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Broken-Symmetry Sum Rules from Superconvergent Dispersion Relations*

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Generalized superconvergent sum rules are derived for reactions of the type $P+P\rightarrow P+V$, where P and V are pseudoscalar and vector mesons, respectively. The sum rules are of such a form that a natural range exists, over which saturation by a limited number of resonances is appropriate and outside of which negligible contribution to the sum rules is made. Results include $SU(6)$-type relations such as $m_\rho^2-m_\pi^2=m_{K^*}^2-m_K^2$, $m_\rho=m_\omega$.

SUPERCONVERGENT sum rules have been used to relate high-energy parameters, such as those associated with Regge trajectories and with low-lying resonances, by means of equations of the form[1]

$$\int_0^N \nu^n \operatorname{Im}F(\nu,t)=\sum \frac{\beta N^{\alpha+n+1}}{\Gamma(\alpha+1)(\alpha+n+1)}. \qquad (1)$$

In the derivation of such expressions, the assumption is made that the Regge expansion is an accurate representation of the function F for $\nu\geq N$. In the early superconvergence paper of de Alfaro *et al.*,[2] the information about the Regge terms inferred from the high-energy data was used to determine direct-channel resonance parameters. Saturating the superconvergence relations is related to assuming that the right-hand side of Eq. (1) is negligible with respect to the individual contributions to the left-hand side. This resonance saturation by a few low-lying states has been found in some cases to reproduce the results of various symmetry schemes.[3–5] (Of course, complete saturation of superconvergence relations for a finite range of momentum transfer t is generally only possible if the spin of the particles is not limited.[5]) Furthermore, superconvergence relations for pseudoscalar-meson (P) vector-meson (V) scattering have been studied by Venturi[4] in the approximation of saturation with nonets of V and P. Nonet symmetry breaking at the vertices was related to nonet symmetry breaking in the masses.

We are concerned here with the derivation of sum rules from superconvergent dispersion relations, especially for the case of broken $SU(3)$ symmetry. We will discuss processes of the kind $P+P\rightarrow P+V$ only. These processes have been studied by Ademollo *et al.* (ARVV),[6] using techniques somewhat similar to these used here. They found simple algebraic equations for the parameters of the leading Regge trajectories which are assumed to be exchanged. However, their results are not unambiguous, since some of the parameters treated as constants emerged at the end with a functional dependence on t. Furthermore, ARVV are mainly interested in the exact SU(3) symmetric limit.

The techniques used here will differ from those of ARVV mainly in the use of generalized superconvergence relations which are so contrived as to make saturation by a limited number of resonances appropriate over a natural, well-defined range, and so as to make the contribution to the integrals outside this range negligible. Most previous work in superconvergent dispersion relations, such as saturation by a few resonances, had been used with little or no justification other than hindsight. Moreover, here we are explicitly interested in the broken symmetry case while the emphasis in ARVV is placed on the exact symmetry limit. Finally, our results will not contain an ambiguity of the sort mentioned above, but will show that coupling constants will not turn out to be t dependent.

We should mention that Pakvasa and Papastamatiou[7] have given a derivation of some meson mass formulas by means of superconvergent dispersion relations. They assume that the asymptotic behavior of $PV\rightarrow PV$ amplitudes are given by $SU(3)$ symmetric Regge formulas; they furthermore assume $SU(3)$ symmetry for form factors, and finally treat the case of $PV_1\rightarrow PV_1$, where V_1 is the unitary singlet vector meson. The approach employed by us is free of such symmetry assumptions and makes no use of scattering amplitudes involving fictitious states such as V_1. Of course, we do not obtain, with our limited input, as much information as do Pakvasa and Papastamatiou.

The results that we are able to obtain here by studying reactions of the type $P+P\rightarrow P+V$ include the $SU(6)$ relations $m_\rho^2-m_\pi^2=m_{K^*}^2-m_K^2$ and $m_\rho=m_\omega$. It should be emphasized that for these results just mentioned we make no use of group theoretic arguments.

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They are consequences of dynamical assumptions about asymptotic behavior of scattering amplitudes

We consider the class of reactions

$$P^{\alpha}(p_1)+P^{\beta}(p_2)\to P^{\gamma}(p_3)+V^{\delta}(q)\,, \tag{2}$$

where α, β, γ, δ are $SU(3)$ indices. The scattering amplitude is of the form

$$\begin{aligned} T^{\alpha\beta\gamma\delta}(\nu,t)&=\epsilon_{\mu\nu\rho\sigma}e_{\mu}p_{1\nu}p_{2\rho}p_{3\sigma}A^{\alpha\beta\gamma\delta}(\nu,t)\,,\\ \nu&=\tfrac{1}{4}(p_1+q)(p_2+p_3)=\tfrac{1}{4}(s-u)\,,\\ t&=(p_1-q)^2\,, \end{aligned} \tag{3}$$

where e_μ is the polarization vector of the V meson. Conservation laws and Bose statistics imply that the intermediate states in any channel must have $J^{PC}=1^{--}$, 2^{++}, 3^{--}, $\cdots$. In Regge terminology, the states lie on the usual vector and tensor trajectories.

