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Phenomenological Multichannel ND^{-1} Analysis of πN Scattering*

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We investigate a multichannel ND^{-1} parametrization for the π-N partial-wave amplitudes. Using a two-channel, one-input-pole form of the model, we present fits to all the interesting π-N partial waves up to pion laboratory kinetic energy $\sim$1.4 BeV. Reasonable fits are obtained for most of the partial waves. An exceptionally good fit is obtained for the P_{33} partial wave, in which the N_{33}^* resonance appears as a quasi-bound state with respect to the higher-mass inelastic channel. Applications of the model are discussed.

I. INTRODUCTION

PHASE-SHIFT analyses[1–5] of pion-nucleon scattering have revealed many interesting features such as highly inelastic resonances. The principal difficulty encountered in attempting to obtain phase shifts for pion laboratory kinetic energy $T_L \gtrsim 1.5$ BeV is that the number of sets of phase shifts which produce acceptable fits to the data grow rapidly with increasing energy. Thus the method of interpolating from one energy to the next plays a crucial role in obtaining the phase shifts. Energy-independent analyses are done, and two general approaches have been used to eliminate the ambiguities:[6] The Berkeley group[5] defines a distance between solutions and joins those "closest" together but makes no use of the dynamics of π-N scattering. The analysis of Lovelace and co-workers[3] uses partial-wave dispersion relations (parametrized in a polynomial expansion) to produce very smooth phase shifts from the single energy solutions.

The purpose of this article is to investigate a multichannel ND^{-1} parametrization for the partial-wave dispersion relations which we believe is more efficient and flexible than that used by Lovelace since it contains the correct analytic structure of the inelastic cuts. Furthermore, the results can be dynamically interpreted as well as directly applied to the detailed analysis of other experiments, e.g., the inelastic πN experiments and pion photoproduction.

The partial-wave amplitudes are represented by a multichannel ND^{-1} formalism with the interaction given by simple poles. For each partial wave, we use as few poles and channels as the data will allow. Ball, Shaw, and Wong[7] showed that two channels and a simple pole were sufficient to produce a reasonable fit to the rather complicated features of the P_{11} partial wave. We will show that reasonable fits up to moderate energies can also be obtained for most partial waves (with the notable exceptions being the S waves) with two channels and one pole. We believe that the addition of one or two more inelastic channels would produce really good fits for all the partial waves.

One of the basic requirements for the success of this method of parameterizing a partial-wave amplitude is that the inelastic scattering channels can be well-represented by quasi-two-body channels such as σ-N,[8] π-N_{33}^*, ρ-N, or actual two-body channels such as η-N and K-Y. At present the only partial wave in which production is known to be dominated by a single channel is S_{11} in which η production accounts for most of the inelastic cross section.[9] However, we emphasize that in order for the ND^{-1} formalism to give an accurate representation all the "important" inelastic channels should be considered even if in the energy region of interest the channel may be closed (see, e.g., the analysis of the N_{33}^* in Sec. III). Considerable experimental effort is being directed at analyzing the pion-nucleon inelastic processes in terms of two-body channels.[10] In the absence of this information we will represent the inelastic scattering in each partial wave by a single phenomenological channel. Furthermore, for simplicity, we will use a single-interaction pole.

The two-channel, one-pole N/D parametrization (that we use for the calculations described in Sec. III) is a simple function of only five parameters. Three parameters are directly related to the strength of the interaction in the two elastic amplitudes and the production amplitude, the input-pole position is related to

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[1] L. Roper, Phys. Rev. Letters **12**, 340 (1964); L. Roper and R. Wright, Phys. Rev. **138**, B921 (1965).

[2] R. Bransden, P. O'Donnell, and R. Moorhouse, Phys. Letters **11**, 339 (1964); Phys. Rev. **139**, B1556 (1965).

[3] C. Lovelace, in Proceedings of the Conference on πN Scattering, 1967 (to be published); CERN Report No. Th-837, 1967 (unpublished); A. Donnachie, A. Lea, and C. Lovelace, Phys. Letters **19**, 146 (1965).

[4] P. Bareyre, C. Bricman, and G. Villet, Phys. Rev. **165**, 1730 (1968); P. Bareyre, C. Bricman, A. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965).

