\&\quad-(1+2\alpha)/(1+\alpha)\mu M_\alpha y^{-\alpha}+O(y^{\alpha-1}),\quad 0<\alpha<1.\end{aligned} \quad \text{(A14)}$$

For $\alpha=0$ and 1, one has

$$I_0{}^{(-)}=2/y^2J_1{}^{(+)}=-\mu+O(1/y^2). \quad \text{(A15)}$$

Formulas for power function $N_\alpha(\nu/\mu)^\alpha$ are obtained by using the relation

$$\int_0^\infty\frac{x^\alpha}{x+y}dx=-\frac{\pi y^\alpha}{\sin\pi\alpha},\quad -1<\mathrm{Re}\alpha<0, \quad \text{(A16)}$$

and analytic continuation in α. The corresponding quantities are defined by replacing $P_\alpha(\nu_L/\mu)$ by $N_\alpha(\nu_L/\mu)$, denoted by

$$\hat{I}_\alpha{}^{(-)}=\hat{J}_\alpha{}^{(-)}+(1-\alpha)/(1+\alpha)\hat{K}_\alpha{}^{(-)}.$$

They are calculated easily and read

$$\hat{I}_\alpha{}^{(-)}=-\mu N_\alpha/(1+\alpha)+O(1/y^2),\quad -1\leq\alpha<1. \quad \text{(A17)}$$