Electromagnetic Mass Differences, Equal-Time Commutators, and Oscillating Spectral Functions*

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(Received 13 May 1968)

Motivated by its implication of a divergent pion electromagnetic mass shift δm^2, we study some apsects of the Bjorken limit relating the high-q_0 behavior of the amplitude $T(q;p)=-i\int dx\, e^{-iq\cdot x}\langle p|T(J(x)J'(0))|p\rangle$ to the equal-time commutators $[\partial_0{}^n J, J']$. Our aim is to find ways to avoid an infinite δm^2 without altering the usual current-current commutation-relation models which have $[\dot{J},J']\neq 0$. It is noted that, contrary to what is usually assumed, perturbation-theoretic T products rarely vanish at high q_0. We rigorously show that when $T(q;0)$ is the pion-photon scattering amplitude at $p=0$ and admits a spectral representation $\int da\, \sigma(a)(q^2+a)^{-1}$ (including an arbitrary number of subtractions and an arbitrary additive polynomial), the corresponding mass shift $\delta m_0{}^2$ can be finite only if the moment $I=\int da\, \sigma(a)$ is either zero or does not exist; in the latter case, $\sigma(a)$ must change sign infinitely often in any neighborhood of $a=+\infty$. The nonexistence of I corresponds to the presence of ambiguities in the commutator $[\dot{J},J']$, although, as we emphasize, the commutator *can* be given a consistent nonvanishing value with $\delta m_0{}^2$ remaining finite. As an interesting example of this possiblility, we consider the function $T(q;0)=\exp[-(q^2)^{1/2}]$, which gives $\delta m_0{}^2<\infty$ and I undefined, but which can correspond to any value of $[\dot{J},J']$, provided that the equal-time limit is taken in a suitable way. It is argued that such an exponential falloff for the forward photohadron scattering amplitude M is a likely consequence of an exponentially falling nucleon electromagnetic form factor, a behavior which is consistent with experiment and a number of theoretical ideas. Additional support for such behavior for M comes from the observation that this type of behavior might be expected to hold in theories with asymptotic (on-shell) hadronic scattering amplitudes of the form, e.g., $\exp[-(-t)^{1/2}]$, and the considerable theoretical and experimental evidence in favor of such behavior is reviewed. Finally, we discuss the extension of our analysis to the case where $p\neq 0$ and Schwinger terms are present. In particular, an exponential falloff for $T(q;p)$ *implies* an oscillating spectral function and strongly suggests that I is undefined. Some purely mathematical results concerning the existence of spectral representations when no explicit assumption is made about the asymptotic behavior of the spectral function are also obtained; these may be useful in other contexts.

I. INTRODUCTION

A DISTURBING feature of the current-algebra approach is that,[1,2] when using the Bjorken limit,[1] one appears to get divergent expressions for the electromagnetic mass shifts of hadrons, because the q-number part of the equal-time commutator (ETC) $E_{\mu\nu}(\mathbf{x}) =[\dot{J}_\mu(0,\mathbf{x}),J_\nu(0)]$, where J_μ is the electromagnetic current, does not vanish in any of the familiar models.[3] The purpose of this paper is to discuss this difficulty and to suggest that it may indeed be only apparent. In particular, we wish to point out that there may exist a remarkable connection between (i) the requirement of a finite mass shift, (ii) the probable ambiguity in the definition of $E_{\mu\nu}$, and (iii) the experimental indications that the electromagnetic form factor of the nucleon, $G(q^2)$, is an exponentially falling function of momentum transfer. This connection is based on the following observations:

(a) If $E_{\mu\nu}$ is ambiguous, then the moment I of the spectral function σ associated with $M_{\mu\mu}(q;p)$, the trace of the forward photohadron scattering amplitude $M_{\mu\nu}(q;p)$, will, in general, not exist.

(b) If I does not exist, it seems to be necessary that σ change sign infinitely often (oscillates), in order that the mass shift be finite.

(c) An oscillating σ is precisely what is expected on the basis of an exponentially falling $G(q^2)$ or, more generally, any $G(q^2)$ which decreases more rapidly than an inverse power of q^2.

Under these circumstances the current-algebraic implication that $\delta m^2=\infty$ is invalid, since a nonvanishing $E_{\mu\nu}$ can result from a suitable equal-time (ET) limit even though δm^2 is finite.

Although in this paper our emphasis will be on the deductive approach outlined above, a possible and, we believe, convincing inductive approach is the following.[4] There is considerable experimental and theoretical evidence, which we review in Sec. VI, that the nucleon electromagnetic form factor $G(q^2)$ is an exponentially decreasing function of q^2, viz.,

$$G(q^2)\sim\exp[-a(q^2)^{1/2}],\quad q^2\to\infty. \tag{1.1}$$

Then, if $G(q^2)$ is polynomially bounded, the spectral

* Supported in part by the U. S. Air Force Office of Scientific Research under Grant No. AFOSR 68-1453.

[1] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[2] M. B. Halpern and G. Segrè, Phys. Rev. Letters **19**, 611 (1967); **19**, 1000(E) (1967); G. C. Wick and B. Zumino, Phys. Letters **25B**, 479 (1967).

[3] These include the quark model (Ref. 1), the algebra of fields [T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967)], the $U(2)\otimes U(2)$ σ model [M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960)], and the $U(3)\otimes U(3)$ σ model [M. Lévy, *ibid.* **52A**, 23 (1967)]. An exception is a model recently introduced by T. D. Lee [Phys. Rev. **171**, 1731 (1968)] based on a set of intermediate vector bosons; here $E_{\mu\nu}$ is a c number. See also, J. Bjorken and R. Brandt, Phys. Rev. this issue, **177**, 2331 (1968).

[4] A more complete account of this approach can be found in R. Brandt and J. Sucher, Phys. Rev. Letters **20**, 1131 (1968).

function $\text{Im}G(q^2)$ *necessarily* oscillates, i.e., changes sign infinitely often in any neighborhood of $q^2=+\infty$. One would expect similar behavior for the form factors of other hadrons and hence, as we show in Sec. VI, for the quantity $F(-q^2,\nu)\equiv M_{\mu\mu}(q;p)$ and the associated spectral function σ. Such behavior for F certainly implies finite hadronic mass shifts, but seems to be in conflict with the usual models of current algebra, which appear to give

$$F(-q^2,\nu)\sim I/\pi q^2{}_0\,,\quad \nu=q\cdot p/m\,, \tag{1.2}$$

with I proportional to a (nonvanishing) difference of matrix elements of the ETC $[\dot{J}_\mu,J_\mu]$. However, if σ is oscillatory, then it is unlikely that I exists, so that (1.2) will be incorrect. Furthermore, as we show in Sec. V, an ambiguous I corresponds precisely to an ambiguous commutator $[\dot{J}_\mu,J_\mu]$ in the sense that the ET limit depends on how this limit is taken. In particular, a nonvanishing commutator can result from a suitable ET limit. Since, as we show in Sec. V, such ambiguous commutators are expected in the usual models, we can resolve the apparent conflict of these models with the requirement of finite mass shifts.

We now outline the contents of the following sections. In Sec. II we review the Bjorken analysis relating the high-q_0 behavior of the amplitude $T(q;p)=-i\int dx \times e^{-iq\cdot x}\langle p|T(J(x)J'(0))|p\rangle$ to the ETC's $[\partial_0{}^nJ,J']$ and its implication that the pion electromagnetic mass shift δm^2 is infinite in theories for which

$$\langle\pi^+|[\partial_0J_\mu,J_\mu]|\pi^+\rangle-\langle\pi^0|\cdots|\pi^0\rangle$$

does not vanish. This is the case for all of the usual models. We review the arguments that only $T(q;0)$ need be considered and introduce the spectral representation $T(q;0)=F(-q^2)=\pi^{-1}\int da\,\sigma(a)(q^2+a)^{-1}$ and the moment $I=\int da\,\sigma(a)$ corresponding to $[\dot{J}_\mu,J_\mu]$.

In Sec. III we show, by explicitly considering a pion field in perturbation theory, that $T(q;p)\sim 1/q_0{}^2$ for $q_0\to\infty$ only if $[\dot{J}_\mu,J_\nu]$ is well defined (free-field case). Otherwise (interacting-field case), the high-q_0 behavior of T is different.

By considering the abstract definitions of δm^2 and I when $p=0$ and no Schwinger terms are present, we prove in Sec. IV some theorems relating δm^2 and I. Our main result is that δm^2 can be finite only if I is either zero or does not exist; in the latter case $\sigma(a)$ must change sign infinitely often in any neighborhood of $a=+\infty$. This result is shown to remain true even if the above spectral representation is modified by subtractions and an additive polynomial. We illustrate these remarks with some examples, an especially simple one being $F_1(-q^2)\propto\exp[-(q^2)^{1/2}]$, which gives $\delta m^2<\infty$ and I undefined.

In addition, Sec. IV contains a variety of mathematical results concerning the existence of spectral representations when no explicit assumption is made about the asymptotic behavior of $\sigma(a)$, as well as a number of related results which may be useful in other contexts.

In Sec. V we show that an ambiguous I corresponds precisely to an ambiguous ETC $[\dot{J},J]$ in the sense that the ET limit depends on how this limit is taken. We show for the examples of Sec. IV that a nonvanishing ETC can result from a suitable ET limit even though $\delta m^2<\infty$. Since the original ambiguity in the ET limit corresponds to ambiguities in local-field products, and since the usual models involve such ambiguous field products, we conclude that these models do *not* imply a divergent δm^2.

In Sec. VI, we show that if the nucleon electromagnetic form factor decreases more rapidly than an inverse power—a possibility which is not inconsistent with experiment and consistent with a variety of theoretical ideas—then it is very likely that $\sigma(q'^2,\nu)$, the spectral function associated with $F(-q^2,\nu)$, oscillates. Indirect support for such oscillation is found from an examination of high-energy amplitudes describing hadron-hadron collisions.

In Sec. VII we show how a finite δm^2 can be obtained even if operator Schwinger terms are present. This is illustrated in a simple model. We also show that such Schwinger terms need not invalidate previous results based on the usual current algebra, soft-pion theory, and the Weinberg sum rules.

A summary and concluding discussion is given in Sec. VIII. The Appendices A–D contain some purely mathematical results needed in Secs. IV and V.

II. BJORKEN LIMIT AND PION ELECTROMAGNETIC MASS SHIFT

A. Bjorken Analysis

In an important paper,[1] Bjorken has analyzed a possible connection between the covariant forward pion-photon scattering amplitude[5]

$$M_{\mu\nu}{}^{(\delta)}(q;p)\ (\delta=+,0,-)\,,$$

the time-ordered product

$$T_{\mu\nu}{}^{(\delta)}(q;p)=-i\int dx\,e^{-iq\cdot x}\langle p,\delta|T(J_\mu(x)J_\nu(0))|p,\delta\rangle, \tag{2.1}$$

and the "Schwinger term"

$$S_{\mu\nu}{}^{(\delta)}(q;p)\equiv M_{\mu\nu}{}^{(\delta)}(q;p)-T_{\mu\nu}{}^{(\delta)}(q;p)\,. \tag{2.2}$$

Here $|p,\delta\rangle$ refers to a π^δ state of momentum p, and $J_\mu(x)$ is the electromagnetic current operator. Bjorken advanced the following three specific proposals:

(i) M and T have the *same* absorptive parts[6]

$$\rho_{\mu\nu}(q;p)=\int dx\,e^{-iq\cdot x}\langle p|J_\mu(x)J_\nu(0)|p\rangle \tag{2.3}$$

[5] Bjorken's analysis can be made for any hadronic states and current operators. We emphasize the pion-photon case for convenience and concreteness.

[6] We suppress the charge index δ whenever this does not lead to confusion.

and

$$\bar{\rho}_{\mu\nu}(q;p)=\int dx\, e^{+iq\cdot x}\langle p|J_\mu(0)J_\nu(x)|p\rangle. \quad (2.4)$$

(ii) T can be written in the unsubtracted form

$$T_{\mu\nu}(q;p)=\int_0^\infty \frac{dq_0'}{2\pi}\left[\frac{\rho_{\mu\nu}(q_0',\mathbf{q};p)}{q_0-q_0'}-\frac{\bar{\rho}_{\mu\nu}(q_0',-\mathbf{q};p)}{q_0+q_0'}\right]. \quad (2.5)$$

(iii) The leading high-q_0 behavior of T is obtained by expanding (2.5) in powers of $1/q_0$:

$$T_{\mu\nu}(q;p)\sim\langle p|\left\{\int d\mathbf{x}\left(\frac{1}{q_0}[J_\mu(0,\mathbf{x}),J_\nu(0)]-\frac{i}{q_0^2}[\dot{J}_\mu(0,\mathbf{x}),J_\nu(0)]+O(1/q_0^3)\right)e^{-i\mathbf{q}\cdot\mathbf{x}}\right\}|p\rangle. \quad (2.6)$$

Bjorken observes that (2.6) will be valid in any approximation in which the intermediate-state sums implicit in (2.3) and (2.4) are truncated.

In the above circumstances, the role of S (a polynomial in q_0) is to make up for the noncovariance of T so that the sum $M=T+S$ *is* covariant. Since, by (2.6), T vanishes for $q_0\to\infty$, S can be identified as the part of M which survives in this limit.

Let us record the fact that proposal (i) will in fact be valid in any field theory in which the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas hold and the T product (2.1) is well defined. The scattering amplitude is then given by[7]

$$M_{\mu\nu}(q;p)=-i\int dx\, e^{-iq\cdot x}\langle p|T(J_\mu(x)J_\nu(0))+\delta(t)[J_\mu(x),\dot{A}_\nu(0)]|p\rangle, \quad (2.7)$$

so that one has

$$S_{\mu\nu}(q;p)=-i\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[J_\mu(0,\mathbf{x}),\dot{A}_\nu(0)]|p\rangle, \quad (2.8)$$

which is *independent* of q_0. If the individual terms in (2.7) are not well defined [i.e., subtractions are required in (2.5)], then S will only differ from (2.8) by a polynomial in q_0.

B. Pion Electromagnetic Mass Shift

To second order in e, the electromagnetic contribution to the difference of the squared masses of π^+ and π^0 is given by[8]

$$\delta m^2=\frac{-ie^2}{2(2\pi)^4}\int d^4q\, D_{\mu\nu}(q)\Delta M_{\mu\nu}(q;p), \quad (2.9)$$

where

$$D_{\mu\nu}(q)=\frac{1}{q^2+i\epsilon}\left(\delta_{\mu\nu}-\kappa\frac{q_\mu q_\nu}{q^2}\right) \quad (2.10)$$

and

$$\Delta M(q;p)=M^{(+)}(q;p)-M^{(0)}(q;p). \quad (2.11)$$

The quantity (2.9) is independent of the gauge constant κ since $M_{\mu\nu}$ satisfies the gauge-invariance condition

$$q_\mu M_{\mu\nu}(q;p)=0, \quad (2.12)$$

valid because $p^2=-m_\pi^2$.