[5] C. Johnson and H. Steiner, in Proceedings of the Conference on πN Scattering, 1967 (to be published); C. Johnson, Lawrence Radiation Laboratory Report No. UCRL-17683, 1967 (unpublished).

[6] See the first papers in Refs. 3 and 5.

[7] J. Ball, G. Shaw, and D. Wong, Phys. Rev. **155**, 1725 (1967).

[8] We will refer to an $I=0$, S-wave $\pi\pi$ enhancement as a σ.

[9] See, e.g., J. Ball, Phys. Rev. **149**, 1191 (1966).

[10] A. H. Rosenfeld and P. Soding, in Proceedings of the Conference on πN Scattering, 1967 (to be published); Lawrence Radiation Laboratory Report No. UCRL-18154, 1968 (unpublished).

the range of the interaction, and the fifth parameter is the threshold of the phenomenological channel chosen to correspond to the effective threshold of the inelastic scattering.

The analytic properties of the partial-wave amplitude are reasonable, having the correct threshold behavior and being analytic everywhere in the complex energy plane except for the physical cut and the input pole. Furthermore, the amplitude is decomposed into N and D which is convenient for applications to electromagnetic and weak processes. The fact that the analytic structure of the amplitudes are reasonable allows one to search for nearby poles on unphysical sheets of the scattering amplitude and thereby distinguish true resonances from "kinematic bumps".

This model can readily be extended to include more two-body channels and/or more input poles. Although we obtain reasonable fits to most of the partial waves, this extension is necessary to quantitatively fit the detailed features found in the phase-shift analyses. Based on our calculations, we think that in general it is more important to consider additional inelastic channels rather than more poles and that the addition of one or two more inelastic channels would produce good fits for all the partial waves. This extension is certainly feasible (in terms of introducing new parameters) since there are many data points in each partial wave. Furthermore, we expect that in the near future, the shift analyses will be extended to higher energies and more detailed knowledge of the production processes will become available.

In Sec. II, we will formulate the model. Section III contains a comparison between the calculations using our model and several of the more interesting phase shifts. In particular we note that an exceptionally good fit is obtained for the P_{33} partial wave in which the N_{33}^* resonance appears as a quasibound state of the higher-mass inelastic channel.

In Sec. IV we draw some general conclusion from the calculations in Sec. III, and discuss some applications of the model. In particular by considering pion photoproduction, we illustrate how partial-wave amplitudes obtained from a multichannel, energy-dependent phase-shift analysis could be used in calculating nonstrong processes.

II. PARAMETRIZATION OF THE PARTIAL-WAVE AMPLITUDE

Consider a partial-wave amplitude for a system of n-coupled two-body channels. As a function of the total energy W, the ND^{-1} equations are

$$\begin{aligned} f_{ij} &= (2i\rho_i^{1/2}\rho_j^{1/2})^{-1}(S_{ij}-\delta_{ij}) = (ND^{-1})_{ij}\,, \\ N_{ij}(W) &= B_{ij}(W) + \sum_{k=1}^{n} \frac{1}{\pi}\left(\int_{\alpha_i}^{\infty} + \int_{-\infty}^{-\alpha_i}\right) dW' K_{ik}(W,W')N_{kj}(W')\,, \\ K_{ik}(W,W') &= \left[B_{ik}(W') - \frac{W-W_0}{W'-W_0}B_{ik}(W)\right]\rho_k(W')\frac{1}{W'-W}\,, \\ D_{ij}(W) &= \delta_{ij} - \frac{W-W_0}{\pi}\left(\int_{\alpha_i}^{\infty} + \int_{-\infty}^{-\alpha_i}\right) dW' \frac{\rho_i(W')N_{ij}(W')}{(W'-W_0)(W'-W-i\epsilon)}\,, \end{aligned} \tag{1}$$

where ρ_i is the phase-space factor and α_i the threshold for the ith channel. The amplitudes f in the $\pm W$ regions for a given J have the opposite parity. The input generalized potential B (regular in the physical scattering region) is symmetric as required by time-reversal invariance; the solutions f are symmetric and independent of the subtraction point W_0 in D.

