Structure of Phenomenological Lagrangians. I*

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The general structure of phenomenological Lagrangian theories is investigated, and the possible transformation laws of the phenomenological fields under a group are discussed. The manifold spanned by the phenomenological fields has a special point, called the origin. Allowed changes in the field variables, which do not change the on-shell S matrix, must leave the origin fixed. By a suitable choice of fields, the transformations induced by the group on the manifold of the phenomenological fields can be made to have standard forms, which are described in detail. The mathematical problem is equivalent to that of finding all (nonlinear) realizations of a (compact, connected, semisimple) Lie group which become linear when restricted to a given subgroup. The relation between linear representations and nonlinear realizations is discussed. The important special case of the chiral groups $SU(2)\times SU(2)$ and $SU(3)\times SU(3)$ is considered in detail.

1. INTRODUCTION

In most phenomenological field theories the Lagrange density is not an aribtrary function of the fields. The fields transform in some well-defined way under some internal symmetry group [typically chiral $SU(2)\times SU(2)$ or chiral $SU(3)\times SU(3)$], and the Lagrange density consists of a main part which is invariant under this group and of a symmetry-breaking part which is usually assumed to have simple transformation properties under the group. Thus, to study phenomenological field theories, we must study the transformation properties of fields under such groups.

If the fields transform linearly, the classification of all possible field-transformation laws reduces to the standard problem of representation theory. However, for most phenomenological theories,[1] the situation is more complicated: The fields transform linearly only under a certain subgroup of the full group (in the cases cited, this subgroup is the subgroup of parity-conserving transformations). In this paper, we consider exactly such a situation and we show that it is possible to classify all possible nonlinear realizations of an internal symmetry group which become linear when restricted to a given subgroup.[2] In a subsequent paper, we describe the general method for constructing nonlinear Lagrange densities which are invariant under the nonlinear field transformations.[3]

In order to give a useful classification of nonlinear group realizations, one must first find a suitable definition of equivalence of nonlinear realizations. As discussed in Sec. 2, the appropriate definition is suggested by a property of local Lagrangian field theory, namely the independence of the on-shell S-matrix elements from the particular set of local fields in terms of which one expresses the Lagrangian. It is then natural to consider as equivalent two nonlinear group realizations which can be transformed into each other by a fixed nonlinear transformation belonging to a certain rather general class. The physical equivalence of two such nonlinear group realizations is established not only for the exact solution of a Lagrangian theory, but also for the approximation in which one uses, for each particular process, only the tree diagrams which contribute to it (diagrams with no internal loops, or with no integrations over internal lines). Since one customarily restricts oneself to this tree approximation in phenomenological Lagrangian theories, we call it here the phenomenological approximation.[4]

In Sec. 3 we give some relatively simple forms for the nonlinear realizations of a compact Lie group which become linear when restricted to a given subgroup.[5] In Sec. 4 we then show that these are standard forms which

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[1] Nonlinear phenomenological Lagrangians have been the subject of a number of papers. We quote here only a few where references to earlier work can be found: J. Cronin, Phys. Rev. **161**, 1483 (1967); S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); J. Schwinger, Phys. Letters **24B**, 473 (1967); J. Wess and Bruno Zumino, Phys. Rev. **163**, 1727 (1967); Bruno Zumino, Phys. Letters **25B**, 349 (1967); B. Lee and H. T. Nieh, Phys. Rev. **166**, 1507 (1968).

[2] A condensed description of the results of the present paper was given by one of us (B. Z.) in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy*, edited by A. Perlmutter, C. A. Hurst, and B. Kursunoglu (W. A. Benjamin, Inc., New York, 1968). Partial results were presented in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Interscience Publishers, Inc., New York, 1968).

[3] C. G. Callan, S. Coleman, J. Wess, and Bruno Zumino, Phys. Rev., following paper, **177**, 2247 (1969).

[4] Properties of tree diagrams have been studied by K. Symanzik, *Boulder Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1960); R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963); Y. Nambu, University of Chicago Report, 1968 (unpublished). A discussion of the connection between the structure of diagrams and the power in an expansion in the coupling constant can be found in the paper by Lee and Nieh quoted in Ref. **1**.

[5] For the case of $SU(2)\times SU(2)$ and $SU(3)\times SU(3)$, nonlinear realizations equivalent to those given here have been used in the papers quoted in Ref. 1. For $SU(2)\times SU(2)$, the question of their generality has been discussed by S. Weinberg, Phys. Rev. **166**, 1568 (1968).

in a precise sense give the solution of the general classification problem. The generalization to some noncompact groups appears possible, but will not be discussed here.

One usually requires the symmetry-breaking terms of the Lagrangian to belong to particular linear representations of the group in question. Or one may be interested in linearly transforming local fields to describe certain bound states. For this, one must solve the problem of constructing functions of the nonlinearly transforming fields which transform linearly under the group. If one wants to understand the relation between theories which assign fields to linear representations of a group and theories which use nonlinear realizations, one must ask a related question. It becomes necessary to study the possible (nonlinear) equivalence between representations and realizations. These problems are studied in Sec. 4, where we discuss also the question of the possible (nonlinear) equivalence between two different linear representations.

