Branch Points in the Complex Angular Momentum Plane and High-Energy Behavior of Scattering Amplitudes

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The high-energy behavior of scattering amplitudes is investigated by analyzing the contribution from a certain class of Mandelstam branch points in the complex angular momentum plane. We denote symbolically, for example, by nP the branch point generated by n-Pomeranchon exchange. It is shown that the totality of the contributions from the branch points of the type $X+nP$ ($n=1, 2, \cdots$) gives an oscillatory behavior to scattering amplitudes at high energies, and that it results in a simple modification of the one-Regge-pole (X) contribution in the impact-parameter formalism. For the branch points of the type nX ($n=2, 3, \cdots$; $X\neq P$), the concept of an *effective trajectory* is introduced. High-energy proton-proton scattering at 90° is explained by the effective trajectory generated by the branch points of the type $(n-m)P'+m\omega$ ($n=2, 3, \cdots; m=0, \cdots, n$).

1. INTRODUCTION AND SUMMARY

It has been shown by Mandelstam[1] that for a certain class of Feynman diagrams, poles in the complex angular momentum j plane generate branch points.[2] Gribov, Pomeranchuk, and Ter-Martirosyan[3] have investigated the branch points assuming a definite structure of the many-particle unitarity condition for complex j. They have obtained the discontinuities on the branch cuts corresponding to the formation of several Reggeons in the intermediate state in terms of Reggeon production amplitudes. Essentially the same discontinuity formula has been obtained by Gribov,[4] who has investigated the asymptotic behavior of a large class of Feynman diagrams containing exact two-particle amplitudes. It is of great interest to note that the entirely different methods (the Reggeon-unitarity method[3] and the Reggeon-diagram one[4]) lead to the discontinuity formula of the same form.

Recently, the Mandelstam cuts have been discussed in connection with various high-energy phenomena: fixed poles at the nonsense wrong-signature points[5] where dip phenomena occur, the polarization[6] observed in π^-p charge-exchange scattering and the break in the differential cross section of high-energy 90° proton-proton scattering.[7]

The purpose of the present article is to investigate the contribution from a certain class of branch points to scattering amplitudes at high energies. We denote symbolically, for example, by nP the branch point generated by n-Pomeranchon exchange. We consider the following three cases:

(1) the total contribution from the branch points of the type nP ($n=2, 3, \cdots$),
(2) that of the type P' (or ω)$+nP$ ($n=1, 2, \cdots$), and
(3) that of the type $(n-m)P'+m\omega$ ($n=2, 3, \cdots$; $m=0, \cdots, n$).

We have chosen P' and ω, for simplicity of the argument, to represent even- and odd-signature trajectories other than the Pomeranchuk one, which is assumed, throughout the present paper, to pass through $j=1$ at zero energy. The discontinuity formula derived by Gribov[4] is utilized in our analysis.

Our results are the following:

(a) For case (1), whose contribution dominates elastic scattering amplitudes at very high energies ($|t|\ln s\gg 1$, $t<0$), the amplitudes oscillate as

$$\cos[\pi(\alpha_P'(0)|t|\ln s/\ln\ln s)^{1/2}+O(1/\ln\ln s)].$$

This confirms the conclusion of Anselm and Dyatlov[8] who approached the problem in a rather general way. This oscillation is characteristic of the Mandelstam mechanism of generating branch points and there appears no oscillation for the summation of the Amati-Fubini-Stanghellini type of branch points.

(b) The whole contribution of the branch points of the type $X+nP$ ($n=1, 2, \cdots$) results in a simple modification of the one-Regge-pole (X) contribution in the impact parameter formalism. Using this result, it is shown that the Schmid mechanism[9] of generating cross-channel Regge poles is not essentially changed in the presence of branch points.

(c) For case (3), we are led to the concept of an *effective trajectory*, which is the envelope of the branch-point trajectories.

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(d) The 90° proton-proton scattering at high energies is explained by the *effective trajectory* of the branch points of the type $(n-m)P'+m\omega$ $(n=2,\ 3,\ \cdots;\ m=0,\ \cdots,\ n)$.

Although the main point of the item (a) is essentially contained in the work of Anselm and Dyatlov,[8] we present our version in Sec. 2 in order to fix our notation and in order to make clear some detailed points for the sake of completeness. Point (b) is treated in Secs. 3 and 4. In Secs. 5 and 6 we develop points (c) and (d), respectively. Some additional remarks and discussion are given in Sec. 7.