Our model consists of representing the combined effects of the interaction cuts and the $-W$ unitarity cut by a sum of p poles in B. That is, in (1) we neglect the $-W$ region and write

$$B_{ij} = \sum_{r=1}^{p} \frac{g_{ij}^r}{W-W_r}\,. \tag{2}$$

The kernel $K(W,W')$ is degenerate, and the equations can be solved by quadrature. A useful relation gives N in terms of D evaluated at the pole positions W_r:

$$N_{ij}(W) = \sum_{r=1}^{p}\sum_{k=1}^{n} \frac{g_{ik}^r}{W-W_r}D_{kj}(W_r) \equiv \sum_{r=1}^{p} \frac{\gamma_{ij}^r}{W-W_r}\,. \tag{3}$$

In the case where only one interaction pole at W_1 is used (making the subtraction at $W_0=W_1$), the solutions to (1) are

$$\begin{aligned} f_{ij} &= \sum_{k=1}^{n} N_{ik}(D^{-1})_{kj}\,, \\ N_{ij}(W) &= B_{ij}(W) = \frac{g_{ij}}{W-W_1}\,, \quad g_{ij}=g_{ji} \\ D_{ij}(W) &= \delta_{ij} - g_{ij}d_i\,, \\ d_i &= \frac{W-W_1}{\pi}P\int_{\alpha_i}^{\infty} \frac{\rho_i(W')dW'}{(W'-W_1)^2(W'-W)} + \frac{i\rho_i(W)}{(W-W_1)}\theta(W-\alpha_i)\,. \end{aligned} \tag{4}$$

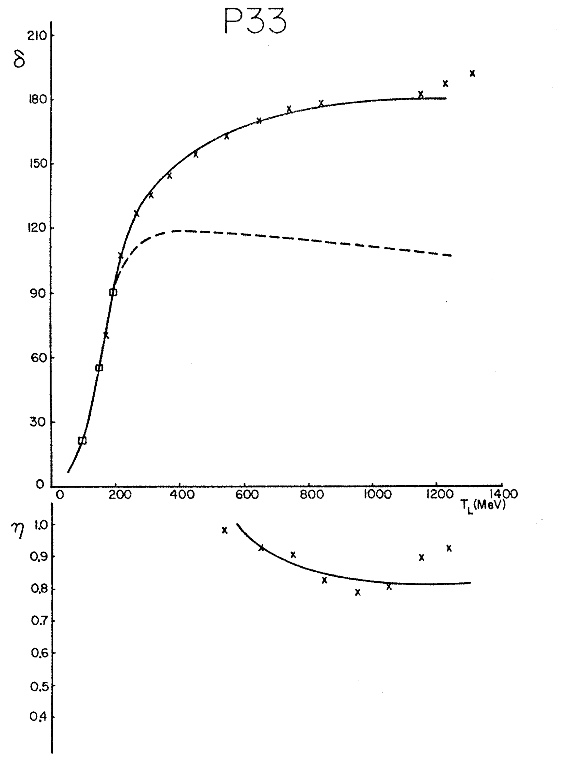

FIG. 1. Calculated values of δ and η for the P_{33} partial wave versus laboratory kinetic energy T_L. In Figs. 1–8, the x's are from the phase-shift analysis of Ref. 3 and the three □'s are the points which the calculated curves are forced to pass through. Here the □'s are $\delta(98\text{ MeV})=21.2°$, $\delta(151)=45.3°$, $\delta(195)=90.0°$. The solid curve is calculated using the two-channel one-pole model with $W_1=6.5$, $\alpha_2=10.7$, $g_{11}=70.15$, $g_{12}=253$, $g_{22}=1005$. The dashed curve is calculated using a single-channel two-pole model with $B=-1157/(W+13)+4700/(W+618)$.

Here, if we use n channels in a given πN partial wave, there are $\frac{1}{2}(n^2+n)$ g's (plus W_0) which we vary to fit the S matrix,

$$S=\eta e^{2i\delta},$$

as a function of energy for a given partial wave $l_{2I,2J}$.

III. ONE-POLE, TWO-CHANNEL FITS TO THE πN PARTIAL WAVES

To illustrate the usefulness of the simple ND^{-1} model described in Sec. II, we consider one input pole in B and approximate all the inelastic channels by a single phenomenological channel, i.e., we use Eqs. (4) with $n=2$.