For fixed-momentum transfer t, the amplitude A is analytic in the cut ν plane with branch points determined by the s- and u-channel thresholds, s_0 and u_0, respectively. The symmetry of the scattering processes implies $s_0=u_0$. Therefore, the cuts in ν start at $\pm\nu_0$, where

$$4\nu_0=2s_0+t-(m_1^2+m_2^2+m_3^2+m_V^2)\,. \tag{4}$$

For example, in the case of $\pi\pi\to\pi\omega$, $s_0=4m_\pi^2$, and $\nu_0=5m_\pi^2-m_\omega^2+t$.

We assume that the asymptotic form for the even-crossing and odd-crossing parts of the amplitude $A^{\alpha\beta\gamma\delta}$ are given by the usual Regge expressions

$$\begin{aligned} A^{\mathrm{even}}(\nu,t)\underset{\nu\to\infty}{\longrightarrow}&\,[\beta_{\alpha\beta\gamma\delta}{}^{V}(t)]/[\sin\pi\alpha_V(t)]\\ &\times[1-e^{-\pi\alpha_V(t)}]\nu^{\alpha_V(t)-1}\\ &\times\alpha_V(t)(\alpha_V(t)+1)\,, \end{aligned} \tag{5}$$

$$\begin{aligned} A^{\mathrm{odd}}(\nu,t)\underset{\nu\to\infty}{\longrightarrow}&\,(\beta_{\alpha\beta\gamma\delta}{}^{T})/[\sin\pi\alpha_T(t)]\\ &\times[e^{-i\pi\alpha_T(t)}+1]\nu^{\alpha_T(t)-1}\alpha_T(t)[\alpha_T(t)+1]\,, \end{aligned} \tag{6}$$

where α_V and α_T are the trajectory functions for the vector and tensor trajectories and β_V, β_T are the analogous residue functions. Note that we have not assumed any $SU(3)$ symmetry for the β's, nor have we assumed factorization of them. The $SU(3)$ indices are just for labeling convenience. The power of ν is reduced because of the helicity flip that is required by the V production.[8] This also gives rise to the multiplicative factor of $\alpha(t)$.

Restricting our attention now to the amplitudes dominated in the t channel by Regge exchange of the vector-meson trajectory, we construct the function $\mathcal{A}$ by

$$\mathcal{A}(\nu,t)\equiv i\nu A(\nu,t)/(\nu^2-\nu_0^2)^{1/2}\,. \tag{7}$$

Then, if t is chosen so that the cuts in ν are separated (i.e., $\nu_0>0$), $\mathcal{A}$ and A will have the same reality properties as the same domain of analyticity. Furthermore, from the above assumptions on the asymptotic form of A, we find

$$\mathcal{A}(\nu,t)\underset{\nu\to\infty}{\longrightarrow} i\nu^{\alpha_V-1}\alpha_V\tan\tfrac{1}{2}\pi\alpha_V\,. \tag{8}$$

Since A is crossing even, $\mathcal{A}$ must be crossing odd and will satisfy a dispersion relation:

$$\mathrm{Re}\mathcal{A}(\nu,t)=\frac{2\nu}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'^2-\nu^2}\frac{\nu'\,\mathrm{Re}A(\nu',t)}{(\nu'^2-\nu_0^2)^{1/2}}\,. \tag{9}$$

We are interested in obtaining expressions that are valid when t is more negative; as we see from Eq. (4), as t grows large and negative, eventually ν_0 becomes negative. This means that the s- and u-channel cuts overlap. In order to study this situation, we employ the device of giving the external vector-meson mass a small imaginary part. This has the effect of separating the cuts. Cauchy's theorem then provides a dispersion relation (for nonforward scattering) when $\nu_0<0$

$$\begin{aligned} \mathrm{Re}\mathcal{A}(\nu,t)=&\frac{2\nu}{\pi}\int_{|\nu_0|}^{\infty}\frac{\nu'\,\mathrm{Re}A(\nu',t)d\nu'}{(\nu'^2-\nu^2)(\nu'^2-\nu_0^2)^{1/2}}\\ &+\frac{\nu}{\pi}\int_{-|\nu_0|}^{|\nu_0|}\frac{\nu'[A_s(\nu',t)+A_u(\nu',t)]}{(\nu'^2-\nu^2)(\nu_0^2-\nu'^2)^{1/2}}\,, \end{aligned} \tag{10}$$

where A_s and A_u are the jumps across the s and u cuts, respectively. We have, of course, assumed that the limit, in which the external vector-meson mass is real, may be taken without introducing any further complications. This limiting process is implicit in the dispersion integrals we write.