Now one expects from (2.6) that δm^2 will be *at least* logarithmically divergent in any theory with a current algebra in which the difference

$$E_p(\mathbf{x})\equiv\langle\pi^+|[\dot{J}_\mu(0,\mathbf{x}),J_\mu(0)]|\pi^+\rangle-\langle\pi^0|\cdots|\pi^0\rangle \quad (2.13)$$

does not vanish. A nonvanishing contribution from the first commutator in (2.6) or from the Schwinger term (2.2) (a polynomial in q_0) would only make the divergence worse. Since, in fact, $E_p\neq0$ in the known theories such as the quark model,[3,9] the algebra of fields,[3] the $U(2)\otimes U(2)$ σ model,[3] or the $U(3)\otimes U(3)$ σ model,[3] it has been concluded[1,2] that δm^2 is divergent in this approach. The same conclusion has been reached within the phenomenological Lagrangian framework.[10] Dispersion-theoretic techniques, on the other hand, appear to give a finite (and numerically accurate) value for δm^2.[11]

If (2.6) is maintained, then the above difficulty is apparently only avoided in a world with vanishing pion mass. For the above models,[3] in fact, (2.13) vanishes in the soft-pion limit $p\to0$ ($m_\pi=0$), as does the contribution of the first commutator in (2.6), so that the integral in (2.9) becomes convergent. A direct calculation[12] of δm^2 in a world with $m_\pi=0$ indeed leads to a finite and accurate result. The usual commutation relations of current algebra[13] together with soft-pion theory lead to

$$(\delta m^2)_{\text{soft}}=\text{const}\int\frac{d^4q}{q^2}\left(\delta_{\mu\nu}-\kappa\frac{q_\mu q_\nu}{q^2}\right)[\Delta_{\mu\nu}{}^V(q)-\Delta_{\mu\nu}{}^A(q)], \quad (2.14)$$

where

$$\Delta_{\mu\nu}{}^V(q)=\int dx\, e^{-iq\cdot x}\langle0|TV_\mu{}^3(x)V_\nu{}^3(0)|0\rangle \quad (2.15)$$

and Δ^A is given by a similar expression with the axial-vector current $A_\mu{}^3$ replacing the vector current $V_\mu{}^3$. The assumption that the corresponding two-point functions of the vector and the axial-vector currents have spectral

[7] We assume, in accordance with all known models, that the other ETC's which might occur in (2.7) are c numbers and hence irrelevant for our purposes.

[8] Riazuddin, Phys. Rev. **114**, 1184 (1959); M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters **2**, 7 (1959); V. Barger and E. Kazes, Nuovo Cimento **28**, 385 (1963).

[9] In Bjorken's quark model, (2.13) actually does vanish, since the ETC has $\Delta I=1$. The corresponding quantity does not vanish for $\Delta I=1$ mass differences such as that between K^+ and K^0 or p and n.

[10] See, for example, G. C. Wick and B. Zumino (Ref. 3); I. S. Gerstein *et al.*, Phys. Rev. Letters **19**, 1064 (1967).

[11] See, for example, H. Harari, Phys. Rev. Letters **17**, 1303 (1966), and references cited therein.

[12] T. Das *et al.*, Phys. Rev. Letters **18**, 759 (1967).

[13] M. Gell-Mann, Physics **1**, 63 (1964).

functions $\rho_V(a)$ and $\rho_A(a)$ obeying (when $m_\pi=0$) the Weinberg[14] sum rules

$$\int da\, a^{-1}[\rho_V(a)-\rho_A(a)]=F_\pi^2 \tag{2.16}$$

and

$$\int da\,[\rho_V(a)-\rho_A(a)]=0 \tag{2.17}$$

then leads to a convergent expression for δm^2. The value $\delta m \simeq 5.0$ MeV is obtained[12] if the spectral functions are approximated by retaining only the ρ and A_1 poles. A similar result is obtained in the phenomenological Lagrangian framework.[15]

The above methods break down, however, as soon as they are extended to the realistic case in which $m_\pi\neq 0$. This is in accordance with the expansion (2.6) and the nonvanishing of (2.13) in these theories. Whereas the phenomenological Lagrangian approach is not expected to be valid in the high-q_0 region of the integration in (2.9), the current commutator (2.13) is supposed to be specifically describing the small-x_0 behavior of $J(x)J(0)$ and hence the large-q_0 properties of $T(q;p)$. Thus the implication of a divergent δm^2 from the current commutators of the usual models is definitely disturbing.

C. Dependence on Pion Momentum

In order to discuss further the role of the pion momentum p in the above considerations, let us consider the amplitude

$$F(-q^2,\nu)\equiv \Delta M_{\mu\mu}(q;p) \tag{2.18}$$

as an analytic function of q^2 and $\nu=q\cdot p/m_\pi$. As emphasized by Cottingham,[16] the analytic structure of F in q_0, which follows from (2.5) together with (2.10), allows one to rotate the q_0 integration contour in (2.9) to iq_0 and obtain

$$\delta m^2=\frac{-i\alpha}{8\pi^3}\int d^4q\,\frac{1}{q^2}F(q^2,\nu)\,,$$
$$q^2=\mathbf{q}^2+q_0^2\,,\quad \nu=(-iq_0p_0+\mathbf{q}\cdot\mathbf{p})/m_\pi\,. \tag{2.19}$$

Next, following Cottingham[16] and Bjorken,[1] we express F in terms of a dispersion integral over ν, assumed to be unsubtracted:

$$F(-q^2,\nu)=\frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'\nu'\,\mathrm{Im}F(q^2,\nu')}{\nu'^2-\nu^2}\,. \tag{2.20}$$

Bjorken has argued that, since $\nu_0\to\infty$ like q^2/m_π, it follows[17] from (2.20) that

$$F(-q^2,\nu)\to F(-q^2,0)\quad \text{as } q^2\to\infty\,. \tag{2.21}$$

Then the divergent part of δm^2 becomes (after a trivial angular integration)[18]

$$(\delta m^2)_{\mathrm{div}}=\frac{\alpha}{8\pi}\int_0^\infty dq^2\,F(-q^2,0)\,. \tag{2.22}$$

The expression (2.22) puts very sharply the implication that $\delta m^2=\infty$ whenever $E(\mathbf{x})\neq 0$, since then, according to (2.6),[19] $F(-q^2,0)\sim 1/q^2$ (by covariance). We note that this does not contradict the result of Das *et al.*,[12] since it has not been assumed here that $m_\pi=0$. The simplicity of Eq. (2.22), and also of the limit of Eq. (2.5) for $\nu\to 0$ (to be discussed in Sec. II D), makes it highly desirable to work at $\nu=0$. This is justified in part by Bjorken's arguments leading to Eq. (2.22) and in part by the fact that our considerations can be extended to the case $\nu\neq 0$. Thus in the following sections we shall largely ignore the p dependence of ΔM. We shall find functions $F(-q^2)$ corresponding to $E(\mathbf{x})\neq 0$, but nevertheless giving $\delta m^2<\infty$. On adding to such functions arbitrary functions $G(q^2,\nu,p^2)$ which give finite contributions to δm^2 and vanishing contributions to $E(\mathbf{x})$, one obtains functions with essentially the same properties as $F(-q^2)$ but with nontrivial p dependence. All this can be done consistently with the soft-pion calculation if, for example, one puts

$$\Delta M \propto m_\pi F(-q^2)+G(q^2,\nu,p^2)\,,$$

with $G(q^2,0,0)\sim(q^2)^{-2}$. Then for $p=0$, $m_\pi=0$, one has $\Delta M\propto G(q^2,0,0)$, which reproduces the soft-pion result, and for general p and $m_\pi\neq 0$ one finds $E(\mathbf{x})\neq 0$, in accordance with the above models, but δm^2 will be finite, in accordance with experiment.

Another way of looking at our $p=0$ analysis is to suppose that we also put $m_\pi=0$. Then (2.22) gives the correct complete mass shift, and the necessary conditions for a finite mass shift, given in Sec. IV, are valid in a world with $m_\pi=0$. Our assumption is, then, that these conditions continue to hold when $m_\pi\neq 0$. This approach avoids possible gauge-invariance ambiguities at $p=0$, $m_\pi\neq 0$, and also allows a unification of the $m_\pi=0$ and $m_\pi\neq 0$ treatments.

We should emphasize that, although in Secs. IV and V we neglect p dependence, the arguments of Secs. VI and VIII are valid on the physical pion mass shell.

D. Spectral Representation

When $p=0$, it is easy to exhibit explicitly the covariance of the representation (2.5) for $T_{\mu\mu}$. The definition (2.1) of $T_{\mu\mu}(q;0)\equiv F(-q^2)$ leads in the usual way to the manifestly covariant (assumed to be un-

[14] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[15] See Ref. 10 and references therein.

[16] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963).

[17] This argument has been sharpened by A. Mueller (private communication), who has shown that (2.21) holds, in the absence of delicate cancellations, even if (for $\Delta I=1$ mass shifts) the dispersion relation for M_1 (see Ref. 18) requires a subtraction, as is suggested by Regge theory.

[18] By (2.22) we mean the following: Write $\Delta M_{\mu\nu}=M_1P_{\mu\nu}^{(1)}+M_2P_{\mu\nu}^{(2)}$, where $P^{(1)}$ and $P^{(2)}$ are the usual gauge-invariant polynomials in p and q, and $M_i=M_i(q^2,\nu;p^2)$. We analytically continue the M_i to $p=0$ and calculate (2.9) there, thus obtaining (2.22), which defines a particular "mass shift at $p=0$."

[19] We first extrapolate (2.6) to $p=0$.

subtracted) spectral representation

$$F(-q^2)=\frac{1}{\pi}\int d\kappa^2\,\sigma(\kappa^2)(q^2+\kappa^2)^{-1}. \qquad (2.23)$$

Alternatively, Eq. (2.23) follows from a Jost-Lehmann-Dyson[20,21] (JLD)-type representation for $T_{\mu\mu}(q;p)$. Equation (2.23) can easily be written in the form (2.5) with

$$\rho_{\mu\mu}(q_0',\mathbf{q};0)=\bar{\rho}_{\mu\mu}(q_0',-\mathbf{q};0)\propto\sigma(q_0'^2-\mathbf{q}^2).$$

Expansion of (2.23) in powers of $1/q^2$ formally gives

$$F(-q^2)\sim I/\pi q^2+O(1/q^4), \qquad (2.24)$$

where

$$I=\int d\kappa^2\,\sigma(\kappa^2). \qquad (2.25)$$

The moment I represents the matrix element

$$\langle p|\int dx\,[\dot{J}_\mu(0,\mathbf{x}),J_\mu(0)]|p\rangle$$

continued to $p=0$ ($m_\pi\neq 0$). [The first commutator in (2.6) cannot contribute to $T_{\mu\mu}$ at $p=0$.]

Let us now assume that $\Delta S_{\mu\mu}=0$. (The existence of Schwinger terms will be considered in Sec. VII and will not change our results.) Then the difficulty referred to above can be neatly expressed in terms of Eqs. (2.22)–(2.25) if we let

$$F(-q^2)\equiv F(-q^2,0)=\Delta M_{\mu\mu}(q;0). \qquad (2.26)$$

The usual models give $I\neq 0$, so that substitution of (2.24) into (2.22) apparently gives $\delta m^2=\infty$.

In the remainder of this paper we shall explore a way of avoiding this difficulty. In Sec. IV A we show that if $I\neq 0$ and $\delta m^2<\infty$, then I cannot exist. In Sec. V we show that the nonexistence of I does not preclude the existence of a nonvanishing ETC $E(\mathbf{x})$ defined by a suitable ET limit.

In the above discussion we have assumed that the representations (2.5) and (2.23) are unsubtracted. In general, however, subtractions are required. In fact, if $\langle p|J_\mu(x)J_\mu(0)|p\rangle$ behaves like $(x_0)^{-n}$ for $x_0\sim 0$, then (2.5) and (2.23) must be replaced with an n-times subtracted representation plus an additive polynomial of degree $n-1$.[22] This case will be fully discussed in Sec. IV B and will be shown not to change our results.

III. BEHAVIOR OF SOME TIME-ORDERED PRODUCTS IN PERTURBATION THEORY

A. General Considerations

In this section we shall use some perturbation-theoretic examples to examine the relationship between the nature of the ETC's occurring in (2.6) and the actual high-q_0 behavior of the time-ordered product (2.1). This analysis is primarily intended to motivate the more general and more rigorous discussion in Sec. IV. It is also of interest in itself, however, since it shows that the usual assumptions concerning time-ordered products are not supported by perturbation theory. In Sec. V we shall present a more thorough discussion of the ETC's in terms of ET limits of ordinary commutators.

Throughout this paper we shall assume that the first ETC in (2.6) gives a vanishing contribution. This is in accordance with the models of Sec. II and the present section. Then, if (2.6) is correct, the high-q_0 behavior of T will be determined by the ETC $[\dot{J}_\mu,J_\nu]$. By locality and temperedness we can write, formally,

$$[\dot{J}_\mu(0,\mathbf{x}),J_\nu(0)]=O_{\mu\nu}(0)\delta(\mathbf{x})+O_{\mu\nu\lambda}{}^{(1)}(0)\partial_\lambda\delta(\mathbf{x})+\cdots \qquad (3.1)$$

in terms of a finite number of field operators $O(x)$, $O^{(1)}(x)$, $\cdots$. For simplicity we shall now assume that our large-q_0 limits are taken for $\mathbf{q}=0$. Then only the first term in (3.1) contributes to (2.6). This restriction is made only to simplify our discussion and inclusion of the other terms in (3.1) would not alter our conclusions. Furthermore, we need only consider the q-number part of $O_{\mu\nu}$ (i.e., the difference of $O_{\mu\nu}$ and its vacuum expectation value), since we are ultimately interested in the difference (2.13).[23]

Now if $O_{\mu\nu}(x)$ is a well-defined operator, one expects from (2.6) that $T_{\mu\nu}$ will behave as $\langle p|O_{\mu\nu}(0)|p\rangle/q_0^2$ for large q_0. In renormalized perturbation theory, however, the ETC (3.1) is rarely well defined and the operator $O(x)$ involves divergent or ambiguous quantities, such as products of local field operators. In this case one would no longer expect that $T\sim\text{const}/q_0^2$. Rather, if

$$\langle p|0|p\rangle\sim c(\Lambda^2),$$

where Λ represents a cutoff with dimension of mass and $c(\infty)=\infty$, then one might expect $T\sim c(q_0^2)/q_0^2$.[24] Thus, if $O(x)$ were sufficiently divergent, one would not even expect T to vanish for large q_0.

B. Free Pion Field

To illustrate this effect, we first consider a *free* pion field $\phi(x)$. The corresponding electric current operator is

$$J_\mu=ie(\partial_\mu\phi^\dagger\cdot\phi-\phi^\dagger\partial_\mu\phi)-2e^2\phi^\dagger\phi A_\mu, \qquad (3.2)$$

where $\phi=\frac{1}{2}\sqrt{2}(\phi_1+i\phi_2)$ and A_μ is the free electromagnetic field operator (we work to order e^2). The general form of $M_{\mu\nu}$ is [see (2.7) and related remarks]

$$M_{\mu\nu}(q;p)=-i\int dx\,e^{-iq\cdot x}\langle p|T(J_\mu(x)J_\nu(0)) +\delta(t)[J_\mu(x),\dot{A}_\nu(0)]|p\rangle, \qquad (3.3)$$

[20] R. Jost and H. Lehmann, Nuovo Cimento **5**, 1598 (1957).
[21] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).
[22] O. Steinmann, Helv. Phys. Acta **36**, 90 (1963).
[23] In any case, c-number contributions to (3.1) only affect the vacuum electromagnetic self-energy.
[24] This is the case, for example, for low-order propagators in perturbation theory, where $c(\Lambda^2)\sim Z^{-1}$.

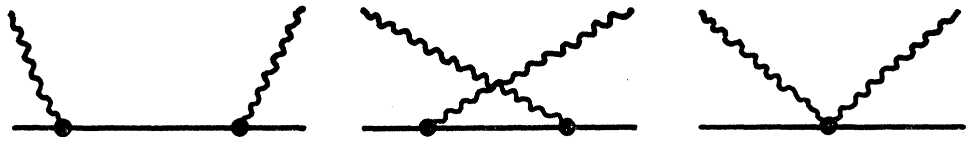

FIG. 1. Diagrams contributing to the photopion scattering amplitude in order $e^2\lambda^0$.

and for the case at hand one has (using the canonical commutation relations)

$$\delta(t)[J_\mu(x),\dot{A}_\nu(0)]=-2ie^2\phi^\dagger(x)\phi(x)\delta_{\mu k}\delta_{k\nu}\delta(x)\,, \quad (3.4)$$

so that [cf. Eq. (2.2)]

$$S_{\mu\nu}(q;p)=-2e^2\langle p|\phi^\dagger\phi|p\rangle\delta_{\mu k}\delta_{k\nu}. \quad (3.5)$$

The relevant current commutator is

$$\delta(t)[\dot{J}_k(x),J_l(0)]=-4ie^2[\partial_k\phi^\dagger(x)\partial_l\phi(x) +\text{H.c.}]\delta(x)+\cdots\,, \quad (3.6)$$

where the omitted terms do not contribute to (2.6) for $\mathbf{q}=0$, because they are either of the form $O(0)\partial\delta(x)$ or of the form $\partial O(x)\cdot\delta(x)$, which satisfies $\langle p|\partial O(x)|p\rangle=0$.