2. PROPERTIES OF NONLINEAR LAGRANGIANS

Let the Lagrangian be known in terms of a set of field variables ϕ:

$$L[\phi]=L_0[\phi]+L_1[\phi],$$

where $L_0[\phi]$ is the Lagrangian of free fields and $L_1[\phi]$ the interaction. If one expresses the field variables ϕ as nonlinear but local functions of another set of field variables χ,

$$\phi=\chi F[\chi], \quad F[0]=1, \tag{1}$$

one can similarly separate the resulting Lagrangian into a free-field term and an interaction

$$L[\chi F[\chi]]=L_0[\chi]+L_2[\chi].$$

The results obtained from the Lagrangian $L[\phi]$ (for instance by means of suitable Feynman rules) can be compared with those obtained from

$$L'[\phi]=L_0[\phi]+L_2[\phi].$$

Clearly the many-particle propagators will in general differ in the two cases when the momenta of the external lines are off the mass shell. However, according to a theorem of relativistic Lagrangian theory valid with rather weak restrictions on L and F, the on-mass-shell S matrices calculated with $L[\phi]$ and $L'[\phi]$ are identical (in making the comparison it may be necessary to introduce appropriate wave-function renormalizations). Without going into the conditions for the validity of this theorem, nor into the precise meaning of the local products of field operators involved, we observe that, in all cases of physical interest, $F[\phi]$ is a local power series in the fields ϕ and $L[\phi]$ is a local power series in the fields ϕ and their derivatives; these should be sufficient restrictions to insure the validity of the theorem. The result has an analog in axiomatic field theory[6] (irrelevance of the choice of a particular set of local interpolating fields). Stated in the simplest possible way, the reason for its validity is that the terms higher than linear in the expansion of the function F do not contribute on the mass shell because they do not contain one-particle singularities.

We have stated the above theorem as a property of the exact solution, but we will now show that it applies also to the S matrix calculated in the phenomenological approximation (tree diagrams).

First we give a simple characterization of the phenomenological approximation. Given any Lagrangian $L[\phi]$, define the Lagrangian

$$L[\phi,a]=a^{-2}L[a\phi],$$

which depends upon the parameter a (clearly the original Lagrangian $L[\phi]$ is recovered for $a=1$). Consider a connected Feynman diagram and denote with E the number of its external lines, with I the number of internal lines, with L the number of loops (equal to the number of internal integrations), with V the number of vertices, and with N_i ($i=1, 2, \cdots, V$) the number of lines attached to the ith vertex. Expanding the Lagrangian in power series, one sees that each vertex carries the power N_i-2 of the parameter a. Therefore, the diagram carries a power P given by

$$P=\sum_{i=1}^{V}(N_i-2).$$

On the other hand, since a line is either an internal line (connecting two vertices) or an external line,

$$\sum_{i=1}^{V}N_i=E+2I.$$

Combining these two equations, we find

$$P=E+2I-2V.$$

Since the number of loops satisfies

$$L=I-V+1,$$

we have finally

$$P=E+2L-2. \tag{2}$$

This formula shows that the smallest power of the parameter a for which a particular process (given E) will occur is

$$P=E-2;$$

this corresponds to diagrams with no loops. The next higher power of the parameter a is

$$P=E+2-2=E$$

[6] R. Haag, Phys. Rev. **112**, 669 (1958); D. Ruelle, Helv. Phys. Acta **35**, 34 (1962); H. J. Borchers, Nuovo Cimento **25**, 270 (1960). For a discussion in the context of Lagrangian perturbation theory, see the contribution by one of us (S. C.) in *Hadrons and Their Interactions*, edited by A. Zichichi (Academic Press Inc., New York, 1968).

and corresponds to diagrams with one loop, and so on. The tree diagrams emerge as the lowest-order term in a systematic expansion.

We now introduce the parameter a also in the formula (1) connecting the fields ϕ to the fields χ and write it as

$$\phi=\chi F[a\chi].$$

Substituting into the Lagrangian, we obtain

$$L[\chi F[a\chi],a]=a^{-2}L[a\chi F[a\chi]],$$

so that the connection between the power of a and the number of lines attached to the vertex is the same as before. For a given process, the same power of a characterizes the tree diagrams for the new Lagrangian as for the old Lagrangian. From the equality of the S matrices calculated from the two Lagrangians using the exact solutions follows the equality of the coefficients of their expansion in a and therefore the equality of the phenomenological approximations, which are the lowest-order terms of this expansion.

In most phenomenological Lagrangians, there are coupling constants which enter in the Lagrangian just like the parameter a introduced above. This fact, however, is not directly relevant to the argument given. One can always introduce the parameter a, as we have done, for power-counting purposes and set it equal to unity at the end of the argument.

The above characterization of the phenomenological approximation permits us, in particular, to understand the agreement often noticed between calculations performed in the phenomenological approximation with nonlinear Lagrangians and general results obtained using current algebra (or the algebra of fields) and conservation or partial conservation equations. To see this, let us first observe that the nonlinear theories are good models for current algebra and therefore their exact solutions must satisfy the general relations deduced from current algebra. If we introduce the parameter a in the Lagrangian in the manner described above, the new currents will also contain this parameter, but the commutation relations between correctly normalized currents will remain the same. However, a partial conservation equation, like that for the axial-vector current A_μ, will contain the parameter explicitly. It is not difficult to see that the parameter a will enter as indicated by the example

$$\partial_\mu A_\mu[\phi,a]=(c/a)\varphi\,, \tag{3}$$

where c is a constant and φ is the pion field. A typical current-algebra relation connects various amplitudes and consists of several terms which may refer to processes involving different numbers of particles (external lines). Since these relations are obtained, using equations such as (3) in conjunction with reduction techniques, it is easy to see that the parameter a will enter into the coefficients in such a way that the sum of the power of a and of the number of particles involved is the same for all terms. From this observation and from Eq. (2) it follows immediately, by expanding the exact relation in a power series in the parameter a, that the tree diagrams satisfy the relation by themselves, and so do the diagrams with one loop, two loops, etc.