2. TOTALITY OF THE CONTRIBUTIONS FROM THE POMERANCHUK SINGULARITIES

The elastic scattering of two spinless, identical particles is considered for simplicity. Our amplitude is normalized as

$$\sigma_{\text{tot}}(s)=\frac{8\pi}{p\sqrt{s}}\,\text{Im}A(s,t=0)\,, \tag{2.1}$$

and the t-channel partial-wave expansion is

$$A(s,t)=\sum_{j=0}^{\infty}(2j+1)f_j(t)P_j(z_t)\,, \tag{2.2}$$

where s and t are the usual Mandelstam invariants. The momentum and the cosine of the scattering angle θ in the s-channel center-of-mass system are denoted by p and z, respectively. The corresponding t-channel quantities are specified by the suffix t.

The discontinuity formula on the branch cut in the complex j plane generated by n-Pomeranchon exchange is given by Gribov[4]:

$$\Delta f_j{}^n(t)=(j-j_n)^{n-2}\pi\sin(\tfrac{1}{2}\pi j)B_nN_{\alpha\cdots\alpha}{}^2(\zeta_\alpha)^{-n}(\bar{\alpha})^{-n+1}\,,\qquad (j\sim j_n) \tag{2.3}$$

where

$$j_n=n\alpha(t/n^2)-n+1\,, \tag{2.4}$$

$$B_n=4\pi^{n+1}(n-1)/n!\,, \tag{2.5}$$

$$\zeta_\alpha=\sin[\tfrac{1}{2}\pi\alpha(t/n^2)]\,, \tag{2.6}$$

$$\bar{\alpha}=\{\alpha'(t/n^2)[\alpha'(t/n^2)+2t\alpha''(t/n^2)/n^2]\}^{1/2}\,,\quad [\alpha'(u)\equiv d\alpha(u)/du\,,\quad \alpha''(u)\equiv d^2\alpha(u)/du^2] \tag{2.7}$$

and the n-Reggeon production amplitude $N_{\alpha\cdots\alpha}$ has n $\alpha(t/n^2)$'s as suffixes, $\alpha(t)$ being the position of the Pomeranchuk trajectory. The discontinuity divided by $2i$ is denoted by $\Delta f_j{}^n(t)$ and j_n is the position of the branch point. The factor due to the identity of the Reggeons in the intermediate state is considered to be included in the Reggeon production amplitude $N_{\alpha\cdots\alpha}$. The amplitude $N_{\alpha\cdots\alpha}$ is real below the production threshold and in particular for $t<0$. The formula (2.3) is derived by the Reggeon diagram technique[4] for $|t|\ll\mu^2$, where μ is the mass of the particle. We assume the formula to be valid also for $|t|\gtrsim\mu^2$. One reason is that the formula is independent of μ and depends on t alone. Another reason is that the form of the expression (2.3) is the same as the one derived by the Reggeon unitarity method[3] which applies also to the region $|t|\gtrsim\mu^2$.

The expression (2.3) for $n=2$ does not satisfy the requirement[10] of unitarity that $\Delta f_j{}^{n=2}(t)$ be singular and vanishing at $j=j_2(t)$. The correct formula may be of the form[3]

$$\Delta f_j{}^{n=2}(t)\simeq c/[\ln(j-j_2)]^2\,, \tag{2.8}$$

which was obtained by Gribov, Pomeranchuk, and Ter-Martirosyan by taking into account the unitarity relation. This form differs from the expression (2.3) in the contribution to the full amplitude only by the factor ln ln$s$ at high energies. As the main contribution to the full amplitude at high energies turns out to come from large n, we take the expression (2.3) also for $n=2$ for the simplicity of the presentation. The form (2.3) correctly reproduces the pole contribution for $n=1$ when we identify $2\pi N_\alpha(t)$ with the reduced residue $\gamma(t)$, which is the usual one multiplied by gamma functions.

Now, the total contribution to the full amplitude (at large s for fixed t) from the Pomeranchuk singularities is expressed by

$$[A(s,t)]_P=\sum_{n=1}^{\infty}\frac{1}{2}\int_{-\infty}^{j_n}\Delta f_j{}^n(t)\frac{e^{-i\pi j}+1}{\sin\pi j}s^jdj\,. \tag{2.9}$$

Strictly speaking, the formula (2.3) loses its validity far away from the branch point. The main contribution to the scattering amplitude, however, comes around $j\simeq j_n(t)$. Therefore we will make no bad estimation if we use the formula (2.3) for the whole integration range. After a straightforward calculation we get

$$[A(s,t)]_P\simeq\sum_{n=1}^{\infty}(-1)^n2\pi^2\bar{\alpha}N_{\alpha\cdots\alpha}{}^2x\ln x\,\frac{1}{n}\left(\frac{\pi}{\bar{\alpha}\zeta_\alpha\ln x}\right)^n x^{at/n}\,, \tag{2.10}$$

where we have set

$$x=e^{-i\pi/2}s\,, \tag{2.11}$$

$$j_n\simeq at/n+1\,, \tag{2.12}$$

and

$$a=\alpha'(0)\,. \tag{2.13}$$

In order to calculate the summation, we make the following approximations:

$$\bar{\alpha}\simeq a\,, \tag{2.14}$$

$$\zeta_\alpha\simeq1\,, \tag{2.15}$$

and

$$N_{\alpha\cdots\alpha}\simeq N_{\alpha(0)\cdots\alpha(0)}\equiv N_n\,. \tag{2.16}$$