Since $\phi(x)$ is a free field, the q-number part of (3.6) is finite and well defined, so that (2.6) implies that $T\sim 1/q_0^2$. This behavior of T can easily be explicitly verified. The diagrams contributing to $M_{\mu\nu}$ are shown in Fig. 1; one finds

$$M_{\mu\nu}(q_0,\mathbf{0};p)=-2e^2\delta_{\mu k}\delta_{k\nu}+(8e^2/q_0^2)[p_\mu p_\nu +(\delta_{\mu 4}p_\nu+\delta_{\nu 4}p_\mu)p_0+\delta_{\mu 4}\delta_{4\nu}p_0^2]+O(1/q_0^4). \quad (3.7)$$

Thus, in view of (3.5), we see that $T_{\mu\nu}=M_{\mu\nu}-S_{\mu\nu}$ is of order $1/q_0^2$, in agreement with (2.6). The coefficient of $1/q_0^2$ is also seen to agree with that predicted by (2.6) and (3.6).

C. Interacting Pion Field

Now let us suppose that a $\lambda(\phi^2)^2$ interaction is introduced. Then, to all orders of λ, the expression (3.2) for J_μ remains unchanged (and finite[25]) and the relations (3.3)–(3.6) remain formally valid. The q-number part of (3.6) now, however, becomes divergent, so that (2.6) is no longer meaningful. For example, in order λ only the diagram in Fig. 2 contributes to (3.6) (between one-pion states) and it gives a quadratic divergence. In this case, in fact, $T_{\mu\nu}$ does *not* vanish for $q_0\to\infty$. To see this, it is sufficient to work to first order in λ. The diagrams contributing to $M_{\mu\nu}$ in this order are shown in Fig. 3. Denoting the contribution of the final (seagull) diagram by $R\delta_{\mu\nu}$ and observing that the contribution of the ETC term (3.5) is $S_{\mu\nu}=R\delta_{\mu k}\delta_{k\nu}$, we can write

$$T_{\mu\nu}=F_{\mu\nu}+R\delta_{\mu 4}\delta_{4\nu}\,,$$

where $F_{\mu\nu}$ is the (covariant) contribution of the first four diagrams in Fig. 3 (plus their crossed counterparts). More explicitly, we have[26]

$$T_{\mu\nu}=A\delta_{\mu\nu}+B[p_\mu q_\nu q^2-(p_\mu q_\nu+q_\mu p_\nu)(p\cdot q)] +Cq_\mu q_\nu+R\delta_{\mu 4}\delta_{4\nu}. \quad (3.8)$$

In order to show that $T_{\mu\nu}(q;p)$ does not vanish for $q_0\to\infty$, it is sufficient to consider the case $\mathbf{p}=\mathbf{q}=0$, $\mu=\nu=1$. Then only the first term in (3.8) is nonvanishing, so that we need only show that $A(q;p)$ does not vanish for $q_0\to\infty$ in this case. Only the first diagram in Fig. 3 contributes to A and this gives, after renormalization,[27]

$$A(q_0,\mathbf{0};m,\mathbf{0})\propto\int_0^1 d\eta(1-\eta)\ln\frac{q_0^2\eta(1-\eta)+m^2}{m^2}\,,$$

which indeed does *not* vanish for $q_0\to\infty$. Correspondingly, $T_{\mu\nu}$ no longer satisfies an unsubtracted spectral representation.

D. Implications

It might be thought that the above analysis is not useful for understanding how a finite mass shift can be obtained, since it may appear that a bad high-q_0 behavior for T can only worsen the divergence of δm^2.

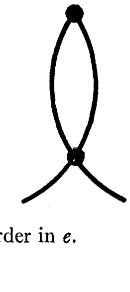

FIG. 2. Diagram contributing to the matrix element $\langle p|\partial_k\phi^\dagger\partial_l\phi|p\rangle$ in order λ. It gives rise to the divergent integral $\int d^4q\,(q^2+m^2)^{-2}\times q_kq_l$.

FIG. 3. Diagrams contributing to the photopion scattering amplitude in order $e^2\lambda$. The remaining diagrams which contribute in this order are obtained from those shown by crossing the photon lines.

[25] Recall that we are working only to second order in e.

[26] Gauge invariance of $M_{\mu\nu}=F_{\mu\nu}+R\delta_{\mu\nu}$ requires that $A+R=B(p\cdot q)^2-Cq^2$, but we shall not need this relation.

[27] In this order A is actually independent of p. We have renormalized this diagram by subtracting at $q=0$. Subtraction at any other point would only change A by a constant.

The first point to be made is, however, that this bad behavior of T can be cancelled by a corresponding behavior of S in (2.2), so that the resulting M has a good behavior. We shall investigate this possibility in Sec. VII and there show that it actually occurs in a perturbation-theoretic example. The second, more important point is that these examples show that an ill-defined ETC can alter the behavior of T (example in Sec. III C) and even can lead to a finite δm^2 (example in Sec. VII). The changed behavior of T is related to the *divergent character* of the ETC (3.1). However, we observe that if this ETC fails to exist in a more subtle way, it might be possible to obtain a finite δm^2 even without Schwinger terms and a bad behavior of T. This possibility will occupy us in Secs. IV and V.

IV. MOMENTS AND "MASS SHIFTS": MATHEMATICAL ASPECTS

Motivated by the expressions (2.22)–(2.25), we now study the connection between the existence of the moment of the spectral function associated with an analytic function $f(z)$ which has a right-hand cut and the convergence of the integral of $f(x)$ in the neighborhood of $x=-\infty$. We first introduce some notation.

Let $\pi(x_0)$ denote the complex z plane with the interval $[x_0,+\infty]$ excluded, and let $f(z)$ be analytic for $z\in\pi(x_0)$. The spectral function $\sigma(x)$, defined by

$$\sigma(x)=(2i)^{-1}[f(x+i0)-f(x-i0)],\quad x\geq x_0 \quad (4.1)$$

is assumed to be integrable in any finite interval $[x_0,x_1]$.[28] We define a "moment function" $I(x)$ via

$$I(x)=\int_{x_0}^{x}\sigma(x')dx',\quad x\geq x_0 \quad (4.2)$$

and a "mass-shift function" $L(x)$ via

$$L(x)=\int_{x}^{0}f(x')dx',\quad x\leq 0. \quad (4.3)$$

We assume, for ease of writing, that $x_0>0$; if not, the upper limit in (4.3) should be changed to any convenient value less than x_0. The moment of σ and the "mass shift" are defined by

$$I=I(\infty)=\lim_{x\to\infty}I(x) \quad (4.4)$$

and

$$L=L(-\infty)=\lim_{x\to-\infty}L(x), \quad (4.5)$$

respectively, provided, of course, that the limits exist. If, for example, the limit (4.4) does not exist, we shall say that $I(\infty)$ does not exist.

We restrict ourselves (at first) to functions $f(z)$, which are simply related to their spectral functions, i.e., satisfy

$$f(z)=\frac{1}{\pi}\int_{x_0}^{\infty}\frac{\sigma(x')}{x'-z}dx', \quad (4.6)$$

with no subtraction required in the integral and no additive entire function appearing on the right-hand side of (4.6). If $f(z)$ satisfies (4.6), we say that $f(z)$ *admits a simple unsubtracted spectral representation* (USR). A simple criterion for (4.6) to hold is the following, which for ease of reference we state as a theorem (for proof, see Appendix A):

Theorem 1. A sufficient condition for $f(z)$ to admit a simple USR is that $f(z)\to 0$ as $|z|\to\infty$, for $0<\arg z<2\pi$, and that $f(x\pm i0)$ is bounded in the interval (x_0,∞).

Note that this condition is not necessary, since, for example, $f(z)=z^p e^{i\sqrt{z}}$ satisfies (4.6) for $0<p<\frac{1}{2}$, but $f(x\pm i0)$ is not bounded as $x\to+\infty$.

A. Case of a Simple USR

1. *Heuristics, Examples, and Some Theorems*

The connection between the behavior of $I(x)$ for large x and the existence of $L(-\infty)$ is most easily seen as follows. We assume at first that for large negative x one can expand the denominator in (4.6) in inverse powers of x, to obtain

$$f(x)=-(\pi^{-1}/x)I(\infty)+\cdots; \quad (4.7)$$

more precisely, on multiplying (4.6) by x, we assume that one may pass to the limit $x=-\infty$ inside the integral, to get

$$B\equiv\lim_{x\to-\infty}xf(x)=-\pi^{-1}I(\infty). \quad (4.8)$$

If B exists, we may write $f(x)=[B+\gamma(x)]/x$, with $\gamma(x)\to 0$ as $x\to-\infty$, so that if $B\neq 0$, the integral (4.5) for $L(-\infty)$ is logarithmically divergent. It follows that if both B and $I(\infty)$ exist and (4.8) holds, then $L(-\infty)$ can exist only if $I(\infty)=0$. [As shown in Appendix B, if $I(\infty)$ exists, a sufficient condition that (4.8) hold is that $(x\ln x)\sigma(x)$ is bounded as $x\to+\infty$.] A useful criterion for determining whether or not $I(\infty)=0$ is given by

Theorem 2. If $f(z)$ admits a simple USR, then $I(\infty)=0$ if and only if $zf(z)$ also admits a simple USR.

The proof is immediate: If $I(\infty)=0$, we find from (4.6), using $(x'-z)^{-1}=z^{-1}[-1+x'(x'-z)^{-1}]$, that

$$zf(z)=\pi^{-1}\int_{x_0}^{\infty}\frac{x'\sigma(x')}{x'-z}dx'. \quad (4.9)$$

This is a simple USR for $zf(z)$, since $x\sigma(x)$ is the associated spectral function. Conversely, if (4.9) holds, then

[28] We shall also assume that $f(z)$ is *real* analytic so that $\sigma(x)=\mathrm{Im}f(x)$ is real; this is the case in the applications and we also thereby simplify the wording in many places. However, from the mathematical point of view there is no actual restriction involved, since we may write $\sigma(x)=\sigma_r(x)+i\sigma_i(x)$ and consider σ_r and σ_i separately.

on dividing by z and subtracting (4.6), we obtain $I(\infty)=0$.[29]

A simple example of a function which admits a simple USR such that $I(\infty)=0$ and $|L(-\infty)|<\infty$ is given by

$$h(z)=(z^{1/2}+ia)^{-2}e^{ibz^{1/2}}, \qquad (4.10)$$

with $a>0$, $b>0$. Since $h(x)\sim|x|^{-1}e^{-b\sqrt{|x|}}$ for $x\rightarrow-\infty$, $L(-\infty)$ exists; that $I(\infty)=0$ may be inferred from theorem 2, since $zh(z)$ admits a USR, by theorem 1.

However, it is not necessary to have $I(\infty)=0$ in order to get $|L(-\infty)|<\infty$. In particular, it may well be that B exists (and is zero) but that $I(\infty)$ does not exist, since the interchange of limits used in arriving at (4.8) may not be legitimate. Consider, e.g.,

$$k_1(z)=e^{i\sqrt{z}}. \qquad (4.11)$$

By theorem 1, $k_1(z)$ satisfies (4.6), with $\sigma(x)=\sin(\sqrt{x})$, but

$$I_1(\infty)=\int_0^\infty \sin(\sqrt{x})dx=2\int_0^\infty \kappa \sin\kappa\, d\kappa$$

does not exist. On the other hand, $L(-\infty)<\infty$, since $k_1(x)=e^{-|x|^{1/2}}$ for $x<0$. More generally, consider

$$k_\beta(z)=(-iz^{1/2})^{\beta-1}e^{iz^{1/2}}, \quad -1<\beta\leq 1 \qquad (4.12)$$

which provides a continuous link between the two possibilities being considered. For $-1<\beta<0$, $I_\beta(\infty)=0$, while for $0\leq\beta\leq 1$, $I_\beta(\infty)$ is not well defined; here

$$I_\beta(\infty)=\int_0^\infty dx\, x^{(\beta-1)/2}\cos(x^{1/2}-\tfrac{1}{2}\beta\pi).$$

In all cases, however, we have $L(-\infty)<\infty$, since $f_\beta(x)$ decreases exponentially as $x\rightarrow-\infty$.[30]

Another class of examples is provided by the following choice of spectral functions (with $x_0=0$); the associated values of $I(x)$ and $L(-\infty)$ are given in each case:

$$\sigma(x)=\sin x, \quad I(x)=1-\cos x, \quad L(-\infty)=\infty; \qquad (4.13a)$$

$$\sigma(x)=\cos x, \quad I(x)=\sin x, \quad L(-\infty)<\infty; \qquad (4.13b)$$

$$\sigma(x)=\sin(\sqrt{x}), \; I(x)=-2(\sqrt{x})\cos(\sqrt{x})+2\sin(\sqrt{x}), \quad L(-\infty)<\infty; \qquad (4.13c)$$

$$\sigma(x)=\cos(\sqrt{x}), \; I(x)=2(\sqrt{x})\sin(\sqrt{x})+2\cos(\sqrt{x})-2, \quad L(-\infty)=\infty. \qquad (4.13d)$$

[29] Theorem 2 may be used to provide a neat indirect proof of the remarkable fact that there exist continuous nonvanishing functions $\sigma(x)$ for which *all* the moments $I_n=\int_{x_0}^\infty x^n\sigma(x)dx$ vanish. Take, e.g., $f(z)=e^{(i-1)z^{1/4}}$, so that $\sigma(x)=e^{-x^{1/4}}\sin x^{1/4}$. Since $g_n(z)=z^nf(z)\rightarrow 0$ as $|z|\rightarrow\infty$ $(n=0, 1, 2, \cdots)$, $g_n(z)$ admits a simple USR; then, by theorem 2 applied to g_n and g_{n+1}, $I_n=0$. The reason for this possibility is that the integration is over an *infinite* interval. For a finite interval such a situation is not possible.

[30] Our examples may give the impression that if $L(-\infty)$ exists and $I(\infty)$ is undefined, then $f(x)$ necessarily decreases very rapidly as $x\rightarrow-\infty$. This is not the case, as shown by the function

$$\int_{\pi/2}^\infty dx'\frac{\sin x'}{x'-z},$$

which is $O(x^{-2})$ for $x\rightarrow-\infty$.

These examples show that if $I(\infty)$ does not exist, $L(-\infty)$ may or may not exist. Moreover, there is no correlation between the boundedness of $I(x)$ and the existence of $L(-\infty)$.

However, a useful criterion for the existence of $L(-\infty)$ is obtainable by the following consideration. Assume that $I(x)/x\rightarrow 0$ as $x\rightarrow+\infty$. Then integration by parts of (4.6) yields

$$f(z)=\pi^{-1}\int_{x_0}^\infty \frac{I(x')}{(x'-z)^2}dx'. \qquad (4.14)$$

On substitution of (4.14) into (4.3) we obtain

$$L(-\infty)=\pi^{-1}\int_{-\infty}^0 dx\int_{x_0}^\infty dx'\frac{I(x')}{(x'-x)^2}dx'. \qquad (4.15)$$

If we assume that the order of integration in (4.15) is reversible, we get

$$L(-\infty)=\pi^{-1}\int_{x_0}^\infty dx'\frac{I(x')}{x'}dx'. \qquad (4.16)$$

This suggests that the existence of the right-hand side of (4.16) may be both a necessary and sufficient condition for the existence of $L(-\infty)$, at least in practice, as a rule of thumb, if not in the strict mathematical sense. Indeed, if we reconsider the examples (4.13), we see that (4.16) diverges logarithmically in cases (4.13a) and (4.13d) and is convergent in cases (4.13b) and (4.13c), in agreement with the last column in (4.13), obtained by direct computation.