From the asymptotic form of $\mathcal{A}(\nu,t)$ given in Eq. (8), we see that for $\alpha_V(t)\leq 0$ the function $\mathcal{A}$ vanishes faster than ν^{-1}. Therefore, for this range of t, the following superconvergence relation must hold:

$$\begin{aligned} 0=&\int_{|\nu_0|}^{\infty}\nu\,\mathrm{Re}A(\nu,t)(\nu^2-\nu_0^2)^{-1/2}d\nu\\ &+\frac{1}{2}\int_{-|\nu_0|}^{|\nu_0|}\nu[A_s(\nu,t)+A_u(\nu,t)](\nu_0^2-\nu^2)^{-1/2}d\nu\,. \end{aligned} \tag{11}$$

The infinite-range dispersion integral converges at the upper limit, since the integrand is asymptotic to $\alpha_V(t)\nu^{\alpha_V(t)-1}\tan\frac{1}{2}\pi\alpha_V(t)$ for $\alpha_V(t)<0$ and is asymptotic to $\nu^{\alpha_1(t)-1}$ for $\alpha_1(t)<\alpha_V(t)=0$, where $\alpha_1(t)$ is the next highest Regge trajectory that contributes to the amplitude $A(\nu,t)$. The contribution from the topmost Regge trajectory vanishes like $\tan\frac{1}{2}\pi\alpha_V(t)$ as $\alpha_V(t)\to 0^-$, since the integration removes the extra factor of $\alpha_V(t)$. It is expected that α_1 and α_V are separated by a finite gap; therefore, the point t_V, defined by

$$\alpha_V(t_V)=0\,, \tag{12}$$

[8] See, e.g., S. Matsuda, Phys. Rev. **169**, 1169 (1968).

which is the endpoint of the range of validity for the superconvergent relation Eq. (11), is singled out as a point at which the contribution of the infinite-range integral should be especially small. Furthermore, it is reasonable to approximate the integrands of both integrals for small ν by the s and u poles of $A(\nu,t)$. In the case of the first integral, this would again lead to the conclusion that the integral is small since the integrand involves $\mathrm{Re}A(\nu,t)$. On the other hand, a pole approximation for the finite-range integral would not, *a priori*, be expected to be small, since the integrand involves essentially $\mathrm{Im}A(\nu,t)$. Finally, the endpoint t_V is probably a better point at which to make a pole approximation to $A(\nu,t)$ than a point further away from the forward direction.

As $|t|$ grows larger, so does ν_0, so does the range of ν over which we will approximate $\mathrm{Im}A$ by poles, and therefore so does the number of poles that we must include. The fewer poles we need, the fewer parameters we must introduce. To illustrate this point, consider again the process $\pi\pi\rightarrow\pi\omega$. The poles in ν that are due to a ρ-meson intermediate state occur at

$$\nu_\rho=\pm\tfrac{1}{4}(2m_\rho^2-m_\omega^2-3m_\pi^2+t)\,,$$

so that

$$\nu_\rho^2-\nu_0^2=\tfrac{1}{4}(m_\rho^2-4m_\pi^2)(m_\rho^2-m_\omega^2+m_\pi^2+t)\,. \qquad (13)$$

Thus, for t only slightly negative, the ρ poles lie in the interval $(-|\nu_0|,|\nu_0|)$. However, the next important pole, which presumably comes from a g-meson intermediate state, will lie outside this interval until t becomes so large and negative that $\alpha_\rho(t)$ is considerably less than zero.

In the $SU(3)$ limit, consider those processes $PP\rightarrow PV$, for which no tensor meson can occur as an intermediate state in any channel. Then we conclude that the point t_V at which the trajectory function $\alpha_V(t)$ for the degenerate octet of vector Regge poles must vanish is determined by

$$t_V+m_V^2-3m_P^2=0\,. \qquad (14)$$

This result agrees precisely with the zero of α_V given by the expression for $\alpha_V(t)$ obtained by ARVV. With the zero of α_V now fixed, we now observe that consideration of the remaining processes for which tensor-meson states can contribute gives no new information, since the tensor-meson poles do not lie in the interval $(-|\nu_0|,|\nu_0|)$. This interval is fixed by putting $t=t_V$, which makes $\nu_0=\frac{1}{2}(4m_P^2-m_V^2)$. The tensor-meson poles occur at $\nu=\pm\frac{1}{2}(m_T^2-m_V^2)$.