These simplifications are allowed *a posteriori* because

[10] J. B. Bronzan and C. E. Jones, Phys. Rev. **160**, 1494 (1967).

for fixed t the high-energy behavior turns out to be dominated by large n. More precisely, the dominant n will be shown to be approximately

$$n \propto (|t| \ln s)^{1/2}, \tag{2.17}$$

and therefore

$$t/n^2 \propto 1/\ln s. \tag{2.18}$$

Thus the argument of $\alpha(t/n^2)$ is close to zero for large s and the expressions (2.14), (2.15), and (2.16) result. Next, the n dependence of N_n must be specified. Here we simply assume the form

$$4\pi^2 N_n^2 \simeq \gamma^2 \rho^{n-1}, \tag{2.19}$$

where

$$\gamma = \gamma(0) \tag{2.20}$$

and ρ is a positive constant due to the reality of N_n. As will become clear below, the result does not depend on ρ in an essential way. Then we have

$$[A(s,t)]_P \simeq \frac{\gamma^2 a}{2\rho} x \ln x \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \left(\frac{\pi\rho}{a \ln x}\right)^n (x^{at})^{1/n}$$

$$\equiv \frac{\gamma^2 a}{2\rho} x \ln x \, F(x,t), \tag{2.21}$$

where

$$F(x,t) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \epsilon^n e^{-\omega/n}, \tag{2.22}$$

$$\epsilon \equiv \pi\rho / a \ln x, \tag{2.23}$$

and

$$\omega \equiv -at \ln x. \tag{2.24}$$

We estimate the function $F(x,t)$ by the Sommerfeld-Watson transform

$$F(x,t) = \tfrac{1}{2} i \int_\Gamma dn \frac{\exp(-n\sigma - \omega/n)}{n \sin \pi n}, \tag{2.25}$$

where

$$\sigma \equiv \ln[(a \ln x)/\pi\rho] \tag{2.26}$$

and Γ is the usual hairpin contour in the clockwise direction. We approximate $\sin\pi n$ by $e^{i\pi n}/2i$ $(-e^{-i\pi n}/2i)$ in the lower (upper) half n plane. This approximation is justified because the saddle points of the integrand $n \simeq [\omega/(\sigma \pm i\pi)]^{1/2}$, whose neighborhoods give the main contribution to the integral, are located far from the real axis. [The quantity in (2.17) is the real part of the saddle points.] Then

$$F(x,t) \simeq \int_{c_1 \atop \text{upper}}^{\infty} dn \frac{1}{n} \exp(i\pi n - n\sigma - \omega/n)$$

$$+ \int_{c_1 \atop \text{lower}}^{\infty} dn \frac{1}{n} \exp(-i\pi n - n\sigma - \omega/n), \tag{2.27}$$

where the integration path in the first (second) term is on the upper (lower) half n plane and c_1 is somewhere between 0 and 1. Next we displace both the paths to the real axis. This procedure is allowed because there appears no singularity during the displacement. Using the integral representation formula for the modified Bessel function (of the third kind) of zero order, K_0, we obtain

$$F(x,t) \simeq \sum_{\pm} 2K_0(2\omega^{1/2}(\sigma \pm i\pi)^{1/2}). \tag{2.28}$$

The asymptotic expansion of $K_0(z)$ for $1 \ll |z|$ is given by

$$K_0(z) \simeq (\pi/2z)^{1/2} e^{-z} [1 + O(1/z)]. \tag{2.29}$$

Thus the asymptotic form of the amplitude $[A(s,t)]_P$ has the following expression:

$$[A(s,t)]_P \simeq C_1(s,t) \exp[-C_2(s,t)] \cos C_3(s,t) + D, \tag{2.30}$$

with

$$C_1(s,t) \simeq -i(\sqrt{\pi}) \frac{\gamma^2 a}{\rho} s \ln s \, (-at \ln s \ln \ln s)^{-1/4} (1 + D_1), \tag{2.31}$$

$$C_2(s,t) \simeq 2(-at \ln s \ln \ln s)^{1/2} (1 + D_2), \tag{2.32}$$

$$C_3(s,t) \simeq \pi(-at \ln s / \ln \ln s)^{1/2} (1 + D_3), \tag{2.33}$$

$$D_1 = O[(\ln \ln s)^{-2}], \tag{2.34}$$

$$D_2 = O[(\ln \ln s)^{-2}], \tag{2.35}$$

$$D_3 = O[(\ln s \ln \ln s)^{-1/2}], \tag{2.36}$$

and D is smaller than the main term by a factor of the order $(\ln \ln s)^{-2}$. These expressions are valid for $|t| \ll s$, $1 \ll \ln s$, and $1 \ll |t| \ln s$.