2. *Main Theorem: Necessary Condition for Finite Mass Shift*

The above heuristic considerations are put on firm footing by the following theorem, which is the main result of this section. We first introduce some terminology: (a) $I(x)$ is *asymptotically non-negative* (*non-positive*) if there exists an A such that $x>A\Rightarrow I(x)\geq 0$ (≤ 0), and (b) $I(x)$ is *asymptotically bounded away from 0* if there exist $\epsilon>0$ and A such that $x>A\Rightarrow|I(x)|>\epsilon$. Note that (b) implies one of the choices in (a), since $I(x)$ is continuous. We then have

Theorem 3. If $f(z)$ satisfies a simple USR and $L(-\infty)$ exists, then $I(x)$ cannot be asymptotically bounded away from zero; in particular, if $I(\infty)$ exists, it must be zero, and if $I(\infty)$ does not exist, $\sigma(x)$ must change sign infinitely often in any neighborhood of $x=+\infty$. However, neither a zero value for $I(\infty)$ nor oscillations in $\sigma(x)$ are sufficient to ensure the existence of $L(-\infty)$.

To keep the logic of the proof clear, we define auxiliary functions $C(x)$ and $K(x)$ via

$$C(x)=I(x)/x, \qquad (4.17)$$

$$K(x)=\int_{x_0}^x \frac{C(x')}{x'}dx', \quad x\geq x_0 \qquad (4.18)$$

and state some lemmas, whose proof is deferred to Sec. IV A 3:

Lemma 1. (i) If $C(\infty)$ exists, its value is zero. $K(\infty)$ exists if and only if $C(\infty)$ exists. (ii) If $I(x)$ is asymptotically non-negative or non-positive, then $K(\infty)$ exists.

Lemma 2. If $K(\infty)$ exists, the integral

$$g_0(z)\equiv\frac{z}{\pi}\int_{x_0}^{\infty}\frac{I(x')}{x'(x'-z)}dx',\quad z\in\pi(x_0) \tag{4.19}$$

exists and defines an analytic function of z with derivative $g_0'(z)=f(z)$.

Lemma 3. If $I(x)$ is asymptotically bounded away from zero, then[31] $g_0(-\infty)=\pm\infty$.

Proof of theorem 3. Define[32]

$$g(z)=\int_{x_0}^{z}f(\omega)d\omega,\quad z\in\pi(x_0) \tag{4.20}$$

with the contour from x_0 to z not cutting the real axis above x_0. Then $g(z)$ is analytic for $z\in\pi(x_0)$ and, as follows readily from (4.20), the discontinuity of $g(z)$ across the cut is $2iI(x)$. Now assume that $I(x)$ is asymptotically bounded away from zero. [This includes the cases $I(\infty)=$real number$\neq 0$ and $I(\infty)=\pm\infty$.] Then lemma 1 implies that $K(\infty)$ exists and lemma 2 implies that $g_0(z)$ exists, with

$$g_0'(z)=f(z). \tag{4.21a}$$

From (4.20),

$$g'(z)=f(z), \tag{4.21b}$$

and from (4.19), the discontinuity of $g_0(z)$ across the cut is $2iI(x)$, as for $g(z)$. Thus $E(z)=g(z)-g_0(z)$ is entire and $E'(z)=0$ by (4.21a) and (4.21b), so that

$$g(z)=g_0(z)+\text{const.} \tag{4.22}$$

Now, from (4.3) and (4.20),

$$L(x)=-g(x)-\int_0^{x_0}f(x')dx', \tag{4.23}$$

so that, from (4.22) and (4.23), $L(-\infty)$ exists if and only if $g_0(-\infty)$ exists. But lemma 3 implies that $g_0(-\infty)=\pm\infty$. Thus, if $L(-\infty)$ exists, $I(x)$ cannot be asymptotically bounded away from zero, and if $I(\infty)$ exists, it must be zero, as claimed.

To proceed further, we write

$$I(x)=\int_{x_0}^{A}\sigma(x')dx'+\int_{A}^{x}\sigma(x')dx', \tag{4.24}$$

and note that if $\sigma(x)$ is asymptotically non-negative (or non-positive), we may choose A so large that the second term in (4.24) is nondecreasing (or nonincreasing). Then either $I(\infty)$ exists or $I(\infty)=\pm\infty$. The latter violates the hypothesis on $I(x)$, so that if $I(\infty)$ does not exist, σ cannot be asymptotically non-negative or non-positive; $\sigma(x)$ can also not be identically zero in some neighborhood of $x=+\infty$, for then I certainly exists. Thus $\sigma(x)$ must change sign infinitely often, as asserted.

That $I(\infty)=0$ is not sufficient for the existence of $L(-\infty)$ is shown by the following counterexample. Take

$$\begin{aligned}&\sigma(x)=x-a, && 0\leq x\leq b\\ &\sigma(x)=\sigma(b)(b/x)(\ln b/\ln x)^2, && b\leq x.\end{aligned} \tag{4.25}$$

With $a=b(1+2\ln b)/(2+2\ln b)$ one finds

$$I(x)=I(b)(\ln b/\ln x),\quad x\geq b.$$

For $x<0$, $g_0(x)$ [Eq. (4.19)] is then proportional to the sum of a bounded function of x and the function

$$\int_b^{\infty}\frac{1}{x'\ln x'}\frac{|x|}{x'+|x|}dx',$$

which is readily seen to behave as $\ln\ln|x|$ for large $|x|$. It follows that $g_0(-\infty)=\infty$, and hence that $L(-\infty)=\infty$. The fact that oscillations in $\sigma(x)$ are not sufficient to ensure a finite $L(-\infty)$ is apparent from examples (4.13a) and (4.13d). This completes the proof.

Note that the criterion for the existence of $L(-\infty)$ based on the convergence or divergence of (4.16) is quite consistent with theorem 3. On the one hand, if $I(x)$ is asymptotically bounded away from zero, e.g., if $I(\infty)=\pm\infty$ or $I(\infty)$ exists but is not zero, then clearly (4.16) is at least logarithmically divergent. On the other hand, the insufficiency of oscillations in $\sigma(x)$ or of $I(\infty)=0$ for getting $L(-\infty)$ finite is also immediately apparent from (4.16): The $I(x)$ corresponding to (4.13a) and to (4.25) evidently lead to a divergent right-hand side of (4.16).

3. *Proof of the Lemmas*

To prove lemma 1, consider the function

$$J_A(x)=\int_A^x\frac{\sigma(x')}{x'}dx',\quad A\geq x_0. \tag{4.26}$$

Since

$$J_A(\infty)=\pi f(0)-\int_{x_0}^{A}\frac{\sigma(x')}{x'}dx',$$

we see that $J_A(\infty)$ exists. On integration by parts, (4.26) becomes, using $I'(x)=\sigma(x)$,

$$J_A(x)=K_A(x)+C(x)-C(A), \tag{4.27}$$

[31] The hypothesis cannot be weakened to include $I(x)$, which are simply asymptotically non-negative or nonpositive, which is sufficient for the existence of $g_0(z)$, by lemmas 1(ii) and 2. For example, by taking for $I(x)$ the positive distribution $\sum_{n=1}^{\infty}n^{-1}\times\delta(x-n)$, one gets $g_0(-\infty)=-\pi^{-1}\sum_{n=1}^{\infty}n^{-2}$, which is finite. An $I(x)$ which is an ordinary function and gives $g_0(-\infty)$ finite may now be obtained by replacing $\delta(x-n)$ with a smooth function which is sharply peaked about $x=n$ $(n=1,2,\cdots)$ and is rapidly decreasing as $x\to\infty$.

[32] We thank Dr. A. Martin for showing us the usefulness of the function $g(z)$ in problems of the kind that we are studying here.

where

$$K_A(x)=\int_A^x \frac{C(x')}{x'}dx'. \quad (4.28)$$

Since $J_A(\infty)$ exists, (4.27) shows that $K_A(\infty)$, and hence $K(\infty)$, exists if and only if $C(\infty)$ exists. However, if $C(\infty)\neq 0$, the first mean-value theorem applied to (4.28) in the form

$$K_A(x)=C(\zeta)\int_A^x \frac{1}{x'}dx'=C(\zeta)\ln\frac{x}{A},$$

with $\zeta\in(A,x)$, shows that $|K_A(x)|\to\infty$ as $x\to\infty$, so that we must have $C(\infty)=0$ if $K(\infty)$ exists. This proves part (i) of lemma 1.

To prove part (ii) note that under the hypothesis we may choose A so large that, for $x>A$, $I(x)$ is either non-negative or non-positive. Then, for $x>A$, $C(x)$ and $K_A(x)$ both have the same sign *and* $K_A(x)$ is monotonic. The existence of $J_A(\infty)$ then implies that these two terms must *separately* have limits for $x\to\infty$, so that $K_A(\infty)$, and hence $K(\infty)$, exists.

To prove lemma 2 note first that, for $z\in\pi(x_0)$, the function $p_z(x')=x'/(x'-z)$ is such that both $\mathrm{Re}p_z(x')$ and $\mathrm{Im}p_z(x')$ are bounded monotonic functions of x' for x' sufficiently large. From the existence of $K(\infty)$ it then follows, by a slight extension of a theorem of Abelian type,[33] that

$$\frac{1}{\pi}\int_{x_0}^\infty \frac{C(x')}{x'}p_z(x')dx' \quad (4.29)$$

also exists. But the integral (4.29) coincides with $g_0(z)$, so that $g_0(z)$ exists. The derivative of $g_0(z)$ is then

$$g_0'(z)=\lim_{\Delta z\to 0}\frac{1}{\pi}\int_{x_0}^\infty \frac{I(x')}{(x'-z)(x'-z-\Delta z)}dx'. \quad (4.30)$$

Since $C(\infty)$ exists, by part (i) of lemma 1, $I(x')/x'$ is a bounded function of x', and it follows, on writing

$$(x'-z)^{-1}(x'-z-\Delta z)^{-1}=(x'-z)^{-2} +\Delta z(x'-z)^{-2}(x'-z-\Delta z)^{-1}, \quad (4.31)$$

that the second term in (4.31) makes no contribution to (4.30). Thus

$$g_0'(z)=\frac{1}{\pi}\int_{x_0}^\infty \frac{I(x')}{(x'-z)^2}dx', \quad (4.32)$$

i.e., we have justified differentiation under the integral sign in (4.19), if the right-hand side of (4.32) exists. This is the case, because the integrand of (4.32) is the same as that of (4.29) with p_z replaced with p_z^2, so that the Abelian-type theorem used above applies. Moreover, on integration by parts, we get

$$g_0'(z)=-\pi^{-1}C(\infty)+f(z).$$

[33] See Appendix C, theorem C2.

This completes the proof, since for the case at hand $C(\infty)$ is zero, by lemma 1.

To prove lemma 3, note that the hypothesis implies that, for some $\epsilon>0$, there exists an $A>x_0$ such that either $x>A\Rightarrow I(x)>\epsilon$ or $x>A\Rightarrow I(x)<-\epsilon$. The conditions for lemma 2 hold, so that $g_0(z)$ exists. On writing, for $x<0$, $g_0(x)$ in the form

$$-g_0(x)=\frac{|x|}{\pi}\int_{x_0}^A \frac{I(x')}{x'(x'+|x|)}dx' +\frac{|x|}{\pi}\int_A^\infty \frac{I(x')}{x'(x'+|x|)}dx', \quad (4.33)$$

we see that the first term in (4.33) is a bounded function of x, while the second term is larger in absolute value than

$$\frac{\epsilon}{\pi}\ln\frac{A+|x|}{A},$$

which diverges as $x\to-\infty$. Thus, as $x\to-\infty$, $g_0(x)\to\pm\infty$ if $I(x)$ is asymptotically bounded away from zero.

B. Extension to Include Subtractions

The purpose of this subsection is to show that, if the spectral representation (4.6) for $f(z)$ must be modified to include subtractions and/or an additive polynomial, the essential conclusions of Sec. IV A 3 remain unchanged. First, it is convenient to record some elementary facts concerning the representation of $f(z)$ in terms of its spectral function $\sigma(x)$. Let n denote a non-negative integer, and let

$$J_{x_0}{}^n(x)\equiv\int_{x_0}^x \xi^{-n}\sigma(\xi)d\xi. \quad (4.34)$$

Then we have

Theorem 4. A necessary and sufficient condition that the integral

$$h_n(z)=\pi^{-1}z^n\int_{x_0}^\infty \frac{\xi^{-n}\sigma(\xi)}{\xi-z}d\xi \quad (4.35)$$

defines an analytic function for $z\in\pi(x_0)$ is that $J_{x_0}{}^{n+1}(\infty)$ exists. [The boundedness of $J_{x_0}{}^n(x)$ is also sufficient.]

Proof. The necessity follows from the relation $[d^nh_n(z)/dz^n]_{z=0}=\pi^{-1}n!J_{x_0}{}^{n+1}(\infty)$ and the sufficiency from theorem C2, with $p(\xi)=\xi(\xi-z)^{-1}$ [and from theorem C1, with $q(\xi)=(\xi-z)^{-1}$].

Suppose that $h_n(z)$ exists for some $n\geq 1$. The "subtractions" in Eq. (4.35) for $h_n(z)$ are said to be *needed* if $h_{n-1}(z)$ does not exist. An immediate consequence of theorem C2 (and C1) is then

Theorem 5. In order that the n (≥ 1) subtractions in (4.35) be needed, it is necessary that $J_{x_0}{}^n(\infty)$ does not exist. [It is also necessary that $J_{x_0}{}^{n-1}(x)$ is unbounded.]

If $h_n(z)$ exists for some $n \geq 0$, we may write

$$f(z) = E(z) + h_n(z), \tag{4.36}$$

where $E(z)$ is entire, and *fix* the representation (4.36) by requiring that all the subtractions in $h_n(z)$ are needed. We shall only consider the case where $E(z)$ is a polynomial[34] $P_m(z)$ of degree m:

$$f(z) = P_m(z) + h_n(z). \tag{4.37}$$

We are now ready to state generalizitions of theorem 3, for the case $n \neq 0$ and for the case $n=0$, $P_m \neq 0$. For the first, more interesting case, we have

Theorem 6. If $f(z)$ admits a spectral representation with n (≥ 1) needed subtractions, including an additive polynomial (which may be zero), and $L(-\infty)$ is finite, then $\sigma(x)$ must oscillate at infinity.

A similar theorem for $n=0$ is

Theorem 7. If $f(z)$ admits a spectral representation with no subtractions and a nonzero additive polynomial and $\sigma(x)$ does not oscillate at infinity, then $L(-\infty)$ is infinite.

The shift in emphasis in the wording of theorem 7 as compared to theorem 6 is related to the fact that, rather unaesthetically, we have not excluded the possibility that, if $n=0$ and $P_m \neq 0$, $L(-\infty)$ may be infinite regardless of the properties of $\sigma(x)$. The same may be true if $n \neq 0$ and $P_m \neq 0$; however, if $n \neq 0$ and $P_m = 0$, $L(-\infty)$ can definitely be finite if $\sigma(x)$ oscillates.[35] We note here also that Martin has shown that, if $n=0$ in (4.36) and $|f(z)|$ is bounded by $A \exp|z|^{1/2-\epsilon}$, and if $I(\infty)$ exists, then $L(-\infty)$ cannot be finite if $I(\infty) \neq 0$.[36] *If $n \geq 1$, $I(\infty)$ cannot exist in any case*, since, by theorem 5, $I(x) = J_{x_0}{}^0(x)$ cannot even be bounded. After these remarks, we consider the proofs of theorems 6 and 7, for the remainder of this subsection.