We now drop the assumption of $SU(3)$ symmetry and consider the various $PP\rightarrow PV$ processes. Applying the method outlined above to the cases of $\pi\pi\rightarrow\pi\omega$, $K\pi\rightarrow K\omega$, and $K\pi\rightarrow\pi K^*$, in which all have their asymptotic form determined by the ρ trajectory $\alpha_\rho(t)$, we find

$$\begin{aligned}\alpha_\rho(t_\rho)&=0\,,\\ t_\rho&=-2m_\rho^2+m_\omega^2+3m_\pi^2\\ &=-2m_{K^*}^2+m_\omega^2+m_\pi^2+2m_K^2\\ &=-m_{K^*}^2+2m_\pi^2+m_K^2\,,\end{aligned} \qquad (15)$$

which implies

$$\begin{aligned}m_\rho^2&=m_\omega^2\,,\\ m_\rho^2-m_\pi^2&=m_{K^*}^2-m_K^2\,,\\ t_\rho&=-m_\rho^2+3m_\pi^2\,.\end{aligned} \qquad (16)$$

From the ω-trajectory-dominated process $K\pi\rightarrow K\rho$ we get

$$\begin{aligned}\alpha_\omega(t_\omega)&=0\,,\\ t_\omega&=t_\rho\,.\end{aligned} \qquad (17)$$

Next we turn to the K^*-trajectory-dominated processes, such as $K\bar{K}\rightarrow\pi\omega$, $K\bar{K}\rightarrow\pi\rho$, and $\bar{K}K^*\rightarrow\pi\pi$. The last-mentioned reaction satisfies our criterion of having the threshold value of ν, given by

$$4\nu_0=6m_\pi^2-m_K^2-m_{K^*}^2+t \qquad (18)$$

negative for t at the point t_{K^*} where $\alpha_{K^*}(t)=0$. The value of t_{K^*} is uncertain, since there is both a direct-channel ρ pole as well as a crossed-channel K^* pole in this case, and so the position of the vanishing of the amplitude depends on the relative magnitudes of the coupling constants. The value of t_{K^*} may be estimated by using $SU(3)$ coupling constants, and the result is that it is not much different from t_ρ. On the other hand, the reactions $K\bar{K}\rightarrow\pi\omega$ and $K\bar{K}\rightarrow\pi\rho$ give a value for ν_0,

$$\nu_0=\tfrac{1}{4}(6m_K^2-m_\pi^2-m_V^2+t)\,, \qquad (19)$$

which is positive. This means that in our approach we obtain no information from these processes, since there is no contribution in Eq. (11) to the finite-range dispersion integral involving $\mathrm{Im}A$. In particular, we do not encounter any contradiction to the results previously derived from consideration of those reactions for which the ρ trajectory is dominant. Finally, we note that those reactions, such as $\pi\eta\rightarrow\pi\rho$, which are dominated at large energies by vector Regge trajectories in the t channel but have no vector-meson poles in the s or u channels, give no further information and thus no contradictions.

We now turn our attention to those reactions of the type $PP\rightarrow PV$ which have their asymptotic form determined by tensor Regge trajectories; this includes, for example, the process $\pi\pi\rightarrow\eta\rho$. In analogy with the above discussion of the vector-trajectory-dominated processes, we construct the function

$$\mathcal{A}(\nu,t)=\frac{i\nu^2A(\nu,t)}{(\nu^2-\nu_0^2)^{1/2}}\,. \qquad (20)$$

As before, if t is chosen so that the cuts in ν are separated (i.e., $\nu_0>0$), then $\mathcal{A}$ and A will satisfy the same reality properties and will enjoy the same analyticity domain. Furthermore, from the assumptions we have made on the asymptotic form of A, Eq. (6), we find

$$\mathcal{A}(\nu,t) \underset{\nu\to\infty}{\longrightarrow} i\nu^{\alpha_T}(\alpha_T+1)[\tan\tfrac{1}{2}\pi(\alpha_T+1)+i]. \qquad (21)$$