The result shows two interesting features. One is the oscillation and the other is the exponential decrease with respect to $(|t| \ln s)^{1/2}$. This confirms the conclusion of Anselm and Dyatlov.[8] First we discuss the oscillatory behavior. The oscillation originates in the sign change of $\Delta f_j^n(t)$ $(j < j_n)$ with respect to n and in the fact that the main contribution comes from $n \propto (|t| \ln s)^{1/2}$. In other words, the dominant n increases as s increases and the discontinuity changes sign as n increases by one unit. Thus the oscillation comes out. In this respect the reality of the production amplitude $N_{\alpha \cdots \alpha}$ and the positiveness of ζ_α for $\alpha \simeq 1$ are important. [See Eq. (2.3).] The branch points found by Amati, Fubini, and Stanghellini (AFS),[2] which in reality are located in the second sheet, do not lead to an oscillatory behavior. This is due to the lack of the sign change of the discontinuity. On the other hand, the exponential decrease with respect to $(|t| \ln s)^{1/2}$ appears also in the amplitude of Amati, Cini, and Stanghellini[11] which takes into account the whole contribution from the AFS branch points.

[11] D. Amati, M. Cini, and A. Stanghellini, Nuovo Cimento **30**, 193 (1963).

Anselm and Dyatlov[8] state that the oscillation due to the sign change of the discontinuities on the branch cuts of the type nP ($n=2, 3, \cdots$) will be observed at presently available energies. It seems to the present author that there may exist still other possibilities. We present our interpretation of the pp large-angle scattering data at high energies in Sec. 6.

3. CONTRIBUTION OF BRANCH POINTS IN THE IMPACT-PARAMETER FORMALISM

Our impact-parameter amplitude[12] $a(b,s)$ is defined by

$$a(b,s)=2\int_0^1 ydy\, J_0(2pby)A(s,t), \tag{3.1}$$

where

$$y=(-t)^{1/2}/2p. \tag{3.2}$$

At high energies

$$a(b,s)\simeq\frac{1}{2p^2}\int_0^\infty vdv\, J_0(bv)A(s,t) \tag{3.3}$$

and

$$a(b,s)\simeq f_j(s) \quad \text{for} \quad 2j+1=2pb, \tag{3.4}$$

where

$$v=(-t)^{1/2}. \tag{3.5}$$

In the first place we show that the function $F(x,t)$ defined in Eq. (2.22) has a simple integral representation containing the Bessel function of zero order J_0.

$$F(x,t)\equiv\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\epsilon^n e^{-\omega/n}$$
$$=(-\epsilon)\int_0^\infty dz\frac{J_0(2\sqrt{(\omega z)})}{e^z+\epsilon}. \tag{3.6}$$

One can verify Eq. (3.6) by expanding the denominator in a power series with respect to ϵ. Using Eq. (3.6) and changing the integration variable, we get from Eq. (2.21)

$$[A(s,t)]_P\simeq-\frac{\pi\gamma^2x}{4a\ln x}$$
$$\times\int_0^\infty bdb\frac{J_0(b\sqrt{(-t)})}{\exp(b^2/4a\ln x)+(\pi\rho/a\ln x)}. \tag{3.7}$$

Therefore,

$$[a(b,s)]_P\simeq i\frac{\pi\gamma^2}{2a\ln x}[\exp(b^2/4a\ln x)+(\pi\rho/a\ln x)]^{-1}. \tag{3.8}$$