Denote the πN channel as 1 and the phenomenological second channel as 2. The phase-space factors are taken to be[11]

$$\rho_i=(k_i/W)^{2l_i+1}, \tag{5}$$

$$k_i^2=[W^2-(m-\mu_i)^2][W^2-(m+\mu_i)^2]/4W^2, \tag{6}$$

with $\mu_1=1$ and μ_2 taken as a parameter. Clearly the choice of the power of W in (5) is arbitrary. However. we investigated other forms and found that the fits to the data were not sensitive to this choice. Also note that in (6) we have decomposed the inelastic channel into a nucleon plus a meson; further, we choose the orbital angular momentum $l_2=l_1$ (except where otherwise noted). These latter assumptions can obviously be dropped when we treat the inelastic channels more realistically,

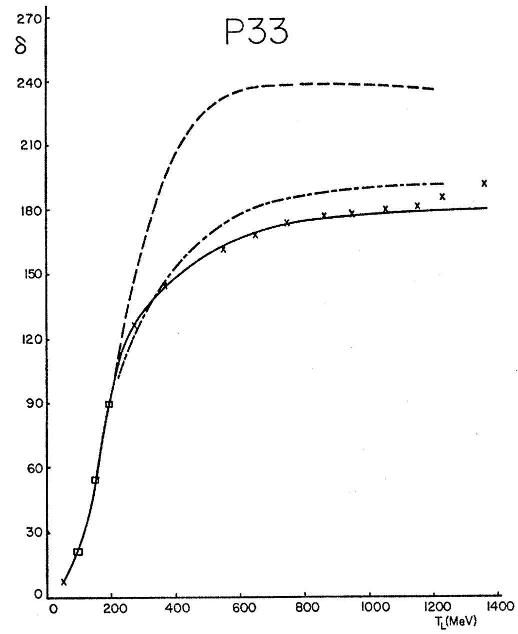

FIG. 2. Variations of model calculations as a function of W_1 for the P_{33} partial wave. The solid curve is the same as in Fig. 1. The dash-dot curve corresponds to the parameters $W_1=6.0$, $\alpha_2=10.7$, $g_{11}=93$, $g_{12}=224$, $g_{22}=928$. The dashed curve corresponds to the parameters $W_1=2.0$, $\alpha_2=10.7$, $g_{11}=229$, $g_{12}=109$, $g_{22}=807$.

i.e., explicitly consider all the "important" two-body and quasi-two-body channels for a given partial wave.[12]

Our calculated fits to the phase-shift analyses were obtained in the following manner. For a given μ_2 and input pole position W_1, the integrals d_i in (4) are calculated. Taking either 3 δ's below the inelastic threshold (so that $\eta=1$), or 1 δ below and δ and η at one energy above the inelastic threshold from the phase-shift analyses, Eqs. (4) were solved for the 3 input couplings g_{ij}. We choose α_2 $(=m+\mu_2)$ to correspond to the inelastic threshold, and varied W_1 until we obtained a reasonable fit. No χ^2 search on the parameters was made.

In Figs. 1–8 we compare our model calculations with the phase-shift analyses for those partial waves which exhibit interesting features for $T_L \lesssim 1.2$ BeV. (Note that the analysis of the P_{11} partial wave was reported in Ref. 6.)

[11] We use units $\hbar=c=m_\pi=1$.

[12] In treating an actual quasi-two-body channel (consisting of particles $a+b$ where b decays strongly into $c+d$) one might use the phase-space factor [instead of Eq. (5)]

$$\rho_2(W)=\int_{m_c+m_d}^{W-m_a} d\omega_{cd}\,\rho_2(W,\omega_{cd})\,|f_{cd\to cd}(\omega_{cd})|^2.$$

The P_{33} partial wave has received the most attention experimentally and theoretically. As seen in Fig. 1, the detailed shape of the P_{33} resonance is easily fit by this model. However, the dominant force is the diagonal interaction in the inelastic channel and the N_{33}^* appears as a quasibound state or Castillejo-Dalitz-Dyson (CDD) effect[13] with respect to the πN channel. A one-channel calculation which approximates the input B by 2 poles is also shown in Fig. 1. This one-channel model does not give a good fit (and we note that the nearby input pole in B is repulsive). Furthermore, the two-channel model yields a satisfactory fit to the inelastic factor η.