3. STANDARD FORM OF NONLINEAR REALIZATIONS

Let G be a compact, connected, semisimple Lie group with n parameters and H a continuous subgroup of G. We denote with V_i $(i=1,2,\cdots,n-d)$ the generators of H and with A_l $(l=1,2,\cdots,d)$ the remaining generators chosen so that V_i and A_l form together a complete set of generators of G, orthonormal with respect to the Cartan inner product. From the familiar properties of the exponentials it follows that, in some neighborhood of the identity of G, every group element $g\in G$ can be decomposed uniquely into a product of the form[7]

$$g=e^{\xi\cdot A}e^{u\cdot V}, \tag{4}$$

where

$$\xi\cdot A=\sum_l \xi_l A_l\,,\quad u\cdot V=\sum_i u_i V_i$$

and ξ_l and u_i are real parameters. (For brevity, we will frequently drop the qualifying phrase "in some neighborhood of the identity" in what follows. However, it should be understood that we are here investigating only the local properties of realizations.) Therefore, for any element $g_0\in G$ one can write

$$g_0e^{\xi\cdot A}=e^{\xi'\cdot A}e^{u'\cdot V}, \tag{5}$$

where

$$\xi'=\xi'(\xi,g_0)\,,\quad u'=u'(\xi,g_0)$$

are functions of the indicated variables which are determined by the structure of the group. Let, further,

$$h:\ \psi\rightarrow D(h)\psi\,,$$

with $h\in H$, be a linear (unitary) representation of the subgroup H. It is immediate that the transformations

$$g_0:\ \xi\rightarrow\xi',\ \ \psi\rightarrow D(e^{u'\cdot V})\psi \tag{6}$$

give a (nonlinear) realization of G. To verify it, just observe that if

$$g_1e^{\xi'\cdot A}=e^{\xi''\cdot A}e^{u''\cdot V},$$

then

$$g_1g_0e^{\xi\cdot A}=e^{\xi''\cdot A}e^{u'''\cdot V},$$

where

$$e^{u'''\cdot V}=e^{u''\cdot V}e^{u'\cdot V},$$

and furthermore that, since D is a representation,

$$D(e^{u'''\cdot V})=D(e^{u''\cdot V})D(e^{u'\cdot V}).$$

[7] This decomposition amounts to a particular parametrization of the left cosets G/H by means of the parameters ξ_l. Any other parametrization would give rise to a treatment completely equivalent from the abstract group-theoretic point of view and would furnish equivalent results. However, the particular parametrization we use here is very convenient because of the simple and well-known properties of exponentials. Alternative parametrizations have been used in concrete applications and can be found, for instance, in the papers quoted in Ref. 1.

Clearly the transformation on ξ can be considered a group realization by itself. The transformation on ψ, on the other hand, is meaningful only together with that on ξ, since u' is a function of ξ and therefore $D(e^{u'\cdot V})$ also is. If $D(h)$ is reducible, the field ψ decomposes by a suitable choice of basis into a set of fields $\psi^{(a)}$ with a smaller number of components. These fields do not mix. We shall refer to Eq. (6), with D written as a fully decomposed representation, as to our fundamental standard form for a realization.

If g_0 belongs to the subgroup H, $g_0=h$, one can write

$$g_0e^{\xi\cdot A}=he^{\xi\cdot A}=he^{\xi\cdot A}h^{-1}h.$$

Since we have chosen our generators to be orthonormal, it follows that

$$e^{\xi'\cdot A}=he^{\xi\cdot A}h^{-1}$$

and therefore

$$e^{u'\cdot V}=h.$$

In this case, the transformation $\xi\rightarrow\xi'$ is a linear transformation

$$\xi'=D^{(b)}(h)\xi,$$

where $D^{(b)}(h)$ is a linear representation of H determined uniquely by the structure of G [for example, if G is $SU(n)\times SU(n)$, and H is the diagonal $SU(n)$ subgroup, $D^{(b)}$ is the adjoint representation of $SU(n)$; if G is $SU(3)$ and H is the usual isospin-hypercharge subgroup, $D^{(b)}$ is the direct sum of the isospin-$\frac{1}{2}$, hypercharge-1 representation and its conjugate]. Furthermore, in this case u' is evidently independent of ξ and, therefore, the transformation

$$\psi\rightarrow D(e^{u'\cdot V})\psi=D(h)\psi$$

is also linear. We see that, when restricted to the subgroup H, the group realization (6) becomes a linear representation.

In Sec. 4 we shall study the general nonlinear realizations of a compact, connected, semisimple Lie group G, which have the property of being linear when restricted to a continuous subgroup H of G. We shall demonstrate, by choosing suitable coordinates, that any manifold on which these realizations are induced is equivalent to one which transforms according to our standard form given in Eqs. (5) and (6). This standard form has the important property that the space of the parameters ξ_l is transitive under the group transformations.

We wish to point out now that there is a special case in which the form of the transformation on ξ can be somewhat simplified. It is the case[8] in which the group G admits the automorphism R: $g\rightarrow R(g)$ such that

$$V_i\rightarrow V_i,\quad A_l\rightarrow -A_l.$$

(This is, for instance, the case for chiral groups: The parity operator induces an automorphism which changes the sign of the axial-vector generators.) Applying the automorphism to the relation

$$g_0e^{\xi\cdot A}=e^{\xi'\cdot A}e^{u'\cdot V},$$

one obtains

$$R(g_0)e^{-\xi\cdot A}=e^{-\xi'\cdot A}e^{u'\cdot V}.$$

One can eliminate u' from these two equations with the simple result

$$g_0e^{2\xi\cdot A}R(g_0^{-1})=e^{2\xi'\cdot A}.\qquad(7)$$

In this form one can verify directly that the transformation on ξ is a realization of the group and that it becomes linear when restricted to the subgroup.