Since A is crossing odd, $\mathcal{A}$ must be crossing odd and will satisfy a dispersion relation:

$$\mathrm{Re}\mathcal{A}(\nu,t)=\frac{2\nu}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'^2-\nu^2}\frac{\nu'^2\,\mathrm{Re}A(\nu',t)}{(\nu'^2-\nu_0^2)^{1/2}}. \qquad (22)$$

Using the same trick to extend t to the left we may obtain an equation similar to that obtained for the crossing-even case above, Eq. (10). Then, from the asymptotic form of $\mathcal{A}(\nu,t)$ given in Eq. (21), we see that for $\alpha_T(t)\leq-1$ the function $\mathcal{A}$ vanishes faster than ν^{-1}. Therefore, for this range of t, the following superconvergence relation must hold:

$$0=\int_{|\nu_0|}^{\infty}\nu^2\,\mathrm{Re}A(\nu,t)(\nu^2-\nu_0^2)^{-1/2}d\nu$$
$$+\frac{1}{2}\int_{-|\nu_0|}^{|\nu_0|}\nu^2[A_s(\nu,t)+A_u(\nu,t)](\nu_0^2-\nu^2)^{-1/2}d\nu. \qquad (23)$$

Thus everything is similar to the crossing-even case, with $\alpha_V(t)$ replaced by $\alpha_T(t)+1$ and with an extra factor of ν in the integrand. We conclude that the point t_T, defined by

$$\alpha_T(t_T)=-1, \qquad (24)$$

which is the endpoint of the range of validity for the superconvergence relation Eq. (23), is singled out as a point at which the contribution of the infinite-range integral should be especially small. We use the same approximations here as above, namely, dropping the infinite-range integral and using a pole approximation for A in the integral $(-|\nu_0|,|\nu_0|)$. There are two significant points of difference between the present situation and the vector-trajectory case. The most obvious one is that here we are considering trajectories in the left-half complex-angular-momentum plane. We have ignored any possible complications that might arise from the existence of singularities other than the ordinary Regge pole described by $\alpha_T(t)$. Cuts and essential singularities may well contribute as we approach $\alpha_T=-1$. Moreover, the continuation in t to the point t_T is a stronger assumption than we made previously if only because it is a much greater distance away from the forward direction. The second point of difference between the present case and that discussed previously is that we no longer will have only vector-meson poles contributing to the finite-range integral. We have continued in t so far that $|\nu_0|$ is now large enough that the tensor-meson poles most likely fall in the interval $(-|\nu_0|,|\nu_0|)$. Consequently, any results obtained along these lines will involve coupling constants and masses of both vector mesons and tensor mesons. Since the experimental situation regarding the octet of tensor mesons is uncertain at present, we will only consider the theoretically simpler $SU(3)$-symmetry limit here.

If the tensor trajectory is degenerate with the vector trajectory, that is, $\alpha_V=\alpha_T$, and if we approximate both by straight lines, then we would expect $\alpha_T=-1$ at $t=-3m_V^2+6m_P^2$ from our discussion above of $\alpha_V(t)$. Then we would find

$$4\nu_0=-4m_V^2+11m_P^2. \qquad (25)$$

The position of the vector-meson pole ν would be given by

$$4\nu_V=-2m_V^2+3m_P^2. \qquad (26)$$

The tensor-meson mass is, of course, fixed by the value of t for which $\alpha_T(t)=2$, and so

$$m_T^2=3(m_V^2-m_P^2), \qquad (27)$$

which implies that the position of the tensor-meson pole ν_T is given by

$$\begin{aligned}4\nu_T&=+2m_V^2-3m_P^2\\&=-4\nu_V.\end{aligned} \qquad (28)$$

Therefore we have the result that $\nu_V^2=\nu_T^2$; furthermore, we easily find

$$\nu_0^2-\nu_V^2>0, \qquad (29)$$

so that both the vector and tensor poles lie in the range $(-|\nu_0|,|\nu_0|)$. With the same approximation for α_V we may estimate the position ν_3 of the next most likely pole to contribute, that of the 3^- octet. The result is that this pole occurs at $4\nu_3=6m_V^2-3m_P^2$ which is outside the interval $(-|\nu_0|,|\nu_0|)$. Thus we need only consider vector- and tensor-meson poles. Using this information to evaluate the sum rule Eq. (23), we obtain

$$g_{VPV}g_{VVP}/g_{PPT}g_{PVT}=2(2m_V^2-3m_P^2), \qquad (30)$$

where we have adopted the definitions of ARVV for the coupling constants g_{PPV}, etc. Because of the equality of ν_V and ν_T all our kinematic factors have disappeared and we have found the ratio of coupling constants to be as given by ARVV if their relation is evaluated at t_T.