At first sight the second term $\pi\rho/a\ln x$ in the bracket in (3.8) might seem to be negligibly small compared with the first term $\exp(b^2/4a\ln x)$. In reality this is not the case. The first term oscillates because of the phase of x. The expression (3.8) without the term $\pi\rho/a\ln x$ is just the one-Pomeranchuk contribution when we neglect a t dependence of $\gamma^2(t)\zeta_{\alpha(t)}^{-1}$. When we take into account the t dependence of $\gamma^2(t)\zeta_{\alpha(t)}^{-1}$ in the form

$$\gamma^2(t)\zeta_{\alpha(t)}^{-1}=\gamma^2e^{r^2t}, \tag{3.9}$$

we have, for the one-Pomeranchuk term,

$$[a(b,s)]_{P\text{-pole}}=i\frac{\pi\gamma^2}{2a\ln x}$$
$$\times\int_0^\infty b'db'\exp[-(b')^2/4a\ln x]F(b',b), \tag{3.10}$$

with

$$F(b',b)=\frac{1}{2r^2}\exp\{-[b^2+(b')^2]/4r^2\}I_0(bb'/2r^2), \tag{3.11}$$

where I_0 is the modified Bessel function (of the first kind) of zero order. For $bb'/2r^2\gg1$

$$F(b',b)\simeq(4\pi bb'r^2)^{-1/2}\exp[-(b-b')^2/4r^2]. \tag{3.12}$$

In any case, the applicability of (3.8) is limited by conditions corresponding to those for the formula (3.7) or, which are the same things, the ones for (2.21). They are $|t|\ll s$, $1\ll\ln s$, and $1\ll|t|\ln s$. The energies presently available do not meet these conditions. At higher energies, where the contribution from branch points of the type nP with large n dominates, the formula (3.8) will become valid.

4. MODIFICATION OF THE P'- OR THE ω-TRAJECTORY CONTRIBUTION BY THE POMERANCHUK SINGULARITIES

In this section we examine the totality of the contributions from the branch points of the type P' (or ω)$+nP$. We denote the trajectory function of P' (or ω) by $\beta(t)$. The discontinuity formula for the present case is obtained by slightly modifying the one given by Gribov[4]:

$$\Delta f_j^n(t)=(j-j_n)^{n-1}\pi\sin[\tfrac{1}{2}\pi(j-\tfrac{1}{2}(P-1))]$$
$$\times B_{n+1}'N_{\beta\alpha\cdots\alpha}{}^2\zeta_\beta{}^{-1}\zeta_\alpha{}^{-n}(\tilde\beta)^{-1}(\tilde\alpha)^{-n+1}, \quad (j\sim j_n) \tag{4.1}$$

where

$$j_n=\beta(t_0)+n\alpha(t_1)-n, \tag{4.2}$$

$$\tilde\beta=\{\beta'[\beta'+2(\alpha')^2\beta''t/(\alpha'+n\beta')^2]\}^{1/2}, \tag{4.3}$$

$$\tilde\alpha=\{\alpha'[\alpha'+2(\beta')^2\alpha''t/(\alpha'+n\beta')^2]\}^{1/2}, \tag{4.4}$$

$$\zeta_\beta=\sin\tfrac{1}{2}\pi\beta(t_0), \quad \text{for} \quad P' \tag{4.5a}$$

$$=\cos\tfrac{1}{2}\pi\beta(t_0), \quad \text{for} \quad \omega' \tag{4.5b}$$

$$\zeta_\alpha=\sin\tfrac{1}{2}\pi\alpha(t_1), \tag{4.6}$$

$$P=+1, \quad \text{for} \quad P' \tag{4.7a}$$

$$=-1, \quad \text{for} \quad \omega \tag{4.7b}$$

$$t_0=[\alpha'/(\alpha'+n\beta')]^2t, \tag{4.8}$$

[12] T. Adachi and T. Kotani, Progr. Theoret. Phys. (Kyoto) Suppl., Extra Number, 316 (1965); W. N. Cottingham and R. F. Peierls, Phys. Rev. **137**, B147 (1965).

and

$$t_1=[\beta'/(\alpha'+n\beta')]^2 t. \tag{4.9}$$

The Reggeon production amplitude is denoted by $N_{\beta\alpha\cdots\alpha}$. In the above formulas the suppressed argument of α and α' (β and β') is t_1 (t_0). B_n' has a slightly complicated expression, but it is nearly equal to B_n of Eq. (2.5) for large n.

By the same argument as that given in Sec. 2, we are led to the expression

$$[A(s,t)]_{P'(\omega)+P}\simeq e^{i\pi(P-1)/4}\frac{a\gamma^2}{2\rho\zeta_d}x^d\ln x \times\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n+1}\left(\frac{\pi\rho}{a\ln x}\right)^{n+1}x^{act/(a+nc)}, \tag{4.10}$$

where

$$c=\beta'(0), \tag{4.11}$$

$$d=\beta(0), \tag{4.12}$$

and

$$\zeta_d=\sin\tfrac{1}{2}\pi d \quad \text{for} \quad P' \tag{4.13a}$$

$$=\cos\tfrac{1}{2}\pi d \quad \text{for} \quad \omega. \tag{4.13b}$$

Approximating

$$ac/(a+nc)\simeq a/(n+1),$$

which is allowed because the main contributions come from large n, we have

$$[A(s,t)]_{P'(\omega)+P}\simeq e^{i\pi(P-1)/4}\frac{a\gamma^2}{2\rho\zeta_d}x^d\ln x\, F(x,t), \tag{4.14}$$