4. CLASSIFICATION OF ALL NONLINEAR REALIZATIONS

We shall now try to construct all possible nonlinear field transformation laws under a group and show that the nonlinear group realizations of the previous section give a standard form for the general case. We begin by phrasing the problem in a slightly more abstract form. Let M be an n-dimensional real analytic manifold. Let G be a compact, connected, semisimple Lie group which is realized as a group of transformations on M. In equations

$$g:\ x\rightarrow T_gx,$$

where we use the symbol x to denote a point on the manifold as well as the real n-vector formed by the coordinates of the point in some coordinate system. We assume that T_gx is an analytic function of both g and x.

If we identify the fields of phenomenological field theory with some particular set of coordinates on the manifold, the problem of finding all possible field transformation laws under a group is equivalent to finding all possible ways of realizing the group as transformations on a manifold. The advantage of formulating the problem in this way is that the passage from one set of fields to another, which as we have shown in Sec. 2 has no effect upon the physical predictions of a phenomenological theory, becomes the passage from one set of manifold coordinates to another, which has no effect upon the geometrical problem. The analyticity assumptions in the manifold problem, which are rather strong from a purely geometric viewpoint, are necessary because of the power-series expansions which occur in the field theory.

However, the field problem has some further special features. General coordinate transformations are not allowed: as seen from Eq. (1), a change of coordinates must leave the origin of coordinates unchanged. Therefore we will assume that there is a special point O on the manifold, which we call the origin, and we will allow only coordinate systems such that the origin is always represented by the zero vector. Because the fields are ultimately used only in power-series expansions, there is no need for us to attempt to characterize the action of the group globally; it suffices to study it

[8] This case has been considered in detail by L. C. Biedenharn (private communication).

only in a neighborhood of the origin of M. Also, by the usual properties of connected Lie groups, we can likewise restrict our attention to the neighborhood of the identity in G.

There may be elements of the group which leave the origin invariant. Their totality forms a subgroup H of G. The subgroup H is called the stability group of the origin and could in particular consist of the identity element alone or, as another extreme case, of the entire group G. For simplicity we assume that H is continuous. Our main problem is the following: we assume that the group G and the subgroup H are given and we wish to find the most general way of realizing them on the manifold M. As we shall see immediately, this problem is equivalent to that of finding all possible nonlinear realizations of G which become linear when restricted to the subgroup H. This equivalence is demonstrated by the following lemma.

Linearization lemma. Let H be that subgroup of G consisting of all elements which leave the origin unchanged; that is to say,

$$T_h 0 = 0$$

for every $h \in H$. Then there exists a set of coordinates, valid in some neighborhood of the origin, such that, in these coordinates,

$$T_h y = D(h) y$$

for every $h \in H$, where $D(h)$ is a linear representation of H.

Proof: By the continuity of the transformation T_g, the group H is closed, and therefore compact. Therefore we can find a neighborhood of the origin that is invariant under the action of H. Let us choose this neighborhood to lie within a single coordinate patch, and let x be the coordinates associated with this patch. If we expand $T_g x$ in power series, we find

$$T_h x = D(h) x + [x^2], \tag{8}$$

where $D(h)$ is evidently a linear representation of H and $[x^2]$ denotes terms which are at least quadratic in x. Now let us define n functions y by

$$y = \int_H dh\, D^{-1}(h) T_h x, \tag{9}$$

where dh is the invariant measure and the group integral is normalized so that

$$\int_H dh = 1.$$

The y's are analytic functions of the x's; indeed, by Eq. (8),

$$y = x + [x^2].$$

Hence the Jacobian determinant $|\partial y/\partial x|$ is equal to 1 at the origin, and we can use the y's as new coordinates in some neighborhood of the origin. Under the action of h_0, an arbitrary element of H,

$$h_0:\quad y \to \int dh\, D^{-1}(h) T_h T_{h_0} x$$
$$= \int d(hh_0) D^{-1}(hh_0 h_0^{-1}) T_{hh_0} x$$
$$= D(h_0) y.$$

This proves the lemma.

This lemma is often useful in itself. Given a nonlinear transformation law, it gives a simple test for linearizability. Also, if the transformation law can be linearized, Eq. (9) provides an explicit formula for the new fields that will do the job.

We now proceed to consider the main problem. We introduce a complete and orthonormal set of generators V_i, A_l as described in Sec. 3, that is, V_i are the generators of the subgroup H and A_l the additional generators of G. We shall again make use of the unique decomposition of a group element given by Eq. (4), and also of Eq. (5), in which we occasionally omit the subscript zero.

In the following, for the sake of conciseness of notation, we shall occasionally denote the transformation T_g of the manifold M associated with a particular group element by the symbol of the group element itself. Thus for $T_g x$ we shall write also gx.

Let us now denote with N the submanifold of M consisting of all points of the form $T_g 0$ (or $g0$). We can associate to each set of parameters ξ_l the point of the submanifold N given by $e^{\xi\cdot A}0$. It is clear that, in some neighborhood of the origin, there is only one set of parameters for each point of N; therefore we may use the real numbers ξ_l as a set of coordinates for N in that neighborhood. The transformation properties of N are completely determined, as seen from

$$g(e^{\xi\cdot A}0) = e^{\xi'\cdot A} e^{u'\cdot V} 0 = e^{\xi'\cdot A} 0.$$

Here we have used the fact that the transformation associated with an element of the subgroup H leaves the origin invariant.