Proof of theorem 6. From theorem 4 we infer that

$$J_A{}^{n+1}(x) = \int_A^x \xi^{-n-1}\sigma(\xi)d\xi \tag{4.38a}$$

has a limit as $x \to \infty$; we define, analogously to (4.17) and (4.18) or (4.28),

$$C^n(x) = x^{-n-1}I(x) \tag{4.38b}$$

and

$$K_A{}^n(x) = \int_A^x \xi^{-1}C^n(\xi)d\xi. \tag{4.38c}$$

Now, if $\sigma(x)$ does not oscillate, $I(x)$ is monotonic for large x and since, as just mentioned, $I(\infty)$ cannot exist, we must have

$$I(\infty) = \pm\infty. \tag{4.39}$$

Then $I(x)$ is, *a fortiori*, asymptotically non-negative or non-positive and it follows as in the proof of lemma 1, on integration by parts of (4.38a), that both $K_A{}^n(\infty)$ and $C^n(\infty)$ exist, with $C^n(\infty)=0$. By the same arguments used in the proof of lemma 2, the function

$$g_n(z) \equiv \pi^{-1}\int_{x_0}^\infty Q_n(\xi,z)I(\xi)d\xi \tag{4.40}$$

exists and is analytic for $z \in \pi(x_0)$; here

$$Q_n(\xi,z) = z^{n+1}\xi^{-n-1}(\xi-z)^{-1}. \tag{4.41}$$

Differentiating (4.40) with respect to z, using the relation

$$\frac{\partial}{\partial z}Q_n(\xi,z) = -\frac{\partial}{\partial \xi}Q_{n-1}(\xi,z),$$

and integrating by parts, we obtain, using $I(x_0)=0$ and $C^n(\infty)=0$ to drop the surface term,

$$\frac{\partial}{\partial z}g_n(z) = h_n(z). \tag{4.42}$$

It follows from (4.20), (4.37), and (4.42) that

$$g(z) = \bar{P}_{m+1}(z) + g_n(z), \tag{4.43}$$

where $\bar{P}_{m+1}(z)$ is a polynomial of degree $m+1$.

We may write $g_n(z)$ [Eq. (4.40)] in the form

$$g_n(z) = g_n{}^{(1)}(z) + g_n{}^{(2)}(z), \tag{4.44}$$

where

$$g_n{}^{(1)}(z) = \pi^{-1}\int_{x_0}^A Q_n(\xi,z)I(\xi)d\xi \tag{4.45}$$

and

$$g_n{}^{(2)}(z) = \pi^{-1}\int_A^\infty Q_n(\xi,z)I(\xi)d\xi. \tag{4.46}$$

Since $Q_n(\xi,z)$ may be rewritten as

$$Q_n(\xi,z) = -\xi^{-n-1}\sum_{j=0}^{n-1} z^{n-j}\xi^j + z\xi^{-1}(\xi-z)^{-1},$$

we see that, correspondingly,

$$g_n{}^{(1)}(z) = R_n(z) + b(z), \tag{4.47}$$

where $R_n(z)$ is a polynomial of degree n and $b(x)$ is bounded as $x \to -\infty$. From (4.43), (4.44), and (4.47) we have

$$g(z) = \bar{R}_p(z) + b(z) + g_n{}^{(2)}(z), \tag{4.48}$$

where $\bar{R}_p(z)$ is a polynomial of degree $p = \max(m+1, n)$.

We now prove that (4.39) implies that $g(x)$ [and hence, via (4.23), $L(x)$] diverges as $x \to -\infty$; it suffices to show that there can be no cancellation between the divergent behavior of $\bar{R}_p(x)$ and $g_n{}^{(2)}(x)$ in (4.48) as

[34] We do not consider the more general possibility of an entire function as an additive term, since T-products are unique up to a polynomial; i.e., if the T-product is not immediately well defined as an element of $\mathcal{S}'$, it requires a subtracted spectral representation and is ambiguous to within a polynomial. See Ref. 22; W. Güttinger, Fortschr. Physik **14**, 483 (1966), and references therein.

[35] A simple example is $f(z) = ze^{i\sqrt{z}}$, for which $n=1$, $P_m=0$, and $\sigma(x) = x\sin(\sqrt{x})$.

[36] A. Martin (private communication). The proof is based on use of a Phragmen-Lindelöf theorem.

$x\to-\infty$. We note that we may choose A so large that $I(\xi)\geq I(A)>0$ for $\xi\geq A$, if, say, (4.39) holds with the plus sign. Then, for $x<0$, (4.46) may be written in the alternative forms

$$g_n^{(2)}(x)=(-1)^{n+1}\alpha(\eta)\eta^n \tag{4.49}$$

or

$$g_n^{(2)}(x)=(-1)^{n+1}\beta(\eta)\eta^{n+1}, \tag{4.50}$$

where $\eta\equiv|x|$, and

$$\alpha(\eta)=\pi^{-1}\int_A^\infty \frac{I(\xi)}{\xi^{n+1}}\frac{\eta}{\xi+\eta}d\xi, \tag{4.51}$$

$$\beta(\eta)=\pi^{-1}\int_A^\infty \frac{I(\xi)}{\xi^{n+2}}\frac{\xi}{\xi+\eta}d\xi. \tag{4.52}$$

Now the integrands in (4.51), with η regarded as an index, constitute a family of positive functions which increase monotonically to $I(\xi)/\xi^{n+1}$ as $\eta\to\infty$, so that by a standard theorem one may pass to the limit $\eta=\infty$ under the integral sign. Since, as we shall show below,

$$K_A^{n-1}(\infty)=\int_A^\infty \frac{I(\xi)}{\xi^{n+1}}d\xi=+\infty,$$

it follows that

$$\alpha(\eta)\to\infty,\quad \eta\to\infty. \tag{4.53}$$

Similarly, since the integrands in (4.52) decrease monotonically to zero,

$$\beta(\eta)\to 0,\quad \eta\to\infty. \tag{4.54}$$

From (4.48), (4.49), and (4.53) we see that, if $p\leq n$, $g(x)$ diverges like $\alpha(\eta)\eta^n$ as $x\to-\infty$, whereas, if $p\geq n+1$, using (4.48), (4.50), and (4.54), we see that $g(x)$ diverges like η^p as $x\to-\infty$. An entirely similar argument holds if $I(\infty)=-\infty$. It follows, as in the proof of theorem 3, that $\sigma(x)$ must change sign infinitely often, if $L(-\infty)$ is to be finite.

To prove the assertion that $K_A^{n-1}(\infty)$ is infinite when σ does not oscillate, we replace n by $n-1$ in Eqs. (4.38) and integrate (4.38a) by parts to get

$$J_A^n(x)=C^{n-1}(x)-C^{n-1}(A)+nK_A^{n-1}(x). \tag{4.55}$$

Now assume that (4.39) holds, say, with the plus sign. Then, for sufficiently large A, both $K_A^{n-1}(x)$ and $J_A^n(x)$ are monotonic increasing. Since, by theorem 5, $J_A^n(\infty)$ cannot exist (otherwise n subtractions would not be necessary), then $J_A(\infty)=+\infty$, so that, if $K_A^{n-1}(\infty)$ exists, we must have $C^{n-1}(\infty)=+\infty$, from Eq. (4.55). But then, by the mean-value theorem,

$$K_A^{n-1}(x)=C^{n-1}(\zeta)\ln(x/A)$$

for some $\zeta\in(A,x)$, so that $K_A^{n-1}(\infty)=+\infty$ and the assumption that $K_A^{n-1}(\infty)$ exists is contradictory. Thus $K_A^{n-1}(\infty)$ does not exist and, since $K_A^{n-1}(x)$ is monotonic, $K_A^{n-1}(\infty)=+\infty$; this completes the proof.

Proof of theorem 7. If $\sigma(x)$ does not oscillate, either (i) $I(\infty)=\pm\infty$, so that, *a fortiori*, $I(x)$ is asymptotically non-negative or non-positive, or (ii) $I(\infty)$ exists, so that, *a fortiori*, $I(x)$ is bounded. In case (i), lemma 2 assures the existence of $g_0(z)$ [Eq. (4.19)]. In case (ii), the boundedness of $I(x)$ implies that $g_0(z)$ certainly exists. As in the proof of theorem 6, for $x<0$, up to an additive function of x, $L(x)$ is a linear combination of a polynomial $\bar{P}_{m+1}(x)$ and, in this case, a term $\beta_0(\eta)\eta$, where $\eta=|x|$ and

$$\beta_0(\eta)=\int_A^\infty \frac{I(\xi)}{\xi(\xi+\eta)}d\xi.$$

Now, if (i) holds, $\beta_0(\eta)\to 0$ as $\eta\to\infty$, as in theorem 6. If (ii) holds, with $|I(x)|<M$, then $|\beta_0(\eta)|<M\eta^{-1}\times\ln[(A+\eta)/A]$, so that again $\beta_0(\eta)\to 0$ as $\eta\to\infty$. Hence the $\beta_0(\eta)\eta$ term cannot cancel the (at least linear) divergence coming from $\bar{P}_{m+1}(x)$ as $x\to\infty$, and $L(-\infty)$ is infinite, as asserted.

V. CONNECTION WITH EQUAL-TIME LIMITS

In this section we shall explore the relation between I being undefined and the existence and uniqueness of the ETC (2.13). We shall show that an ambiguous I corresponds exactly to an ambiguous ETC in the sense that the ET limit depends on how this limit is taken. We specifically show for the examples discussed in Sec. IV that one can have $\delta m^2<\infty$ with a nonvanishing ETC defined by a suitable ET limit. The original ambiguity in the ET limit corresponds to the ambiguities in products of field operators at the same point. Since the models discussed in Sec. II all have the ETC equal to such ambiguous field products, we shall conclude that they do not in fact imply a divergent δm^2. This is just the result that we have been after.

Let us define the commutator function (for general x_0, $\mathbf{x}$)

$$E(x)\equiv\langle\pi^+|[\dot{J}_\mu(x),J_\mu(0)]|\pi^+\rangle-\langle\pi^0|\cdots|\pi^0\rangle. \tag{5.1}$$

It follows from Eqs. (2.1) and (2.23) that[37]

$$E(x)=\int_0^\infty da\,\sigma(a)\dot{\Delta}(x;a), \tag{5.2}$$

where $\Delta(x;a)$ is the usual mass $\sqrt{a}$ free-field commutator function. The ETC $E(\mathbf{x})$ corresponding to (5.1) is obtained by smearing $E(x)$ with a smooth function $f_n(t)$ and taking the limit $n\to\infty$ in which $f_n(t)\to\delta(t)$. This procedure has been discussed in detail for arbitrary spectral functions $\sigma(a)$.[38] The result is that

$$E(\mathbf{x})=\sum_{r=0}^\infty K_r(\nabla^2)^r\delta(\mathbf{x}), \tag{5.3}$$

[37] As before, we imagine that (5.1) is first continued to $p=0$. The T-product (2.23) and the commutator (5.1) then have the same spectral function.

[38] R. A. Brandt, Phys. Rev. **166**, 1795 (1968).

where

$$K_r = \lim_{n\to\infty} K_{r,n},$$

with[39]

$$K_{r,n} = -\frac{(-1)^r}{r!}\int da\,\sigma(a)\left(\frac{\partial}{\partial a}\right)^r \hat{f}_n(a^{1/2}).$$

Here $\hat{f}_n(\kappa)$ is the Fourier transform of $f_n(t)$, so that

$$\hat{f}_n(\kappa) \underset{n\to\infty}{\longrightarrow} 1. \tag{5.4}$$

Since we assume that $E(x)$ is tempered, $\sigma(a)$ is polynominally bounded, so that only a finite number of terms will contribute in (5.3).

We need only consider the case in which just the term with $r=0$ is present in (5.3), so that

$$E(\mathbf{x}) = -K\delta(\mathbf{x}), \tag{5.5}$$

with

$$K = \lim_{n\to\infty} K_n, \quad K_n = \int da\,\sigma(a)\hat{f}_n(a^{1/2}). \tag{5.6}$$

If the limit in (5.6) exists for each sequence and is independent of the sequence used, the ET limit of (5.1) is unique and has the value (5.5). We shall then say that the ETC is unique. If, on the other hand, (5.6) is sequence-dependent, then (5.1) has no unique ET limit and the ETC is ambiguous. In this case some further restriction must be imposed on the sequences $\{f_n\}$ in order that the ETC be meaningful.

Now, we see from (5.6) that K_n is simply a regularization of the moment $I=\int da\,\sigma(a)$. This is the root of the connection between an ambiguous moment and an ambiguous ETC. Indeed, if I is well defined with $\sigma(a)$ well behaved, then $K=I$ for any sequence $\{\hat{f}_n\}$, and, conversely, if the limit (5.6) exists for each sequence and is sequence-independent, then $I=K$ is well defined. Thus we see that I is well defined if and only if the ETC is unique.

Our next task is to show that, for the examples considered in Sec. IV A which gave $\delta m^2<\infty$ and I not existing, the I's are ill defined in the above sense. Consider, for example, the function

$$F(z) = iz^{-1/2}e^{iz^{1/2}} = \pi^{-1}\int_0^\infty da\,\frac{a^{-1/2}\cos a^{1/2}}{a-z}. \tag{5.7}$$

As observed in Sec. IV, the corresponding moment

$$I = \int_0^\infty da\,a^{-1/2}\cos a^{1/2} = 2\int_0^\infty d\kappa\,\cos\kappa \tag{5.8}$$

does not exist. The regularized moment is

$$K_n = \int_0^\infty da\,a^{-1/2}\cos a^{1/2}\,\hat{f}_n(a^{1/2}) = 2\int_0^\infty d\kappa\,\cos\kappa\,\hat{f}_n(\kappa). \tag{5.9}$$

[39] Actually, it is the even part $\frac{1}{2}[\hat{f}_n(\kappa)+\hat{f}_n(-\kappa)]$ of $\hat{f}_n$ which occurs in the expression for $K_{r,n}$.

It is clear that for any real number c there exists a sequence $\{\hat{f}_n(\kappa;c)\}$ of functions in $\mathcal{S}$, converging to 1 in $\mathcal{S}'$, for which the limit of (5.9) is c. Simply choose $\hat{f}_n(\kappa;c)$ to be C^∞ and to satisfy

$$\begin{aligned} \hat{f}_n(\kappa;c) &= 1, && 0\le\kappa<2\pi n \\ &= h(\kappa;c), && 2\pi n\le\kappa\le 2\pi(n+\tfrac{1}{4}) \\ &= 0, && 2\pi(n+\tfrac{1}{4})<\kappa\le\infty \end{aligned} \tag{5.10}$$

with $h(\kappa;c)$ such that

$$2\int_0^{\pi/4} d\kappa\,h(\kappa;c)\cos\kappa = c.$$

It might be thought that the above functions (5.10) are rather special, in that they work because of the unrealistic uniform periodicity of $\cos\kappa$. More generally, it might appear more natural to regularize with testing functions $g_\alpha(\kappa)$ such that $g_\alpha(\kappa)\to 1$ in $\mathcal{S}'$ for $\alpha\to\infty$ continuously. Let us therefore exhibit such a sequence $\{g_\alpha(\kappa;c)\}$ for which

$$2\lim_{\alpha\to\infty}\int d\kappa\,\cos\kappa\,g_\alpha(\kappa;c) = c.$$

A suitable one is

$$g_\alpha(\kappa;c) = e^{-\kappa/\alpha} + \frac{c}{2\sqrt{\pi}}\frac{\cos\kappa}{\cos^2\kappa+\alpha^{-1}}e^{-(\kappa-\alpha)^2}. \tag{5.11}$$

That (5.11) satisfies the stated properties is proved in Appendix D. Similar remarks can be made concerning the other examples of Sec. IV.