where the function $F(x,t)$ is the same as the one in Secs. 2 and 3. The asymptotic behavior of the amplitude $[A(s,t)]_{P'(\omega)+P}$ is almost the same as that of the amplitude $[A(s,t)]_P$. Projecting the expression (4.14) on the impact parameter amplitude, we have

$$[a(b,s)]_{P'(\omega)+P}\simeq -\exp\left[\tfrac{1}{2}i\pi\left(\frac{P-1}{2}-d\right)\right]\frac{\pi\gamma^2 s^{d-1}}{2a\zeta_d\ln x} \times[\exp(b^2/4a\ln x)+(\pi\rho/a\ln x)]^{-1}. \tag{4.15}$$

Here we comment on the Schmid mechanism[9] of generating crossed-channel trajectories. Kugler[13] has made the suggestion that a crossed-channel linear ($t\leq 0$) trajectory generates a direct-channel trajectory $j(s)$ behaving $j(s)\sim\sqrt{s}$ for large s. We show that this mechanism is not changed in the presence of branch points. For large s we have

$$[a(b,s)]_{P'(\omega)+P}\simeq -\exp\left[\tfrac{1}{2}i\pi\left(\frac{P-1}{2}-d\right)\right]\frac{\pi\gamma^2 s^{d-1}}{2a\zeta_d\ln s} \times\left\{\exp\left[\frac{b^2}{4a\ln s}-i\left(\tfrac{1}{2}\pi\frac{b^2}{4a\ln^2 s}+\varphi\right)\right]+\frac{\pi\rho}{a\ln s}\right\} \times|\exp(b^2/4a\ln x)+(\pi\rho/a\ln x)|^{-2}, \tag{4.16}$$

where

$$\varphi=O(1/\ln^4 s).$$

Taking into account the expression (3.4), we find that the Argand diagram shows a circle for

$$-\left[\tfrac{1}{2}\pi\frac{b^2}{4a\ln^2 s}+\varphi+\tfrac{1}{2}\pi(d-\tfrac{1}{2}(P-1))+\pi\right] =\tfrac{1}{2}\pi\pm 2n\pi, \tag{4.17}$$

that is to say, for

$$j\sim(\sqrt{s})\ln s. \tag{4.18}$$

We note that the presence of the term $\pi\rho/a\ln s$ does not change the essential feature of the mechanism.

5. AN EFFECTIVE TRAJECTORY

In this section we investigate the contribution from the branch points of the type $(n-m)P'+m\omega$ with $n=2,3,\cdots$, and $m=0,1,\cdots,n$. For simplicity we consider the case of the exchange degeneracy. The discontinuity on the branch cut of the type $(n-m)P'+m\omega$ is given by

$$\Delta f_j^{n,m}(t)=(j-j_n)^{n-2}\pi\sin[\tfrac{1}{2}\pi(j+m)] \times B_n N_{n,m}{}^2\zeta_1{}^{-n+m}\zeta_2{}^{-m}(\bar{\beta})^{-n+1}, \tag{5.1}$$

with

$$j_n=n\beta(t/n^2)-n+1 \simeq ct/n+n(d-1)+1, \tag{5.2}$$

$$\zeta_1=\sin[\tfrac{1}{2}\pi\beta(t/n^2)], \tag{5.3}$$

and

$$\zeta_2=\cos[\tfrac{1}{2}\pi\beta(t/n^2)]. \tag{5.4}$$

The amplitude for $(n-m)P'+m\omega$ production is denoted by $N_{n,m}=N_{n,m}(t/n^2)$, and $\bar{\beta}=\bar{\beta}(t/n^2)$ is defined in the same way as in (2.7). The contribution to the full amplitude from the branch point of the type $(n-m)P'+m\omega$ is given by

$$A_{n,m}(s,t)=\frac{1}{2}\int_{-\infty}^{j_n}\Delta f_j^{n,m}(t)\frac{e^{-i\pi j}+(-1)^m}{\sin\pi j}s^j dj. \tag{5.5}$$

Substituting the expression (5.1) into (5.5), we have

$$A_{n,m}(s,t)\simeq(-1)^n 2\pi^2\bar{\beta}N_{n,m}{}^2 x\ln x\frac{1}{n}\left(\frac{\pi}{\bar{\beta}x^{1-d}\ln x}\right)^n \times\zeta_1{}^{-n+m}\zeta_2{}^{-m}x^{ct/n}e^{-i\pi m/2}. \tag{5.6}$$