The sensitivity of the P_{33} solution to the position W_1 of the input pole in B is shown in Fig. 2. The value[11] $W_1 = 6.5$ gives an excellent fit. Recall that in the static model,[14] the position of the input pole due to nucleon exchange (which dominates the πN forces in the P_{33} wave) is at 6.7. We caution that this sensitivity of the fit to W_1 has been exaggerated by fitting the 3 δ's at low energies. The two-channel, one-pole calculations in Figs. 1–8 all have the pole position $W_1 > 0$. We note that in situations where $W_1 < 0$, the physical interpretation of these pole positions is not as clear.

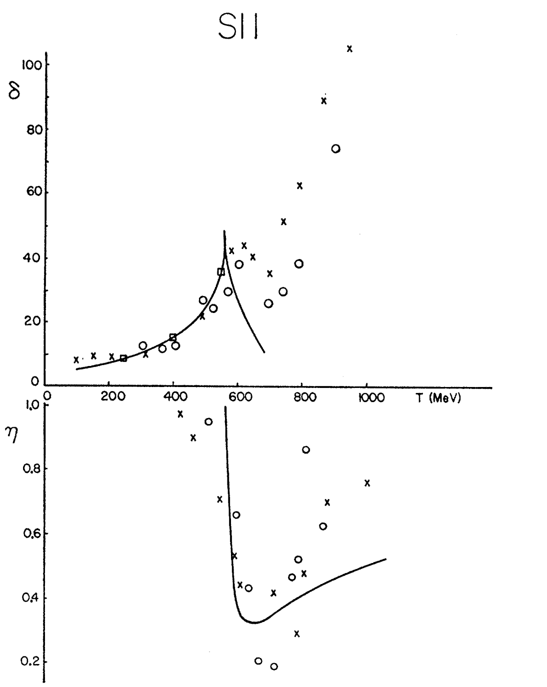

FIG. 3. Calculated values of γ and η for the S_{11} partial wave versus T_L. The □'s are $\delta(250) = 9.5°$, $\delta(400) = 15.0°$, $\delta(550) = 35.0°$. The O's (in Figs. 3 and 8) correspond to the phase-shift analysis of Ref. 4. The parameters for the solid curve are $W_1 = 5.0$, $\alpha_2 = 10.6$, $g_{11} = 0.5$, $g_{12} = 13.6$, $g_{22} = 17.9$.

It is likely, however, that with enough input poles in the πN channel, we would find solutions which would fit all the present data in such a manner that the P_{33} resonance did not appear as a quasibound state with respect to a higher mass channel (as in the above calculation). We can only appeal to "simplicity." Furthermore, even if the physical interpretation of the P_{33} partial wave obtained from the two-channel calculation is essentially correct, we cannot conclude that the Chew-Low bootstrap results[14] were totally fortuitous. The bootstrap equations only give relations among the residues. Thus even though the forces determining the position of the N_{33}^* might come from a second channel, the forces in the πN channel might have considerable effect in determining the residue.

The two-channel one-pole fits to the S_{11}, S_{31}, D_{13}, D_{15}, F_{15}, and F_{37} partial waves are given in Figs. 3–8. The poorest fits are for the S waves; clearly more inelastic channels would greatly improve the results. Before making some general conclusions (in the next section), we make a few comments on some of the individual cases.

S_{11}: The inelastic channel was taken to be ηN since it is known that it dominates the reaction cross section for this partial wave. Here we again stress that in order to be an accurate representation, all the important