We now introduce $n-d$ other coordinates for M, which we assemble into a real vector ψ. Thus a point of M (in some neighborhood of the origin) has as coordinates a pair (ξ,ψ). The points $(\xi,0)$ lie on N, and we have

$$g(\xi,0) = (\xi'(\xi,g),0), \tag{10}$$

where $\xi'(\xi,g)$ is given by Eq. (5). It follows directly from the linearization lemma that we can choose the coordinates (ξ,ψ) so that H acts linearly. According to Eq. (10), the ensuing linear representation of H is reducible and, since the group H is compact and the representation can be made orthogonal by a suitable choice of coordinates, it can always be written in the fully reduced form

$$e^{u\cdot V}(\xi,\psi)=(D^{(b)}(e^{u\cdot V})\xi,D(e^{u\cdot V})\psi),$$

where D is some linear (orthogonal) representation of H of dimension $n-d$ and $D^{(b)}$ is the representation induced on ξ by

$$e^{\xi'\cdot A}=e^{u\cdot V}e^{\xi\cdot A}e^{-u\cdot V}.$$

We now attempt to introduce new coordinates $(\xi,\psi)^*$ by the equation

$$(\xi,\psi)^*=e^{\xi\cdot A}(0,\psi).$$

It follows that

$$(0,\psi)^*=(0,\psi),$$

furthermore, from Eq. (10), we see that

$$(\xi,0)^*=(\xi,0).$$

These equations define an allowable set of coordinates in some neighborhood of the origin, since they define an analytic mapping of the pairs $(\xi,\psi)^*$ into M, whose Jacobian determinant does not vanish at the origin.

But now we know everything about the action of G in a neighborhood of the origin. For

$$\begin{aligned}g(\xi,\psi)^*&=ge^{\xi\cdot A}(0,\psi)^*\\&=e^{\xi'\cdot A}e^{u'\cdot V}(0,\psi)^*\\&=e^{\xi'\cdot A}(0,D(e^{u'\cdot V})\psi)^*\\&=(\xi',D(e^{u'\cdot V})\psi)^*,\end{aligned}$$

where u' and ξ' are given by Eq. (5). In practice we do not want to use the representations D of the group H in their real, orthogonal form but rather in their unitary form. The transformation to the unitary form can be achieved by combining real fields into a complex field when the corresponding representation is not truly real.

Thus, by a proper choice of coordinates, any realization of G as a group of analytic transformations on M may be brought into the form (6) in some neighborhood of the origin of M and of the identity in G. Equation (6) is therefore the solution to the problem we set ourselves at the beginning of this section, to construct all possible nonlinear field transformation laws.

5. RELATIONS BETWEEN LINEAR AND NONLINEAR TRANSFORMATIONS

Let us consider a manifold (ξ,Ξ) spanned by a field Ξ_α which transforms according to a linear irreducible (unitary) representation $\mathfrak{D}(g)$ of the group G,

$$\Xi_\alpha\to\Xi_\alpha'=\sum_\beta\mathfrak{D}_{\alpha\beta}(g)\Xi_\beta,$$

and by the field ξ which transforms into ξ' given by Eq. (5). We show that one can go over to fields which transform by our basic standard form, Eq. (6). Just define

$$\Psi_\alpha=\sum_\beta\mathfrak{D}_{\alpha\beta}(e^{-\xi\cdot A})\Xi_\beta.$$

This relation is an allowed change of fields in the sense of Eq. (1). The field Ψ_α transforms into

$$\begin{aligned}\Psi_\alpha'&=\sum_\beta\mathfrak{D}_{\alpha\beta}(e^{-\xi'\cdot A})\Xi_\beta'\\&=\sum_\beta\mathfrak{D}_{\alpha\beta}(e^{-\xi'\cdot A}g)\Xi_\beta\\&=\sum_\beta\mathfrak{D}_{\alpha\beta}(e^{-\xi'\cdot A}ge^{\xi\cdot A})\Psi_\beta\\&=\sum_\beta\mathfrak{D}_{\alpha\beta}(e^{u'\cdot V})\Psi_\beta,\end{aligned}$$

where we have made use once more of Eq. (5). Naturally, when restricted to the subgroup H, the representation $\mathfrak{D}(g)$ becomes a representation $\mathfrak{D}(h)$ of H. Therefore the transformation law of the field Ψ_α is exactly of the standard form given by Eq. (6). Now, $\mathfrak{D}(h)$ may be reducible, in which case the field Ψ_α breaks up into a set of fields $\psi_\alpha^{(a)}$ with a smaller number of components

$$(\xi,\Psi)\sim(\xi,\sum_a\oplus\psi^{(a)}).$$

In this case, the apparent additional physical connection between the larger number of components of the field Ξ must be illusory, since, as we have seen in Sec. 2, the physical consequences of the theory must be the same whether one uses the fields ξ and Ξ or the fields ξ and $\psi^{(a)}$. In a Lagrangian theory employing the fields ξ and Ξ this can be seen from the possibility of constructing a sufficient additional number of invariants which, when added to the Lagrangian, destroy the "spurious" relations between amplitudes. In any case, the dimensionality of the possible multiplets is that of the irreducible linear representations of the subgroup H and not that of the irreducible linear representations of G. Our standard form (6), with the representation D fully reduced, brings this most clearly into light.[9] To avoid misunderstandings, however, let us emphasize that in a Lagrangian theory one may well choose to use the linearly transforming field Ξ and not to include in the Lagrangian the additional invariants which purely group-theoretic considerations would allow. In this case the particular choice of the Lagrangian will give stronger physical results, which are not, however, consequences of group invariance alone.