We conclude from the above analysis that the "scattering amplitude" (5.7) corresponds to an ETC (2.13) which is completely ambiguous. We can define (2.13) by means of any sequence (5.11) that we like without affecting any observable quantities. The mass shift, which only depends on the well-defined function (5.7), will be finite, while $E(\mathbf{x})=-c\delta(\mathbf{x})$, with c arbitrary. In this case one will not, of course, have $F(z)\sim c/z$ if $c\neq 0$, and there is no connection between (2.13) and the high-z behavior of F.

It is perhaps worthwhile to reemphasize the insensitivity of observable quantities to a sequence such as (5.11) used to define the ETC (2.13). In practice, ETC's are always used in connection with relations of the form

$$k_0\int dx\,e^{-ik\cdot x}TA(x)B(0) = i\int dx\,e^{-ik\cdot x} \times\{T\dot{A}(x)B(0)+\delta(x_0)[A(\mathbf{x},0),B(0)]\}. \tag{5.12}$$

On the mass shell, the left-hand side of (5.12) is well defined and independent of which regularized θ functions $\theta_n(t)$ are used to define the T product.[40] This need

[40] K. Hepp, Commun. Math. Phys. 1, 95 (1965).

not, however, be the case for the individual terms on the right-hand side. The ETC term can be defined by *any* sequence $f_n(t)\to\delta(t)$, provided that the T-product term is defined by the corresponding sequence $\theta_n(t)=\int_{-\infty}^{t}d\tau \times f_n(\tau)$. It might be that some sequences are more convenient to use than are others, say, for purposes of saturation. But these considerations are independent of the high-z behavior of F, for which only the *existence* of sequence dependence is relevant.

We would now like to argue that the models of Sec. II do, in fact, formally give ambiguous expressions for the ETC (2.13), thus invalidating the implication that they give $\delta m^2=\infty$, according to our analysis above. The point is that the ETC's are given in terms of local products of field operators, such as $\phi(x)\phi(x)$. Such expressions are *always* ambiguous, as follows from the Källen-Lehmann representation, or ET commutation relations, or perturbation theory. For our present purposes, this ambiguity can best be thought of as a dependence of the quantity $\lim_{\xi\to 0}\phi(x)\phi(x+\xi)$ on the way in which the $\xi\to 0$ limit is taken. More precisely, if $r_n(\xi)\to_{n\to\infty}\delta^{(4)}(\xi)$ in $\mathcal{S}'$, then

$$\langle\alpha|\phi(x)\phi(x)|\beta\rangle=\lim_{n\to\infty}\int d\xi\,\langle\alpha|\phi(x)\phi(x+\xi)|\beta\rangle r_n(\xi) \tag{5.13}$$

will depend on the sequence used. Even for free scalar fields, a sequence $\{r_n\}$ can be chosen to give *any* value to *any* (diagonal) matrix element (5.13).

The intimate connection between the above $\xi\to 0$ limit and the previous ET $t\to 0$ limit has been extensively discussed in Ref. 38, where it was shown that ETC's $[J(x),J'(x')]_{x_0=x_0'}$ can be calculated as limits of $[J(x;\xi),J'(x';\xi')]_{x_0=x_0'}$ for ξ, $\xi'\to 0$, where $J(x;\xi)$ is a nonlocal expression which converges to $J(x)$ for $\xi\to 0$. Such calculations give expressions of the form $\lim_{\xi\to 0}\times Z(\xi)\phi(x)\phi(x+\xi)$ for the ETC. For the $[J,J]$ commutators considered in Ref. 38, these expressions were rather well behaved, but this need not be the case in general, especially for $[\dot{J},J]$ commutators.

In summary, then, we have seen that the models of Sec. II give ambiguous field products for the ETC (2.13), that this corresponds to ambiguities in the corresponding ET limit, and that this in turn corresponds to a nonexisting moment I. But this latter circumstance is precisely the *necessary* condition for the mass shift to be *finite* when the ETC does not vanish. We therefore conclude that these models do *not* imply that $\delta m^2=\infty$, even if the ETC's (2.13) that they involve are taken to be nonvanishing. We do not claim to have shown that δm^2 is finite in these models, but only that its divergence cannot be concluded from Eq. (2.6) and the nonvanishing of (2.13).

Finally, we note, on the basis of the example (4.25), that the vanishing of the ETC $[\dot{J},J]$ in a given model seems, by itself, *not* to be sufficient to guarantee a finite δm^2.

VI. EVIDENCE FOR OSCILLATIONS

Having indicated why an oscillating spectral function is necessary in order that $\delta m^2<\infty$ when $I\neq 0$, we proceed to investigate the empirical evidence for this behavior.

Let M denote the forward Compton amplitude for $\gamma+h\to\gamma+h$, where h is a hadron, and let σ denote the associated spectral function, with M regarded as an analytic function of q^2, $q\cdot p$ being fixed. The possibility that σ oscillates receives direct support from experimental and theoretical work on high-energy electron-proton scattering, which yields information on the electromagnetic form factor of the proton. This is considered in Sec. VI A. In Sec. VI B we find indirect support for this possibility from properties of high-energy hadron-hadron scattering and a possible connection between the Bjorken and high-energy limits suggested by the JLD representation.

A. Direct Evidence

Experiments on electron-proton scattering at high energies indicate that the electromagnetic form factor of the nucleon $G(q^2)$ decreases very rapidly as the momentum transfer q^2 tends to $-\infty$. A good fit to the data can in fact be obtained with an *exponentially falling* form factor[41,42]

$$G(q^2)\propto\exp[-a(q^2)^{1/2}],\quad q^2\to+\infty\,. \tag{6.1}$$

Likewise, the form (6.1) fits the N^*NY transition form-factor data.[42] There is by now a variety of theoretical work related to the understanding of this behavior:

(a) The form (6.1) arises naturally in a model which correctly describes the rapid decrease of the high-energy, large-angle differential cross section for p-p scattering.[43]

(b) If $G(q^2)$ is analytic in the cut q^2 plane and bounded by $\exp(b|q^2|^{1/2-\epsilon})$ (with b and ϵ positive), the form (6.1) is the *maximal* rate of decrease possible. It is then predicted by a principle of "minimal interaction."[44]

(c) The maximal rate of decrease permitted for $G(q^2)$ in strictly local field theory, a generalization of field theory to include fields which may not be tempered, is given by (6.1).[45]

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[41] For a review, see S. D. Drell, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 85.

[42] For a recent experimental result, see D. H. Coward *et al.*, Phys. Rev. Letters **20**, 1292 (1968); these authors find a best fit with $G\sim\exp[-a(q^2)^{1/4}]$, which also requires an oscillating ImG. We emphasize (6.1) for convenience only; see also J. Harte, Phys. Rev. **171**, 1825 (1968); **171**, 1832 (1968). The well-known dipole form $G\sim(q^2)^{-2}$ also fits the $NN\gamma$ data very well. The recent experimental results [D. Imrie, C. Mistretta, and R. Wilson, Phys. Rev. Letters **20**, 1074 (1968)] on neutral pion electroproduction are also consistent with an exponentially falling transition form factor $\Gamma_{N^*N\gamma}$. The dipole form does *not* fit these data. Thus the exponential form factors seem to be the only ones consistent with the usual notions of universality.

[43] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

[44] A. Martin, Nuovo Cimento **37**, 671 (1965).

[45] A. M. Jaffe, Phys. Rev. Letters **17**, 661 (1966).

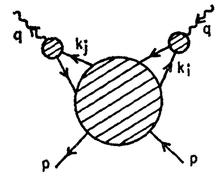

FIG. 4. General (non-seagull) diagram describing photohadron scattering in order e^2. The external photons are attached to arbitrary hadron lines h_i and h_j.

(d) The form (6.1) is consistent with the expected asymptotic behavior of the solutions of a nonlinear Bethe-Salpeter equation, incorporating the bootstrap idea, whereas a wide variety of other possible forms is *not*.[46] The form (6.1) has also been obtained as a consequence of a (highly simplified) model of an "infinitely composite" nucleon.[47]

The above survey suggests that (6.1) may indeed correctly describe $G(q^2)$ as $q^2 \to +\infty$. However, if (6.1) holds and if $G(q^2)$ is polynomially bounded in the complex q^2 plane, then, as pointed out by Wu and Yang[43] and emphasized by Martin,[44] the spectral function $\rho(q^2) = [G(q^2+i0) - G(q^2-i0)]/2i$ *necessarily* changes sign infinitely often.

Thus, it seems fair to say that an oscillating spectral function is a strong possibility for at least one function of physical interest: the electromagnetic form factor of the nucleon. Furthermore, this possibility is of direct import for our problem. First, one would then expect similar behavior for the spectral functions associated with the electromagnetic form factors of other hadrons, such as the pion. Second, and more to the point, if off-mass-shell effects are not dominant, one then expects that the Compton amplitude for, say, $\gamma+\pi \to \gamma+\pi$, has a high-$q^2$ behavior which includes a term precisely of the type envisaged by the example (4.11) and, correspondingly, gives rise to an oscillating spectral function.

Consider the contribution $M_{\mu\nu}^{(ij)}$ to $M_{\mu\nu}(q;p)$ arising from all Feynman diagrams in which the photon is absorbed by a virtual hadron h_i with momentum k_i and electromagnetic vertex function $\Gamma_\mu^{(i)}(-q, k_i)$ and emitted by a hadron h_j with momentum k_j' and a vertex function $\Gamma_\nu^{(j)}(q,k_j')$. (See Fig. 4.) If, for large q, it is permissible to replace each $\Gamma^{(i)}$ with its value on the hadron mass shell, then, using (6.1),

$$M_{\mu\nu}^{(i,j)}(q,p) \sim e^{-(a_i+a_j)(q^2)^{1/2}}, \qquad (6.2)$$

up to a multiplicative function $N_{\mu\nu}^{(i,j)}(q,p)$ whose dependence on q is unknown, but which is unlikely to cancel the exponential exhibited on the right-hand side of (6.2).

For $i=j$, there may also be "seagull" contributions, e.g., if h_i has spin zero. These terms are directly proportional to $g_{\mu\nu}$ and so not immediately expressible as integrals involving electromagnetic form factors. However, they are algebraically related to $M_{\mu\nu}^{(i,j)}$ by gauge invariance. Alternatively, we note that if in the computation of the mass shift we use the appropriate gauge [$\kappa=4$ in Eq. (2.10)], these terms do not contribute at all. Thus, we have, effectively, "transition" vertex functions being implicitly included:

$$M_{\mu\nu}(q,p) = \sum_{i,j} M_{\mu\nu}^{(i,j)}(q,p). \qquad (6.3)$$

From (6.2) and (6.3) we conclude, in light of the preceding discussion, that the indications that the spectral function σ oscillates are indeed quite strong.[48]

The reader may wonder whether constraints on the off-mass-shell vertex functions due to gauge invariance can imply divergences in the mass-shift expression independently of current algebra. To see that this is not the case, note that the Cottingham[16] formula expresses the mass shift exclusively in terms of on-shell form factors concerning which gauge invariance says nothing. All our arguments can, in fact, be presented in terms of the Cottingham expression. Furthermore, note that we do not require that M itself be exponentially decreasing,[49] but only that an exponentially decreasing term survive in (6.3) so that the spectral function continues to oscillate. For example, if $M = ae^{iz^{1/2}} + b(z^{1/2}+i)^{-4}$, then the spectral function σ_a corresponding to the first term oscillates, although σ_b, the one corresponding to the second term, does not. In this case the mass shift is finite and the moment $I = I_a + I_b$ is undefined, since I_a is undefined and I_b is zero.

B. Indirect Evidence

More indirect evidence for the possibility of an oscillating σ comes from an examination of this possibility for the spectral functions associated with the strong-interaction amplitudes describing hadron-hadron collisions. Again a number of items are suggestive here: (a) high-energy, large-angle p-p scattering, (b) Regge theory, (c) general features of dispersion theory, and (d) a possible connection between the high-q_0 behavior and the high-s (on-mass-shell) behavior of scattering amplitudes.

(a) Large-angle, high-energy p-p scattering can be fitted[50] with a form

$$d\sigma/d\Omega \sim e^{-bk_\perp}, \qquad (6.4)$$

where $b>0$ and $k_\perp = k \sin\theta$, with k and θ referring to the c.m. system.

As shown by Martin,[44] under a reasonable set of assumptions this rate of decrease is the maximal allowed. Analogous to the case of the electromagnetic

[46] J. Harte, Phys. Rev. **165**, 1557 (1968), and references cited therein.

[47] J. D. Stack, Phys. Rev. **164**, 1904 (1967).

[48] The experimentally observed (Ref. 42) rapidly falling behavior for $\Gamma_{N^*N\gamma}$ is just what one would expect from a similar behavior for $\Gamma_{NN\gamma}$ if our assumption that off-mass-shell effects do not wipe out the exponential decrease of hadron form factors is correct.

[49] We thank Dr. M. Halpern and Dr. J. Harte for discussions of this point.

[50] G. Cocconi *et al.*, Phys. Rev. **138**, B165 (1965); J. V. Allaby *et al.*, Phys. Letters **23**, 389 (1966); **24B**, 156 (1967).

form factor, (6.4) is then strongly suggestive of an oscillating spectral function $\rho_t(s,t)=(2i)^{-1}[A(s,t+i\epsilon)-A(s,t-i\epsilon)]$, where $A(s,t)$ is the scattering amplitude.

(b) The contribution of a Regge trajectory $\alpha(t)$ to $A(s,t)$ has the form $s^{\alpha(t)}$. If, as is usually assumed, $\alpha(t)$ is real analytic with a nearest branch point at $t=t_0$, $s^{\alpha(t)}$ is analytic with the discontinuity across the cut given by $[\exp \mathrm{Re}\alpha(t)] \sin[\mathrm{Im}\alpha(t) \ln s]$, which oscillates if $\mathrm{Im}\alpha(t)$ is not too slowly varying.

More interestingly, the repeated exchange of the "Pomeranchuckon" is expected to give an exponential function of $(t)^{1/2} \ln s$,[51] while still more complicated aspects of Regge dynamics are suggestive of an exponential function of $(t \ln s)^{1/2}$.[52] In either case we are dealing with an exponential function of the square root of an invariant variable, just as in the prototype example (4.11) where z corresponds to $-q^2$, and, correspondingly, with oscillating contributions to the spectral function.

(c) Independent of the above, from the general viewpoint of dispersion theory the spectral functions associated with $A(s,t)$ might be *expected* to oscillate, as emphasized by Eden,[53] since the point at infinity is an accumulation point of the branch-point singularities associated with the opening of inelastic channels.

(d) The above discussion is concerned with the asymptotic behavior of the on-shell amplitude $A(s,t;q^2=-m^2, p^2=-M^2)\equiv A(s,t)$ with respect to the variables s and t. More relevant for our problem would be knowledge of the asymptotic behavior of a forward scattering amplitude $M(q;p)=A(s,0;q^2,p^2=-M^2)$, with $s=-(q+p)^2$, as the components of q become large. No theoretical work appears to have been done on this question. However, it is possible that $R(r)\sim B(r)$, where $A(s,t)\to R(s)$ as $s\to\infty$ for fixed t and $M(q;p)\to B(q_0^2)$ as $q_0\to\infty$. Such a relation would follow from the JLD[20,21] representation with a smooth spectral function.[54] Even if the relation is not valid, the above survey still indicates that oscillations in s actually occur. It is then not unlikely that they also occur in q_0^2.