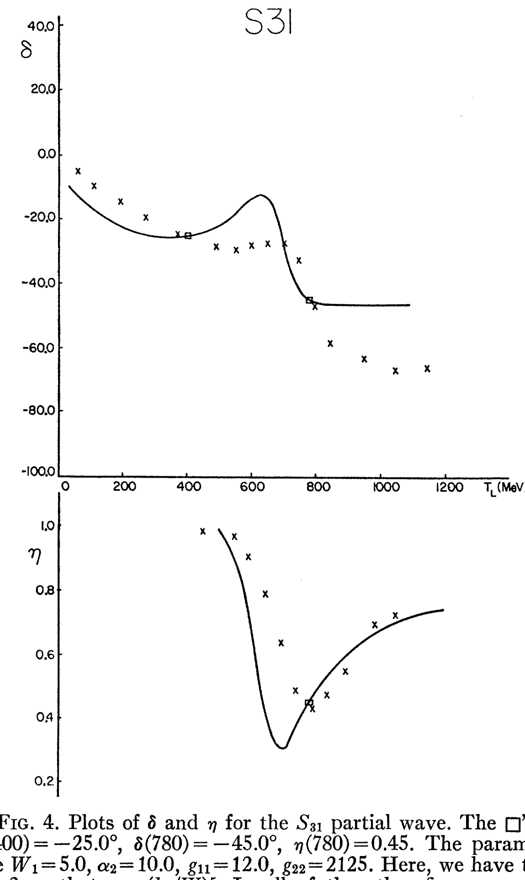

FIG. 4. Plots of δ and η for the S_{31} partial wave. The □'s are $\delta(400) = -25.0°$, $\delta(780) = -45.0°$, $\eta(780) = 0.45$. The parameters are $W_1 = 5.0$, $\alpha_2 = 10.0$, $g_{11} = 12.0$, $g_{22} = 2125$. Here, we have taken $l_2 = 2$ so that $\rho_2 = (k_2/W)^5$. In all of the other figures, we have taken $l_2 = l_1$.

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[13] See, e.g., M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters 14, 270 (1965).

[14] G. Chew, Phys. Rev. Letters 9, 233 (1962); F. Low, *ibid.* 9, 279 (1962).

inelastic channels must be explicitly included even if they are closed in the energy region of interest.

S_{31}: The angular momentum l_2 of the inelastic channel was taken to be 2 so that $\rho_2=(k_2/W)^5$. Even though all our fits were poor, this choice for l_2 seemed to be the best. (For all the other partial waves, the fits are given for $l_2=l_1$). If we attempted to fit the phase-shift analysis δ at $-60°$ instead of the $-45°$ point, our calculated curve had a jump of $\sim\pi$ (and took on the value $+120°$).

D_{13}: The model calculations here were quite sensitive to the position of the inelastic threshold.

IV. CONCLUSIONS

In general, we observe that, with the exception of the S_{11} and S_{31} partial waves, reasonably good fits to the phase-shift analyses can be obtained up to moderate energies with the simple two-channel, one-pole ND^{-1} model. We would like to inject a note of caution concerning the procedure of fitting results of phase-shift analyses: The analyses of different groups 1–5 are certainly not always in agreement (see, e.g., the experimental points from Refs. 3 and 4 in Figs. 3 and 8). Note that errors in δ become large when η is small. Furthermore, as emphasized by Lovelace,[6] the errors on the phase shifts in different partial waves are correlated and thus it is difficult to interpret the meaning of an error associated with a given phase shift. Thus, in principle, one should really use our model directly in an analysis of the observables.

The extension of the calculations to include more input poles and more channels is easily done. We believe (as in the case of the P_{33} partial wave discussed above) that the most effective way to extend the present calculations is to explicitly consider more inelastic channels for each partial wave. We think that really good fits could be obtained for all the partial waves by treating up to 3 (or 4) inelastic channels. They would correspond to the actual 2-body (e.g., ηN) or quasi-two-body inelastic channels (e.g., σN, πN_{33}^*, ρN) that were important in the particular partial wave. This would then involve determining 10 (or 15) input-interaction couplings g_{ij} (plus the pole position W_1). This is not unreasonable since in a given partial wave up to $T_L\sim 1600$ MeV, there are more than 40 values of η and δ to fit.[1–5]