It is clear that the above construction can be inverted to construct a field transforming linearly like Ξ_α as a function of the field ξ_l and of one or more fields such as

[9] As a special case of this result, we recall that, in nonlinear phenomenological theories, the pion is described as a three-dimensional multiplet and it is not necessary to introduce the σ field which would be needed to complete the four-dimensional linear representation of $SU(2)\times SU(2)$. This does not mean, however, that one cannot introduce the σ into the theory, if experimental evidence requires one to do so. The σ, being an isoscalar, would be described as an invariant under the full group $SU(2)\times SU(2)$. The strength of its couplings to other fields will be unrelated to that of the couplings of the pion to the same fields. Nevertheless, invariance under the group imposes well-defined restrictions on the σ couplings.

the $\psi_\alpha^{(a)}$. One is thus led to pose the general problem of finding those functions f_α of ξ and ψ which transform according to a linear representation of the group G:

$$g_0:\quad f_\alpha(\xi,\psi)\to f_\alpha(\xi',\psi')=\sum_\beta D_{\alpha\beta}(g_0)f_\beta(\xi,\psi).$$

We observe that it is sufficient to study the case in which f_α is a linear function of ψ:

$$f_\alpha(\xi,\psi)=\sum_r F_{\alpha r}(\xi)\psi_r,$$

since the general case can be reduced to this one by first solving the Clebsch-Gordan problem for the subgroup H and the tensor products of the ψ's. We write therefore

$$\sum_r F_{\alpha r}(\xi')\psi_r'=\sum_{\beta,s} D_{\alpha\beta}(g_0)F_{\beta s}(\xi)\psi_s.$$

We show first that the representation D is not arbitrary: when restricted to the subgroup H it must reduce and contain as one of its components the representation $R(h)$ under which ψ transforms by the subgroup H. To see this, write, for $g_0=h$,

$$\sum_{r,s} F_{\alpha r}(\xi')R_{rs}(h)\psi_s=\sum_{\beta,s} D_{\alpha\beta}(h)F_{\beta s}(\xi)\psi_s.$$

In this equation set $\xi=0$, which implies $\xi'=0$, since $\xi=0$ is a fixed point for transformations of the subgroup H. We find

$$\sum_{r,s} F_{\alpha r}(0)R_{rs}(h)\psi_s=\sum_{\beta,s} D_{\alpha\beta}(h)F_{\beta s}(0)\psi_s.$$

From Schur's lemma it follows that either $F_{\alpha r}(0)=0$ or $D(h)$ contains the irreducible representation $R(h)$. The first alternative is impossible because $F_{\alpha r}(0)=0$ implies $F_{\alpha r}(\xi)=0$ for all ξ; this follows immediately from the transitivity of the submanifold N considered in Sec. 4.

Vice versa, if D reduces and contains the representation $R(h)$ when restricted to the subgroup H, then there exist functions $F_{\alpha r}(\xi)$ such that $f_\alpha(\xi,\psi)=\sum_r F_{\alpha r}(\xi)\psi_r$ transform linearly according to the representation D. Indeed, consider the unique decomposition

$$g=e^{\xi\cdot A}e^{u\cdot V}$$

which implies, in a suitable basis,

$$D(g)=D(e^{\xi\cdot A})D(e^{u\cdot V})=D(e^{\xi\cdot A})\begin{pmatrix}R(e^{u\cdot V}) & 0\\ 0 & \tilde{D}(e^{u\cdot V})\end{pmatrix}.$$

Now

$$\begin{aligned}D(g_0)D(g)&=D(g_0g)=D(g_0e^{\xi\cdot A}e^{u\cdot V})\\&=D(e^{\xi'\cdot A}e^{u'\cdot V}e^{u\cdot V})=D(e^{\xi'\cdot A})D(e^{u'\cdot V}e^{u\cdot V})\\&=D(e^{\xi'\cdot A})\begin{pmatrix}R(e^{u'\cdot V}e^{u\cdot V}) & 0\\ 0 & \tilde{D}(e^{u'\cdot V}e^{u\cdot V})\end{pmatrix}.\end{aligned}$$

These formulas show that

$$\sum_\beta D_{\alpha\beta}(g_0)D_{\beta r}(e^{\xi\cdot A})=\sum_s D_{\alpha s}(e^{\xi'\cdot A})R_{sr}(e^{u'\cdot V})$$

and therefore the functions

$$F_{\alpha s}(\xi)=D_{\alpha s}(e^{\xi\cdot A})$$

have the desired property

$$\begin{aligned}\sum_s D_{\alpha s}(e^{\xi'\cdot A})\psi_s'&=\sum_{s,r} D_{\alpha s}(e^{\xi'\cdot A})R_{sr}(e^{u'\cdot V})\psi_r\\&=\sum_{\beta,r} D_{\alpha\beta}(g_0)D_{\beta r}(e^{\xi\cdot A})\psi_r.\end{aligned}$$

The same argument shows that, if the representation D reduces and contains the representation $R(h)$ m times when restricted to the subgroup, then one can construct m sets of linearly independent functions $F_{\alpha r}(\xi)$.

As a special case, let the representation $R(h)$ be the identity representation and replace ψ by a constant. The above theorem gives then a characterization of those linear representations of G for which one can find functions of ξ alone which transform according to that representation, as well as a method for finding such functions. It is also easy to see that, if there is a larger subgroup H_1 of G, containing H, and if the representation D is reducible and contains the identity also when it is taken on H_1, then the functions $D_{\alpha 1}(e^{\xi\cdot A})$ do not really depend on all ξ_l, but only on those which are not associated with generators A_l which belong to the larger subgroup H_1. These results can be easily applied to the case of $SU(n)\times SU(n)$. The representations which contain the identity when restricted to the diagonal $SU(n)$ subgroup are those of the form $(v,\bar{v})$, where v denotes an irreducible representation of $SU(n)$. Functions of the field ξ_l only cannot transform linearly by any other representation.