We rewrite Eq. (5.6) into the following form:

$$A_{n,m}(s,t)\simeq(-1)^n C_{n,m}x\ln x\frac{1}{n}\left(\frac{\pi}{x^{1-d}\ln x}\right)^n x^{ct/n}, \tag{5.7}$$

with

$$C_{n,m}=C_{n,m}(t/n^2) =2\pi^2 N_{n,m}{}^2(\bar{\beta})^{1-n}\zeta_1{}^{-n+m}\zeta_2{}^{-m}e^{-i\pi m/2}. \tag{5.8}$$

[13] M. Kugler, Phys. Rev. Letters 21, 570 (1968).

We assume that the summation $\sum_{m=0}^{n} C_{n,m}$ can be written in the form

$$\sum_{m=0}^{n} C_{n,m} \simeq C\rho^n e^{if(n)} \quad \text{for} \quad 1 \ll n, \tag{5.9}$$

where C and ρ are positive constants, and $f(n)$ is a real function of n. The approximation that C, ρ and $f(n)$ are independent of t is justified *a posteriori*, because the main contribution to the full amplitude turns out to come from

$$n_0 \simeq [-ct/(1-d)]^{1/2}, \tag{5.10}$$

and therefore t/n^2 has approximately a constant value

$$t/n^2 \simeq (1-d)/(-c) \sim -1/2. \tag{5.11}$$

For (5.11)

$$\zeta_1 \sim 0$$

and this zero produces a ghost pole. However, this ghost pole is cancelled by a zero of the production amplitude and this situation is taken into account in the expression (5.9). We note that the expression (5.10) for the dominant n is different from the corresponding one (2.17) of the case treated in Sec. 2. The condition $1 \ll n_0$ is translated into the one for t that

$$1 \ll |t|. \tag{5.12}$$

Now, we have

$$A_n(s,t) \equiv \sum_{m=0}^{n} A_{n,m}(s,t)$$

$$\simeq C(-1)^n x \ln x \frac{1}{n}\left(\frac{\pi\rho}{x^{1-d}\ln x}\right)^n x^{ct/n} e^{if(n)}. \tag{5.13}$$

Rewriting the summation $\sum_{n=1}^{\infty} A_n(s,t)$ into a Sommerfeld-Watson integral we have, for $1 \ll s$, the expression

$$[A(s,t)]_{P'-\omega} = \sum_{n=1}^{\infty} A_n(s,t)$$

$$\simeq Cx \ln x \frac{i}{2} \int_{\Gamma} dn \frac{e^{g(n)+ih(n)}}{n \sin \pi n}, \tag{5.14}$$

where

$$g(n) = -\sigma' n - \omega'/n, \tag{5.15}$$

$$\sigma' = (1-d)\ln s, \tag{5.16}$$

$$\omega' = -ct \ln s, \tag{5.17}$$

$$h(n) = f(n) + \tfrac{1}{2}\pi(1-d)n - \tfrac{1}{2}\pi(ct/n), \tag{5.18}$$

and Γ is the usual hairpin contour in the clockwise direction.

Maximizing the function $g(n)$ we have the estimate (5.10). Due to the unknown phase function $h(n)$ in the integrand we can draw no conclusion on the nature of the oscillation of the amplitude $[A(s,t)]_{P'-\omega}$. Taking the contribution around $\mathrm{Re}\, n \simeq n_0$ we have a rough estimate for $1 \ll |t|$:

$$[A(s,t)]_{P'-\omega} \simeq C's \exp\{-2[-c(1-d)t]^{1/2} \ln s\}, \tag{5.19}$$

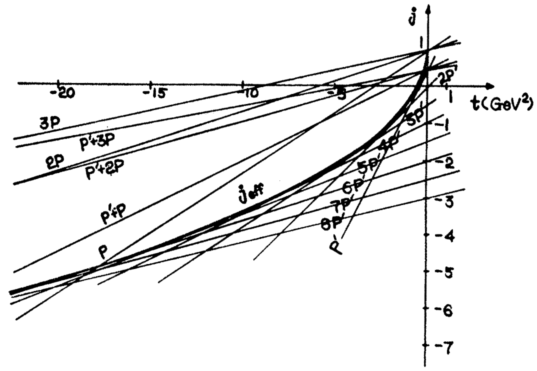

FIG. 1. Plot of pole and branch-point trajectories. The parameters are chosen as $\alpha_P(t) = \frac{1}{3}t+1$ and $\beta_{P'}(t) = t+0.5$. The ω trajectory is assumed to be exchange-degenerate with the P' trajectory. The effective trajectory $j_{\text{eff}}(t)$ is shown.

where C' is a slowing varying function of s and t apart from a possible oscillation. The expression (5.19) is valid for $1 \ll |t|$ and $|t| \ll s$. The first condition comes from (5.12) and the second one from the condition of the usual Regge expansion (5.5).