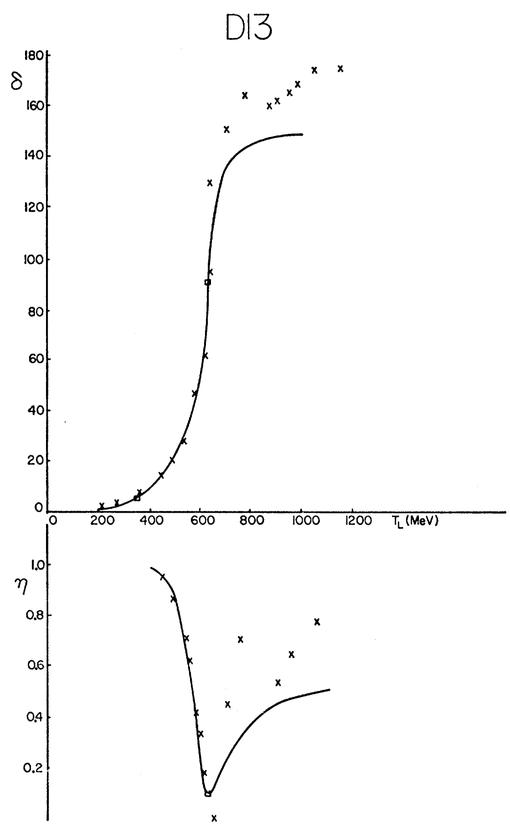

FIG. 5. Plots of δ and η for the D_{13} partial wave. The □'s are $\delta(350)=5.0°$, $\delta(630)=90.0°$, $\eta(630)=0.1$. The parameters are $W_1=5.0$, $\alpha_2=9.8$, $g_{11}=-74$, $g_{12}=1074$, $g_{22}=1659$.

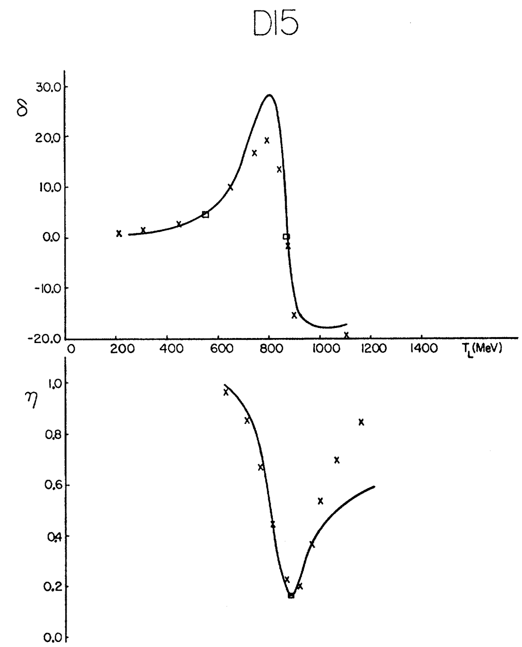

FIG. 6. Plots of δ and η for the D_{15} partial wave. The □'s are $\delta(550)=5.4$, $\delta(870)=0.0$, $\eta(870)=0.16$. The parameters are $W_1=5.0$, $\alpha_2=10.7$, $g_{11}=-51.9$, $g_{12}=699$, $g_{22}=2084$.

This proposed program has the large advantage that it could be directly tied into the analysis of the inelastic πN bubble-chamber data in which one would treat the $2\pi N$ events in terms of quasi-two-body channels. Large numbers of these inelastic events are now being measured.[9]

In addition, the parameters of the model can be interpreted physically, i.e., the g couplings might be compared with theory. Also (as discussed above for the P_{33} partial wave), the dynamical origin of a resonance can be examined, i.e., whether it is mainly due to the forces in the elastic or inelastic channels.[13] We should bear in mind, however, that all such conclusions are very model dependent.

An important theoretical application of the ND^{-1} form which we have suggested is in the calculation of the strong-interaction rescattering effects in weak and electromagnetic processes. To illustrate how this is accomplished, we will consider the problem of pion photoproduction. The unitarity condition satisfied by a particular partial-wave photoproduction amplitude of M is

$$\mathrm{Im} M = f^{+}\rho M\,, \qquad (7)$$

where if f, the strong interaction f matrix, has n channels, M is a column vector with n components representing the photoproduction of each of the n two-body channels. The general problem of including the strong-interaction effects is then the construction of M which satisfies Eq. (7) on the right-hand cuts and has the left-hand singularities given by the input (Born approximation). The particular choice of the interaction terms used to describe photoproduction is not relevant to our discussion and therefore we will assume that for each partial wave we have prescribed functions $B_{ij}(s)$ which contain only the desired left-hand singularities and are analytic elsewhere.