As we have just shown, by taking linear combinations (with ξ-dependent coefficients) of the components of a field which transforms nonlinearly, it is possible to construct a field which transforms linearly. This fact indicates that one may sometimes be able by a similar construction to go from a field transforming according to a certain linear representation to one which transforms according to a different linear representation. More precisely, assume that

$$g:\quad \Xi_\alpha\to\Xi_\alpha'=R_{\alpha\beta}^{(1)}(g)\Xi_\beta. \tag{11}$$

We seek necessary and sufficient conditions for the existence of functions $F_{\mu\beta}(\xi)$ such that the field

$$X_\mu=F_{\mu\beta}(\xi)\Xi_\beta \tag{12}$$

transforms according to

$$g:\quad X_\mu\to X_\mu'=R_{\mu\nu}^{(2)}(g)X_\nu. \tag{13}$$

Here $R^{(1)}$ and $R^{(2)}$ are two irreducible linear representations of G (summation over repeated indices is understood). We show now that the desired transformation functions exist if and only if the product $\bar{R}^{(1)}\times R^{(2)}$ contains in its reduction one of the representations which, as we have found above, can be realized on functions of the field ξ alone.

The condition is necessary. Equations (11)–(13) imply

$$R_{\mu\nu}{}^{(2)}(g)F_{\nu\beta}(\xi)\Xi_\beta=F_{\mu\alpha}(\xi')R_{\alpha\beta}{}^{(1)}(g)\Xi_\beta$$

or

$$\begin{aligned}F_{\mu\alpha}(\xi')&=R_{\mu\nu}{}^{(2)}(g)F_{\nu\beta}(\xi)[R^{(1)}(g)]_{\beta\alpha}{}^{-1}\\&=R_{\mu\nu}{}^{(2)}(g)\bar{R}_{\alpha\beta}{}^{(1)}(g)F_{\nu\beta}(\xi).\end{aligned}$$

The condition is sufficient. Let the product $\bar{R}^{(1)}\times R^{(2)}$ reduce as indicated by the formula

$$\langle\alpha,\bar{1}|\langle\nu,2|=\sum_{\sigma,r}\langle\alpha,\bar{1};\nu,2|\sigma,r\rangle\langle\sigma,r|.$$

By assumption, for at least one value r_0 of r, there exists a set of functions $f_\sigma(\xi)$ such that

$$f_\sigma(\xi')=R_{\sigma\rho}{}^{(r_0)}(g)f_\rho(\xi).$$

On the other hand, as a consequence of the completeness of the Clebsch-Gordan coefficients, we have

$$\bar{R}_{\alpha\beta}{}^{(1)}R_{\mu\nu}{}^{(2)}\langle\beta,\bar{1};\nu,2|\rho,r\rangle=\langle\alpha,\bar{1};\mu,2|\sigma,r\rangle R_{\sigma\rho}{}^{(r)}.$$

If we define

$$F_{\mu\alpha}(\xi)=\langle\alpha,\bar{1};\mu,2|\sigma,r_0\rangle f_\sigma(\xi),$$

we see that

$$\begin{aligned}F_{\mu\alpha}(\xi')&=\langle\alpha,\bar{1};\mu,2|\sigma,r_0\rangle R_{\sigma\rho}{}^{(r_0)}f_\rho(\xi)\\&=\bar{R}_{\alpha\beta}{}^{(1)}R_{\mu\nu}{}^{(2)}\langle\beta,\bar{1};\nu,2|\rho,r_0\rangle f_\rho(\xi)\\&=\bar{R}_{\alpha\beta}{}^{(1)}R_{\mu\nu}{}^{(2)}F_{\nu\beta}(\xi)\\&=R_{\mu\nu}{}^{(2)}F_{\nu\beta}(\xi)[R^{(1)}]_{\beta\alpha}{}^{-1}.\end{aligned}$$

Therefore Eq. (13) follows from Eqs. (11) and (12).

6. EXAMPLES

We illustrate some of the results obtained in the previous sections by considering in some detail the case in which the group G is taken to be the chiral $SU(3)\times SU(3)$ group and the group H the parity-conserving diagonal $SU(3)$ subgroup.

The orthonormal generators of Sec. 3 can be taken now to be the vector and the axial-vector charges. For instance, in the three-dimensional representation of the Dirac quarks one would have

$$V_i=-\tfrac{1}{2}i\lambda_i,\quad A_l=-\tfrac{1}{2}i\gamma_5\lambda_l\quad(i,l=1,2,\cdots,8),$$

where the λ's are Gell-Mann's matrices. There are eight parameters ξ_l which can be put in correspondence with the eight pseudoscalar mesons of $SU(3)$. If we introduce the 3×3 matrix $\xi=\frac{1}{2}\sum_l\xi_l\lambda_l$, then the matrix of the pseudoscalar fields is $a\xi$, where, as it turns out, the scaling factor is given by $1/a=F_\pi$, the pion-decay constant. From Eq. (7) we see that, if we call β_i the parameters of a transformation generated by the vector charges and α_l those of a transformation generated by the axial-vector charges, the pseudoscalar fields transform as given by

$$e^{-2i\gamma_5\xi'}=e^{-i(\gamma_5\alpha-\beta)}e^{-2i\gamma_5\xi}e^{-i(\gamma_5\alpha+\beta)},$$

where $\beta=\frac{1}{2}\sum_i\beta_i\lambda_i$, $\alpha=\frac{1}{2}\sum_l\alpha_l\lambda_l$. There are only eight pseudoscalars, which transform linearly by a vector transformation and nonlinearly by an axial-vector transformation. No relation to a ninth pseudoscalar or to scalar mesons is implied by the group. However, as explained in the preceding section, one can construct functions of the eight-component field which transform linearly under the entire group $SU(3)\times SU(3)$. From the general discussion given there we know that the possible representations are those of the form $(v,\bar{v})$, where v denotes a representation of $SU(3)$. A simple example is afforded by the 18 fields S_0, S_i, P_0, P_l which satisfy

$$(\sqrt{\tfrac{2}{3}})S_0+S+i\gamma_5[(\sqrt{\tfrac{2}{3}})P_0+P]=e^{-2i\gamma_5\xi},$$

where $S=\sum_i S_i\lambda_i$ and $P=\sum_l P_l\lambda_l$. These 18 fields transform according to the representation $(3,\bar{3})$ of $SU(3)\times SU(3)$.