VII. ADDITION OF SCHWINGER TERMS

A. Retention of Finite Mass Shift

We now turn to the interesting case in which the Schwinger term difference

$$\Delta S(q;p)=S^{(+)}(q;p)-S^{(0)}(q;p)\,, \qquad (7.1)$$

defined in Eq. (2.2), does not vanish. Then ΔT will not, in general, be covariant—to the extent of an additive polynomial[55] $P(q;p)$ in q_0. A finite δm^2 could still be obtained, provided that

$$\Delta T(q;p)=P(q;p)+F(q;p)\,, \qquad (7.2)$$

where F is a covariant function of the type discussed in Sec. VI and

$$\Delta S(q;p)=-P(q;p)\,. \qquad (7.3)$$

In this case, the unsubtracted relation (2.5) is unlikely to be correct[22] and the ETC's in (2.6) are even more likely to be ill defined. Nevertheless, the behavior (7.2), (7.3) for ΔT is not unreasonable; it requires ΔM, rather than ΔT, to vanish for $q_0\to\infty$.

The bad high-q_0 behavior for T envisaged above is expected to occur whenever the commutators in (2.6) contain divergent q-number quantities, such as products of local field operators. As indicated in Sec. III, this is the usual case in perturbation theory. In the model discussed in Sec. III, although T did not vanish for $q_0\to\infty$, the cancellation implied by (7.3) did not take place, so that one still had $\delta m^2=\infty$.

A simple case where the cancellation (7.3) does occur can be found in fourth-order quantum electrodynamics, where the time-ordered product $T_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4)$ associated with the photon-photon scattering amplitude $M_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4)$ satisfies[38]

$$T_{\alpha\beta\gamma\delta}(k_1,k_2,0,0)=C_{\alpha\beta\gamma\delta}=\text{const}\neq 0 \qquad (7.4)$$

and (by gauge invariance) one has

$$S_{\alpha\beta\gamma\delta}(k_1,k_2,0,0)=-C_{\alpha\beta\gamma\delta}\,. \qquad (7.5)$$

Thus, whereas the photon electromagnetic mass shift

$$\delta\mu^2=\text{const}\int d^4q(q^2)^{-1}M_{\alpha\alpha\gamma\delta}(q,-q,k,-k)\epsilon_\gamma(k)\epsilon_\delta(k)$$

certainly vanishes, the contribution to $\delta\mu^2$ from the time-ordered product (7.4) is divergent. In this case $T_{\alpha\beta\gamma\delta}$ is given by the sum of a subtracted dispersion integral (which vanishes when $k_3=k_4=0$) and the quantity $C_{\alpha\beta\gamma\delta}$.

The above example and the one in Sec. III involve ill-defined local field products and, as mentioned previously, so do all the models discussed in Sec. II. Thus, in view of the possibility (7.3), the existence of (canonical or noncanonical[56]) Schwinger terms in these models does not affect the arguments of Secs. V and VI for a finite pion electromagnetic mass shift.

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[51] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962); D. Amati, M. Cini, and A. Stanghellini, *ibid.* **30**, 193 (1963); S. Mandelstam, *ibid.* **30**, 1127 (1963); **30**, 1148 (1963); V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosian, Phys. Rev. **139**, B184 (1965); J. C. Polkinghorne, J. Math. Phys. **6**, 1960 (1965).

[52] A. A. Anselm and I. T. Dyatlov, Phys. Letters **24B**, 479 (1967); V. N. Gribov, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (Wiley-Interscience Publishers, Inc., New York, 1967).

[53] R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge University Press, Cambridge, England, 1967), pp. 182–183.

[54] R. Brandt and J. Sucher (unpublished).

[55] We are, as usual, assuming temperedness. See also Ref. 34. T-products cannot, in fact, be defined in general for nontempered theories.

[56] By "noncanonical" Schwinger terms we mean those which arise from the need to define current operators as suitable limits of field products (see Ref. 38), as opposed to those, such as (3.4), which follow simply from canonical commutation relations.

B. Retention of Previous Results

Having shown that a nonvanishing ΔS need not alter our conclusions, it seems interesting to ask at this point how a nonvanishing ΔS can exist without invalidating previous good results. Explicitly, let us assume the validity of the following: (a) usual current algebra; (b) Weinberg sum rules [(2.16) and (2.17)]; (c) soft-pion theory; and (d) $\Delta S_{\mu\nu}(q;p)\neq 0$. In this subsection we shall show, by explicit construction, that (a)–(d) are not inconsistent.

By (a) we mean the commutation relations proposed by Gell-Mann, up to possible Schwinger terms. The sum rule (2.16) is equivalent to the equality

$$\langle 0|[V_4{}^a(0,\mathbf{x}),V_k{}^b(0)]|0\rangle=\langle 0|[A_4{}^a(0,\mathbf{x}),A_k{}^b(0)]|0\rangle.$$

To see the implications of (c), let us consider again the successful calculation of δm^2 in a world with $m_\pi=0$ by Das *et al.*[12] As indicated in Eq. (2.14), from (a) and (c) it was found that

$$\Delta T_{\mu\nu}(q;0)=\text{const}[\Delta_{\mu\nu}{}^V(q)-\Delta_{\mu\nu}{}^A(q)]. \qquad (7.6)$$

Actually, Das *et al.* wrote (2.14) with $\kappa=4$, since in this gauge the contribution of (3.5) vanishes. The form (3.5), however, is highly model-dependent. More importantly, we note that, as a consequence of (2.16), the quantity (7.6) is itself Lorentz-covariant and gauge-invariant. It follows that $\Delta S_{\mu\nu}(q;0)$ is also Lorentz-covariant and gauge-invariant,[57] so that one can write

$$\Delta S_{\mu\nu}(q;0)=(q_\mu q_\nu-q^2\delta_{\mu\nu})W(q^2). \qquad (7.7)$$

Since ΔS is a polynomial in q_0, $W(q^2)$ must be a polynomial in q^2.

Now, if the individual terms in (2.7) are well defined without subtractions, then ΔS will be independent of q_0, so that (7.7) must vanish. We can arrive at essentially the same conclusion even if ΔS is a (nonconstant) polynomial in q_0. Thus, since (2.14) [with (2.17)] gives an accurate value for δm^2, we require that the contribution of $\Delta S_{\mu\nu}(q;0)$ to δm^2 vanish. In view of the form (7.7), however, this requires that $\Delta S_{\mu\nu}(q;0)$ itself vanish. [Conversely, (2.16) is a *consequence* of the vanishing of $\Delta S_{\mu\nu}(q;0)$.]

We thus see that (a)–(c) imply that the commutator (2.13) must vanish in the soft-pion limit $p=0$. But, as a general consequence of gauge invariance, one has[38]

$$\delta(t)[J_k(x),\dot{A}_{l,l}(0)]=\delta(t)[J_k(x),J_0(0)]+c\text{ number}. \qquad (7.8)$$

By expressing $[J_\mu(0,\mathbf{x}),\dot{A}_l(0,\mathbf{x}')]$ as a (finite) sum of δ functions and their derivatives, we conclude from (7.8) that (2.13) will vanish in the soft-pion limit if and only if the difference

$$\langle\pi^+|\delta(t)[V_k{}^3(x),V_4{}^3(0)]|\pi^+\rangle-\langle\pi^0|\cdots|\pi^0\rangle \qquad (7.9)$$

does. To ensure (d), however, we do not want (7.9) to vanish in general.

[57] This is because $M=T+S$ must be gauge-invariant at $p=0$, since $m_\pi=0$.

A simple example of a commutator satisfying the above requirements is

$$[V_4{}^3(0,\mathbf{x}),V_k{}^3(0)]=[A_4{}^3(0,\mathbf{x}),A_k{}^3(0)]$$
$$=\text{const}[V_\mu{}^3(0)V_\mu{}^3(0)-A_\mu{}^3(0)A_\mu{}^3(0)]\partial_k\delta(\mathbf{x})+c\text{ number}. \qquad (7.10)$$

Given (7.10), one can derive (2.16) and, with the aid of (a), (2.17), and assumption (c) one can obtain a good value for δm^2 in the soft-pion limit. Furthermore, one cannot argue that δm^2 is divergent for physical pions, because of the possibilities (7.2) and (7.3).

Although (7.10) is not being presented as a serious proposal (we know of no Lagrangian which gives this commutation relation), we note that its bilinear form is similar to that found in a number of field theories. We also note that the leading singularity of the right-hand side of (7.10) is absent, as a consequence of (2.16). A q-number Schwinger term with some such form seems necessary if one wants to maintain both $E(\mathbf{x})\neq 0$, (7.2), and $\delta m^2<\infty$. It would be interesting to see if a Lagrangian model with the requisite properties could be constructed.

VIII. SUMMARY AND DISCUSSION

A. Summary

The contents of the preceding sections can be divided into a number of somewhat distinct topics, including (a) asymptotic behavior of T-products and commutator ambiguities, (b) finiteness of mass shifts and oscillating spectral functions with nonexistent moments, (c) Schwinger terms and subtractions, and (d) purely mathematical results concerning spectral representations.

(a) In Secs. II A and II B the arguments for a divergent δm^2 in the usual models were reviewed. It was emphasized that even in the absence of Schwinger terms in (2.7) and subtractions in (2.5), these arguments rest on the assumption that the high-q_0 behavior of (2.5) may be inferred from the expansion (2.6). Now, if this expansion is to be meaningful, the expansion coefficients must be well defined, so that the q-number part of commutators of the form $C_n=[\partial_0{}^nJ_\mu(0,\mathbf{x}),J_\mu(0)]$ must be well defined, at least for low values of n $(n=0,1,\cdots)$. However, since in the formal computation of such commutators by use of canonical commutation relations one encounters products of field operators evaluated at the same point, ambiguities are to be expected in the definition of C_n, especially for $n>0$. In Sec. III these remarks were illustrated by study of the electromagnetic current generated by a charged pion field [Eq. (3.2)]. In the absence of strong interaction the behavior $T(q;p)\sim q_0{}^{-2}$ found by direct computation was seen to be in accord with what is expected from (2.6) and (3.6). However, in the presence of a $\lambda\phi^4$ interaction, explicit calculation to lowest order yielded a $T(q;p)$ which no longer vanished as $q_0\to\infty$, corresponding to the fact that the q-number part of the right-hand side of (3.6) is now ill defined, to the same order in perturbation theory. Thus we see that even under very simple circumstances, assumption (2.5) with the expansion (2.6) is not a reliable guide to the high-q_0 behavior of $T(q;p)$. Now, for the example at hand this behavior is in fact *worse* than that inferred from a formal use of (2.5), (2.6), and (3.6), so that no progress appeared to have been made towards getting a finite mass shift. However, it was noted that the high-q_0 behavior of T may also be *better* than that found from a formal use of (2.6) and (3.6). In the above example the breakdown of (2.6) is related to the fact that the integral

$$I_p=\tfrac{1}{2}\int dq_0'q_0'(\rho+\bar{\rho})=\pi\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}E_p(\mathbf{x}), \quad (8.1)$$

with $E_p(\mathbf{x})$ given by (2.13), is infinite. As shown by the examples in Sec. IV, if such integrals fail to converge in a more subtle way, namely, if the limit does not exist even when infinity is admitted as a value, then the asymptotic behavior of $T(q;p)$ can indeed be better than predicted by (2.6). In Sec. V, it was shown that the nonexistence of the integral (8.1) in this sense (considered at $p=0$ for simplicity), or, equivalently, the integral (2.25), corresponds precisely to the ambiguities expected in the definition of an ETC in field theory. That is, a particular regularization of the integral (2.25), followed by a passage to the limit giving a definite value to (2.25), corresponds to a choice of a sequence of "smearing" functions $f_n(t)$ approaching $\delta(t)$, giving a definite value to the limit of (5.2) as $t\to 0$. Such a choice can obviously not determine the high-q_0 behavior of T.

(b) The connection of these ideas with the question of the convergence of δm^2 [Eq. (2.9)] was illustrated by study of $(\delta m^2)_0$ [$=(\delta m^2)_{\text{div}}$], the mass shift at $p=0$,[11] in Secs. II C and II D. This allows the use of the explicitly covariant formulas (2.22) and (2.23) for $(\delta m^2)_0$ and $T_{\mu\mu}(q;0)$ and permits a direct study of the relation between the moment I of the spectral function σ [Eq. (2.25)] and the convergence of (2.22). In Sec. IV, it was shown that if $(\delta m^2)_0$ is finite, it is necessary either that $I=0$ or that I not exist in the sense indicated previously and that this be so whether or not subtractions are needed in (2.23); moreover, if I does not exist, then it is necessary that σ oscillate. The evidence for oscillations was discussed in Sec. VI.

(c) In Sec. VII, the generalization of the preceding results to include Schwinger terms $S_{\mu\nu}$ in $M_{\mu\nu}$ and subtractions in $T_{\mu\nu}$ was discussed. It was pointed out that a $T_{\mu\nu}$ which has bad behavior at high q_0 might be accompanied by an $S_{\mu\nu}$ such that $M_{\mu\nu}=T_{\mu\nu}+S_{\mu\nu}$ has good behavior and an explicit example of this was given from fourth-order quantum electrodynamics. It was also shown that a nonvanishing $\Delta S(q;p)$ could be consistent with previous good results, based on current algebra, the Weinberg sum rules, and soft-pion theory, provided that (7.9) vanishes in the soft-pion limit. An example of a commutator satisfying this requirement was given. It was noted, in passing, that the first Weinberg sum rule, (2.16), in a world with $m_\pi=0$, both implies and is implied by the vanishing of $\Delta S(q;0)$; in other words, (2.16) is equivalent to the gauge invariance of $\Delta T_{\mu\nu}(q;0)$.

(d) In Sec. IV, the connection between the existence of a finite "mass shift" $L(-\infty)$ [Eq. (4.5)] and the behavior of the spectral function $\sigma(x)$ [Eq. (4.1)] and the moment function $I(x)$ [Eq. (4.2)] was explored. The main results were theorems 3, 6, and 7: These may be summarized by the statement that, unless $I(\infty)=0$ [possible only if no subtractions are necessary in a spectral representation of $f(z)$], the condition that $L(-\infty)$ be finite requires that $\sigma(x)$ oscillate at $x=+\infty$. For the case where $f(z)$ admits a simple USR [Eq. (4.60)], rigorous criteria were obtained for having $I(\infty)=0$ (theorem 2) and for $f(x)\sim x^{-1}$ as $x\to-\infty$ (Appendix B), and a simple practical test for determining whether or not $L(-\infty)$ is finite was found [Eq. (4.16)]. A variety of illustrative examples was given [Eqs. (4.10), (4.12), and (4.13)].

In the course of deriving these results a number of auxiliary ones were obtained, which may be useful in other contexts: Thus, a sufficient condition for $f(z)$ to satisfy a simple USR was found, which is weaker than that usually stated (theorem 1). Also obtained were a necessary and sufficient condition that $f(z)$ admit a spectral representation with n subtractions (theorem 4) and a necessary condition that n subtractions be needed (theorem 5); these conditions are more general than those usually given, in that no specific assumption is made for the asymptotic behavior of $\sigma(x)$. Such generalizations are essential if one deals with oscillating $\sigma(x)$ and dispersion integrals which do not converge absolutely. A theorem of Abelian type, described in Appendix C, plays an important role in many of the proofs.