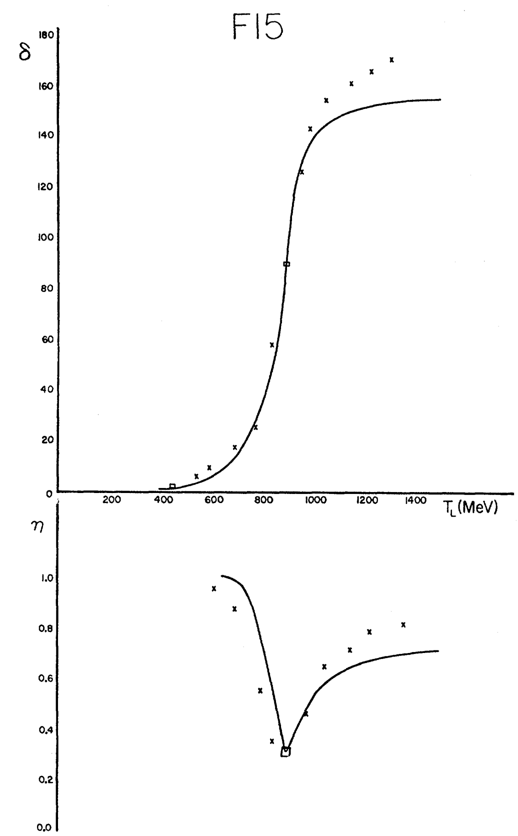

FIG. 7. Plots of δ and η for the F_{15} partial wave. The □'s are $\delta(195)=0.064$, $\delta(900)=90.0$, $\eta(900)=0.3$. The parameters are $W_1=5.0$, $\alpha_2=10.7$, $g_{11}=-401$, $g_{12}=5435$, $g_{22}=12930$.

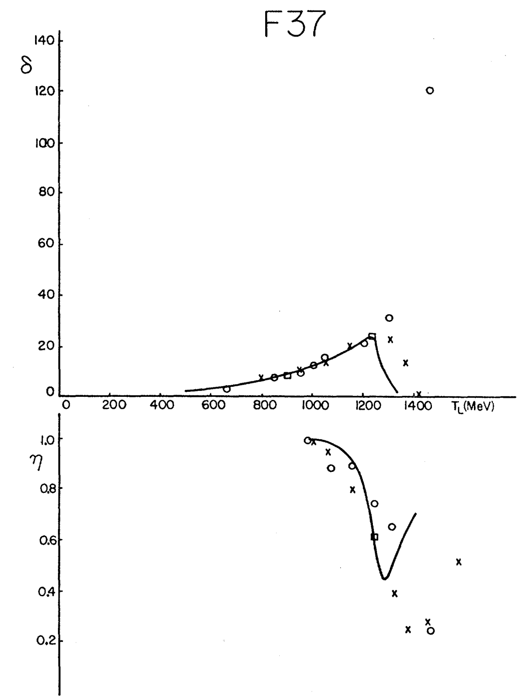

FIG. 8. Plots of δ and η for the F_{37} partial wave. The □'s are $\delta(900)=8.0$, $\delta(1229)=23.4$, $\eta(1228)=0.61$. The parameters are $W_1=5.0$, $\alpha_2=12.1$, $g_{11}=1215$, $g_{12}=1793$, $g_{22}=16118$.

Noting that f may be written as $\rho^{-1}[(D^{*}D^{-1}-1)/2i]$ and writing a dispersion relation for $\tilde{D}(M-B)$, we find, using (7) (to show that $\tilde{D}M$ has no physical cut), that the amplitude M is given by

$$M=B+[\tilde{D}(W)]^{-1}\frac{1}{\pi}\int\frac{\tilde{N}(W')\rho(W')B(W')dW'}{W'-W}. \qquad (8)$$

Owing to the simple form of N given in Eq. (3), we can write M_i in terms of the functions $b_i(s)$,

$$b_i(W)=\frac{1}{\pi}\int_{\alpha_i}^{\infty}dW'\frac{\rho_i(W')B_i(W')}{W'-W}, \qquad (9)$$

as follows:

$$M_i(W)=B_i(W)+\sum_{k=1}^{n}[D^{-1}(W)]_{ki} \times\sum_{j=1}^{n}\sum_{r=1}^{p}\frac{b_j(W)-b_j(W_r)}{W-W_r}\gamma_{ik}{}^{r}. \qquad (10)$$

What functions B one chooses for the coupling of photoproduction channel to a phenomenological channel clearly requires some thought. It is possible that the above expression could be the basis of a phenomenological treatment of photoproduction in which the π-N phenomenology has been correctly included. However, in the extended analysis of the elastic πN partial waves proposed above in which the inelastic channels correspond to actual quasi-two-body channels, reasonable expressions for B could be calculated.