Let us look at the axial-vector transformations for fields other than the pseudoscalars. The formula

$$e^{-i\gamma_5\alpha}e^{-i\gamma_5\xi}=e^{-i\gamma_5\xi'}e^{-iu'}$$

determines $u'=\frac{1}{2}\sum_i u_i'\lambda_i$ as a function of ξ and α. For α infinitesimal, u' is also infinitesimal and one must have

$$u_i'=\sum_l\alpha_l C_{li}(\xi).$$

The functions $C_{li}(\xi)$ can be calculated, for instance, as power series in ξ. Let the infinitesimal vector transformation of an $SU(3)$ multiplet ψ be

$$\delta\psi=i\sum_i\beta_i F_i\psi,$$

where $-iF_i$ are the matrices of the generators in the corresponding irreducible representation. The nonlinear axial-vector transformation is simply given by

$$\delta\psi=-i\sum_i u_i'F_i\psi.$$

Again there is no enlargement of the dimensionality of the $SU(3)$ multiplet in going over to $SU(3)\times SU(3)$. If we use the 3×3 traceless matrix B for the baryon octet, the infinitesimal vector transformation can be written as

$$\delta B=i[\beta,B].$$

In this notation the infinitesimal nonlinear axial transformation for the baryon octet (or any octet other than the pseudoscalar) is

$$\delta B=-i[u',B].$$

The baryons could be required to transform linearly under an axial transformation. For instance, one could consider a baryon matrix B_1 which transforms according to $(1,8)_R+(8,1)_L$ or

$$B_1\rightarrow e^{i\gamma_5\alpha}B_1e^{-i\gamma_5\alpha}.$$

Alternatively, the baryon matrix B_2 could transform by $(\bar{3},3)_R+(3,\bar{3})_L$ or

$$B_2\rightarrow e^{i\gamma_5\alpha}B_2e^{i\gamma_5\alpha}.$$

In the first case the trace of the baryon matrix is invariant and can be set equal to zero. In the second case it is not and we have nine bayons instead of eight. It is easy to verify that the matrix $B_2e^{-2i\gamma_5\xi}$ transforms exactly like B_1. Furthermore the matrices $e^{-i\gamma_5\xi}B_2e^{-i\gamma_5\xi}$ and $e^{-i\gamma_5\xi}B_1e^{i\gamma_5\xi}$ have the same nonlinear transformation law as the matrix B, which, in finite form, is

$$B\rightarrow e^{-iu'}Be^{iu'}.$$

These examples are in agreement with the general theorems of Sec. 5.

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Structure of Phenomenological Lagrangians. II*

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The general method for constructing invariant phenomenological Lagrangians is described. The fields are assumed to transform according to (nonlinear) realizations of an internal symmetry group, given in standard form. The construction proceeds through the introduction of covariant derivatives, which are standard forms for the field gradients. The case of gauge fields is also discussed.

1. INTRODUCTION

THE most convenient way of deriving the physical consequences of the assumptions of (broken) chiral $SU(2)\times SU(2)$ [or $SU(3)\times SU(3)$] is by the method of phenomenological Lagrangians. These Lagrangians consist of a part which is invariant under the field transformations which realize the group and of a symmetry-breaking part which is usually assumed to transform simply under the group. The transformation laws of the fields under the group are in general nonlinear, but they become linear when restricted to the parity conserving $SU(2)$ [or $SU(3)$] subgroup. In the preceding paper,[1] the general form of the field transformation law is given for the general case of a compact, connected, semisimple Lie group. In the present paper, we give the general method for the construction of the invariant part of the Lagrangian. The symmetry-breaking terms in the Lagrangian are usualy assumed to belong to a linear representation of the group. In this case, one can easily construct them as functions of the fields by using the results of Sec. 5 of the preceding paper.

2. COVARIENT DERIVATIVES AND INVARIANT LAGRANGIANS

Our starting point is the analysis of nonlinear realizations of a compact Lie group given by Coleman, Wess, and Zumino. We dispense here with all proofs and definitions and quote only their final result. Let G be a compact, connected, semisimple Lie group and H a continuous subgroup of G. Let V_i and A_l be a complete orthonormal set of generators of G such that V_i are the generators of H. Any element g of G may be decomposed uniquely as a product of the form

$$g=e^{\xi\cdot A}e^{u\cdot V}.$$

A nonlinear realization of G which becomes a linear representation when restricted to the subgroup H is given on coordinates (ξ,ψ) by

$$(\xi,\psi)\rightarrow(\xi',\psi')=g(\xi,\psi),\tag{1}$$

where

$$ge^{\xi\cdot A}=e^{\xi'\cdot A}e^{u'\cdot V}\tag{2}$$

and

$$\psi'=D(e^{u'\cdot V})\psi.\tag{3}$$

Here $D(h)$ is any linear representation of the subgroup H which, if it is reducible, we assume to be written in

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[1] S. Coleman, J. Wess, and Bruno Zumino, preceding paper, Phys. Rev. 177, 2239 (1969). In this paper one can find references to other work, in particular to papers which describe in detail the Lagrangian method as applied to chiral groups.