B. Discussion

A number of further investigations seem desirable in connection with and in continuation of this work:

(i) Some of the links in our chain of argument are rigorous only in the so-called $p=0$ limit; one should try to increase the rigor of the arguments for the $p\neq0$ case. Furthermore, the significance of the $p=0$ limit itself is not completely clear. As remarked in Sec. II, consideration of $(\delta m^2)_0$ [Eq. (2.22)] might suffice for a resolution of the divergence problem, since it may be argued[1] that this is the potentially most divergent part of δm^2. Now, for $m_\pi\neq0$, a variety of definitions of "mass shifts at $p=0$" is possible, apart from the one used in Ref. 1 and adopted here, since off the mass shell ($p^2\neq -m_\pi^2$) the expression (2.9) is gauge-dependent. As mentioned in Sec. II, a possible point of view towards this situation, which unifies the results obtained in this paper and the successful calculation of Das *et al.*, is the following. The general expressions for I, F, and δm^2 in the "real world" W are given, respectively, by (8.1), by

$$F(-q^2,\nu)=\int_0^\infty \frac{dq_0'}{2\pi}\left[\frac{\rho(q_0',\mathbf{q};p)}{q_0-q_0'}-\frac{\bar\rho(q_0',-\mathbf{q};p)}{q_0+q_0'}\right],$$

and by

$$\delta m^2=\text{const}\int d^4q\,\frac{1}{q^2}F(-q^2,\nu).$$

Let us refer to these expressions in a world $\tilde{W}$ with $m_\pi=0$ as $\tilde{I}$, $\tilde{F}$, and $\delta\tilde{m}^2$. We write the quantities evaluated at $p=0$ as $\tilde{I}_0$, $\tilde{F}_0$, and $(\delta\tilde{m}^2)_0$. Since, in $\tilde{W}$, $p=0$ is on the edge of the physical region, we have $(\delta\tilde{m}^2)_0=\delta\tilde{m}^2$, the complete gauge-invariant mass shift in $\tilde{W}$. Then the analysis of Sec. IV A shows that $\delta\tilde{m}^2$ is finite only if $\tilde{I}_0$ is either zero or undefined (calling an infinite value a defined value). Suppose that we now *assume* that the same is true in W, namely, that $\delta m^2<\infty$ is realized by having either $I=0$ or I undefined. Then consistency with the soft-pion calculation and with the usual models is achieved if $(\alpha)\delta\tilde{m}^2$ is finite in $\tilde{W}$ because $\tilde{I}_0=0$ and if $(\beta)\delta m^2$ is finite in W because I is undefined. Of course, these remarks are only speculations and further investigation is needed.

(ii) If the electromagnetic form factor $G(q^2)$ decreases more rapidly than an inverse power as $q^2\to\infty$, then the spectral function $\text{Im}G(q^2)$ necessarily oscillates if also $G(q^2)$ is polynomially bounded for complex q^2. It would be interesting to know to what extent the hypothesis of a polynomial bound can be weakened.

(iii) It would be quite significant if the suggested oscillatory behavior of the spectral function associated with $M_{\mu\nu}(q;p)$ could be found by summation of a suitable infinite set of Feynman diagrams, generated by some Lagrangian model, as is the case for Regge behavior. Since this seems to require consideration of nonplanar graphs,[54] it may be quite difficult, but some further effort in this direction is warranted.

(iv) Another approach to finding the behavior in question by summation of graphs can be based on considerations involving the behavior of the spectral function $\psi(u,a;p)$ in a JLD-like representation for $M_{\mu\mu}(q;p)$. This leads to the problem of an analysis of this type of representation and computation of the spectral function in perturbation theory, again a difficult problem, which is, however, of interest in its own right.

(v) In principle, the ideas invoked here to resolve the difficulty with radiative corrections to the mass operators may be able to resolve similar divergences encountered in the calculation of such corrections to vector weak-decay amplitudes. This appears to require that the ETC $[V_0{}^\alpha,V_i{}^\beta]$ be ambiguous.[58]

(vi) As mentioned in Sec. VII, it would be interesting to see if a Lagrangian model could be found in which a Schwinger term of the form (7.10) is present. Such a term may exist even if it is not formally predicted by the canonical commutation relations.[56]

(vii) We have stressed that the usual models of current algebra give ambiguous field products for the $[\dot{J},J]$ ETC. The nature of the relevant ambiguities has only been studied in perturbation theory and in a few simple soluble models. It is important for many purposes to obtain a better understanding of the behavior of such field products. This behavior should, in particular, be directly related to the high-energy properties of the corresponding theory.

(viii) Finally, we mention the importance of continued experimental investigation of hadronic electromagnetic form factors. The present data[42] are consistent with both a dipole fit and the exponentially decreasing fits that we have stressed. Although our analysis can accommodate either type of fit, our (inductive) arguments are much stronger if the exponential fit is correct. Thus an experimental clarification of this point would be very helpful. Likewise, it is important to obtain further confirmation of exponentially damped high-energy hadron-hadron scattering amplitudes.

Assuming that results of the investigations suggested above do not weaken our arguments, our suggestion that the moment I is ambiguous but not zero, in accordance with the nonvanishing of the ETC $[\dot{J},J]$ in the usual models, seems well founded. We conclude that a determination of whether a particular model of current algebra gives rise to a divergent mass shift cannot be made by simply seeing if $E_{\mu\nu}$, computed formally, does not vanish.

ACKNOWLEDGMENTS

We thank Professor C. H. Woo for many stimulating discussions. We also thank Professor A. Martin for a helpful conversation.

APPENDIX A: PROOF OF THEOREM 1

Consider $f(z)$ analytic in $\pi(x_0)=[z\,|\,z\notin(x_0,\infty)]$ and such that (i) $f(z)\to 0$ as $z\to\infty$, $z\in\pi(x_0)$, and (ii) $f(x\pm i0)$ is bounded for $x\geq x_0$. From (i) we infer that for any positive numbers δ and ϵ (with $\delta<2\pi$) we may choose $R_0=R_0(\delta,\epsilon)$ so large that

$$|f(Re^{i\theta})|<\epsilon,\quad R>R_0,\ \delta\leq\theta\leq 2\pi-\delta. \tag{A1}$$

From (i) and (ii) we infer that $f(z)$ is bounded for all z, i.e., there exists $B>0$ such that

$$|f(Re^{i\theta})|<B,\quad 0\leq\theta\leq 2\pi. \tag{A2}$$

By Cauchy's theorem we may write

$$f(z)=\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta-z}d\zeta+\frac{1}{\pi}\int_{x_0}^{R}\frac{\sigma(x')}{x'-z}dx', \tag{A3}$$

where C_R denotes a circular path of radius $R>|z|$, starting at $\zeta=R+i0$ and ending at $\zeta=R-i0$, and

[58] C. A. Orzalesi and J. Sucher (unpublished).

$\sigma(x)=(2i)^{-1}[f(x+i0)-f(x-i0)]$. The first term in (A3) is bounded in absolute value by

$$\frac{1}{2\pi}\frac{R}{R-|z|}M_R(2\pi,0)\,, \tag{A4}$$

where

$$M_R(\theta_2,\theta_1)\equiv\int_{\theta_1}^{\theta_2}|f(Re^{i\theta})|d\theta\,.$$

From (A1) and (A2) it follows, respectively, that for $R>R_0$

$$M_R(2\pi-\delta,\,\delta)<\epsilon(2\pi-2\delta) \tag{A5}$$

and

$$M_R(2\pi,\,2\pi-\delta)<B\delta\,,\quad M_R(\delta,0)<B\delta\,. \tag{A6}$$

Since

$$M_R(2\pi,0)=M_R(2\pi,\,2\pi-\delta)+M_R(2\pi-\delta,\,\delta)+M_R(\delta,0)\,,$$

we have, from (A5) and (A6), for $R>R_0$

$$|M_R(2\pi,0)|<2\pi\epsilon+B\delta\,.$$

Hence $M_R(2\pi,0)$ tends to zero as $R\rightarrow\infty$ and, by (A4), so does the first term in (A3). Thus

$$f(z)=\lim_{R\to\infty}\frac{1}{\pi}\int_{x_0}^{R}\frac{\sigma(x')}{x'-z}dx'\,,$$

which coincides with Eq. (4.6).

APPENDIX B: THEOREM ON ASYMPTOTIC BEHAVIOR

We prove here the following statement, referred to in Sec. IV:

Theorem. Let $f(z)$ and $I(x)$ be given by

$$f(z)=\frac{1}{\pi}\int_{x_0}^{\infty}\frac{\sigma(x')}{x'-z}dx'\,, \tag{B1}$$

$$I(x)=\int_{x_0}^{x}\sigma(x')dx'\,. \tag{B2}$$

A sufficient condition that

$$B=\lim_{x\to-\infty}xf(x)$$

exists is that $I(\infty)$ exists and that $b(x)=|\sigma(x)|x\ln x$ $(x\geq x_0)$ is a bounded function of x; if the condition holds, then

$$B=-\pi^{-1}I(\infty)\,. \tag{B3}$$

Proof. From (B1) and the relation $x(x'-x)^{-1}=-1+x'(x'-x)^{-1}$ we have, since $I(\infty)$ exists, for $x<0$,

$$xf(x)=-\pi^{-1}I(\infty)+J_1(\eta)+J_2(\eta)\,, \tag{B4}$$

where $\eta=|x|$ and

$$J_1(\eta)=\int_{x_0}^{\eta}\frac{\xi\sigma(\xi)}{\xi+\eta}d\xi\,,\quad J_2(\eta)=\int_{\eta}^{\infty}\frac{\xi\sigma(\xi)}{\xi+\eta}d\xi\,. \tag{B5}$$

We now prove that J_1 and J_2 vanish as $\eta\rightarrow\infty$.

J_1: Since $(\xi+\eta)^{-1}<\eta^{-1}$, we have

$$|J_1(\eta)|<\eta^{-1}\int_{x_0}^{\eta}\xi|\sigma(\xi)|d\eta=\eta^{-1}\int_{x_0}^{\eta}\frac{b(\xi)}{\ln\xi}d\xi\,.$$

For $b(\xi)$ bounded, i.e., $b(\xi)<M$,

$$|J_1(\eta)|<M\eta^{-1}\int_{x_0}^{\eta}\frac{d\xi}{\ln\xi}=M\eta^{-1}(\mathrm{li}\eta-\mathrm{li}x_0)\,,$$

where

$$\mathrm{li}\eta\equiv\int_0^{\eta}\frac{d\xi}{\ln\xi}\,.$$

Since

$$\mathrm{li}\eta=(\ln\eta)[1+O(\eta^{-1})]\,,\quad \eta\rightarrow\infty$$

we see that $J_1(\eta)\rightarrow 0$ as $\eta\rightarrow\infty$.

J_2: On defining

$$H(\eta)\equiv\eta\int_{\eta}^{\infty}\frac{\sigma(\xi)}{\xi+\eta}d\xi \tag{B6}$$

and

$$\Delta(\eta_2,\eta_1)=I(\eta_2)-I(\eta_1)\,,$$

we may write, using $\xi(\xi+\eta)^{-1}=1-\eta(\xi+\eta)^{-1}$,

$$J_2(\eta)=\Delta(\infty,\eta)-H(\eta)\,. \tag{B7}$$

Integrating (B6) by parts, we get

$$H(\eta)=\eta\int_{\eta}^{\infty}\frac{\Delta(\xi,\eta)}{(\xi+\eta)^2}d\xi$$

and, on use of the mean-value theorem,

$$H(\eta)=\tfrac{1}{2}\Delta(\bar{\xi},\eta)\,,\quad \bar{\xi}\in(\eta,\infty)\,. \tag{B8}$$

Since, by the Cauchy convergence criterion applied to (B2), $\Delta(\eta_2,\eta_1)\rightarrow 0$ as η_2 and η_1 become large, we see from (B7) and (B8) that $J_2(\eta)\rightarrow 0$ as $\eta\rightarrow\infty$.

APPENDIX C: SOME THEOREMS OF ABELIAN TYPE

Let $f(x)$ be integrable in any interval (a,ξ) with $\xi>a$ and such that

$$M_a(x)=\int_a^x f(x')dx' \tag{C1}$$

is a bounded function of x, for $x\geq a$. The following theorem, an analog for integrals of an Abelian theorem for series, then holds[59]:

Theorem C1. Let $q(x)$ be positive and monotonically decreasing for $x\geq a$, with $q(x)\rightarrow 0$ as $x\rightarrow\infty$. Let

$$N_a(x)=\int_a^x f(x')q(x')dx'\,.$$

Then

$$N_a(\infty)=\lim_{x\to\infty}N_a(x)$$

exists.

[59] See, e.g., P. Franklin, *A Treatise on Advanced Calculus* (John Wiley & Sons, Inc., New York, 1940), p. 329.

If the integral (C1) actually converges as $x\to\infty$, i.e., if

$$M_a(\infty)=\lim_{x\to\infty} M_a(x) \tag{C2}$$

exists, then we can readily obtain the following modification of theorem C1:

Theorem C2. Let $p(x)$ be a bounded monotonic function of x for $x\geq a$. Then, if $M_a(\infty)$ exists, so does

$$\lim_{x\to\infty}\int_a^x f(x')p(x')dx'. \tag{C3}$$

For proof, note that from the hypothesis on $p(x)$ it follows that $p(\infty)$ exists. Hence one of the functions $p_\pm(x)=\pm[p(\infty)-p(x)]$ has the properties of $q(x)$ assumed in theorem C1, so that

$$W=\lim_{x\to\infty}\int_a^x f(x')[p(\infty)-p(x')]dx' \tag{C4}$$

exists. It follows that (C3) exists, since by (C4) and (C2) it is equal to $p(\infty)M_a(\infty)-W$.

APPENDIX D: PROPERTIES OF $g_\alpha(\kappa;c)$

We have claimed that the family (5.11) has the properties

$$\int_0^\infty d\kappa\,\phi(\kappa)g_\alpha(\kappa;c)\underset{\alpha\to\infty}{\longrightarrow}\int_0^\infty d\kappa\,\phi(\kappa)$$

for every $\phi(\kappa)\in\mathcal{S}$ and

$$2\int_0^\infty d\kappa\cos\kappa\, g_\alpha(\kappa;c)\underset{\alpha\to\infty}{\longrightarrow} c.$$

Since clearly

$$\int_0^\infty d\kappa\,\phi(\kappa)e^{-\kappa/\alpha}\longrightarrow\int_0^\infty d\kappa\,\phi(\kappa)$$

and explicitly

$$\int_0^\infty d\kappa\cos\kappa\, e^{-\kappa/\alpha}=\frac{\alpha}{\alpha^2+1}\longrightarrow 0,$$

we need only show that

$$\int_0^\infty d\kappa\,\phi(\kappa)\frac{\cos\kappa}{\cos^2\kappa+\alpha^{-1}}e^{-(\kappa-\alpha)^2}\longrightarrow 0 \tag{D1}$$

and

$$\int_0^\infty d\kappa\frac{\cos^2\kappa}{\cos^2\kappa+\alpha^{-1}}e^{-(\kappa-\alpha)^2}\longrightarrow \pi^{1/2}. \tag{D2}$$

We first establish (D1). We have

$$\left|\int_0^\infty\cdots\right|\leq\left|\int_0^{\alpha/2}\cdots\right|+\left|\int_{\alpha/2}^\infty\cdots\right|$$

$$\leq\alpha e^{-\alpha^2/4}\int_0^{\alpha/2}d\kappa|\phi(\kappa)|+\alpha\int_{\alpha/2}^\infty d\kappa|\phi(\kappa)|, \tag{D3}$$

and both terms in (D3) approach 0 as $\alpha\to\infty$, since $\phi(\kappa)$ is of fast decrease. Finally, (D2) follows from the dominated convergence theorem[60]:

$$\int_0^\infty d\kappa\frac{\cos^2\kappa}{\cos^2\kappa+\alpha^{-1}}e^{-(\kappa-\alpha)^2}$$

$$=\int_{-\alpha}^\infty d\kappa\frac{\cos^2(\kappa+\alpha)}{\cos^2(\kappa+\alpha)+\alpha^{-1}}e^{-\kappa^2}\longrightarrow\int_{-\infty}^\infty d\kappa\, e^{-\kappa^2}=\pi^{1/2},$$

since

$$V_\alpha(\kappa)\equiv\frac{\cos^2(\kappa+\alpha)}{\cos^2(\kappa+\alpha)+\alpha^{-1}}\theta(\kappa+\alpha)e^{-\kappa^2}\leq e^{-\kappa^2}$$

for all α and $V_\alpha(\kappa)\to e^{-\kappa^2}$ in measure.

[60] See, e.g., P. Halmos, *Measure Theory* (D. Van Nostrand, Inc., New York, 1950), p. 110.