We rewrite (5.19) into the form

$$[A(s,t)]_{P'-\omega} \simeq C' s^{j_{\text{eff}}(t)}, \tag{5.20}$$

with

$$j_{\text{eff}}(t) \equiv 1 - 2[-c(1-d)t]^{1/2}. \tag{5.21}$$

Thus it is found that the totality of the contributions from the branch points of the type $(n-m)P' + m\omega$ has the same contribution as that of the pole whose trajectory is given by (5.21), if a possible oscillation is neglected. We call this trajectory the *effective trajectory* generated by the branch points of the type $(n-m)P' + m\omega$. (Figure 1.) As is easily seen, the *effective trajectory* is the envelope of the pole and the branch-point trajectories:

$$j_n = ct/n + n(d-1) + 1, \quad n = 1, 2, \cdots.$$

This result is intuitively very plausible, as the high-energy behavior is dominated by the right-most singularities. In this respect the twisting trajectory discussed by Srivastava[14] in the case $\alpha(0) \neq 1$ is similar to our *effective trajectory.*

6. HIGH-ENERGY PROTON-PROTON SCATTERING AT LARGE ANGLES

The authors of Refs. 7 and 8 made analyses of high-energy large-angle pp scattering in terms of multiple Pomeranchon exchanges. Here we want to present a tentative alternative. For other approaches see Ref. 15.

[14] Y. Srivastava, Phys. Rev. Letters **19**, 47 (1967).
[15] See, for example, P. G. O. Freund, Nuovo Cimento **56A**, 1087 (1968); T. Yoshida, Kyoto University Report, 1968 (unpublished); I. A. Sakmar and J. H. Wojtaszek, Nuovo Cimento **56**, 92 (1968); M. Greco, Phys. Letters **27B**, 234 (1968); Refs. 24 and 25.

In the first place we briefly justify the complex angular momentum approach to large-angle scatterings. We consider the Mandelstam-Sommerfeld-Watson transform.[16] Then the Regge term is represented by $Q_{-j-1}(-z_t)$ multiplied by the residue and the signature factor. The cosine of the crossed-channel scattering angle z_t satisfies at high energies

$$-z_t \gtrsim 3.$$

From the estimate

$$Q_{-j-1}(-z_t) \simeq \pi^{1/2} \frac{\Gamma(-j)}{\Gamma(\frac{1}{2}-j)} e^{j\ln(-2z_t)}[1+O(1/z_t^2)], \quad (6.1)$$

it follows that the singularities located far to the left make no appreciable contribution. The function $Q_{-j-1}(-z_t)$ decreases nearly one order when j decreases by one unit. Thus we may be allowed to do with a few singularities located to the right. Incidentally, we note that the correct analyticity is always achieved by subtracting from $Q_{-j-1}(\pm z_t)$ the superfluous cuts $(-z_1<z_t<-1$ and $1<z_t<z_2)$, whose contribution can be made arbitrarily small.

Thus we have, for the pole contribution, the form $(-2z_t)^{j(t)}$ multiplied by the residue, the signature factor, and the known kinematical factor. As for the contribution from the branch points of the type $(n-m)P'+m\omega$ discussed in the preceding section, we have the form $C'(-2z_t)^{j_{\mathrm{eff}}(t)}$, because the analysis in that section is unchanged for $|t|\gg 1$ under the replacement $s\to -2z_t$, the reduced residue $\to$ the residue, etc. (The condition $|t|\ll s$ does not apply in the present case because we do not use the usual Regge expansion.) The branch points of the types nP and P'(or ω)$+nP$ will be discussed below.

Next, we summarize the main features of the experimental data[17,18] of large-angle pp scattering. In the present paper we are mainly interested in the gross feature of the data and the break[17] recently observed in the 90° scattering will be briefly discussed in the next section. There are some empirical formulas which fit the data rather well. They are, for example,

$$d\sigma/dt = As^{-2}\exp(-p_\perp/T_1) \quad \text{(Orear[19])}, \quad (6.2)$$

$$d\sigma/dt = \sum_{i=1}^{3} B_i \exp(-a_i\beta^2 p_\perp^2) \quad \text{(Krisch[20])}, \quad (6.3)$$

$$d\sigma/dt = \sum_{i=1}^{2} C_i \exp[-(s\sin\theta)/g_i] \quad \text{(Allaby et al.[17])}, \quad (6.4)$$

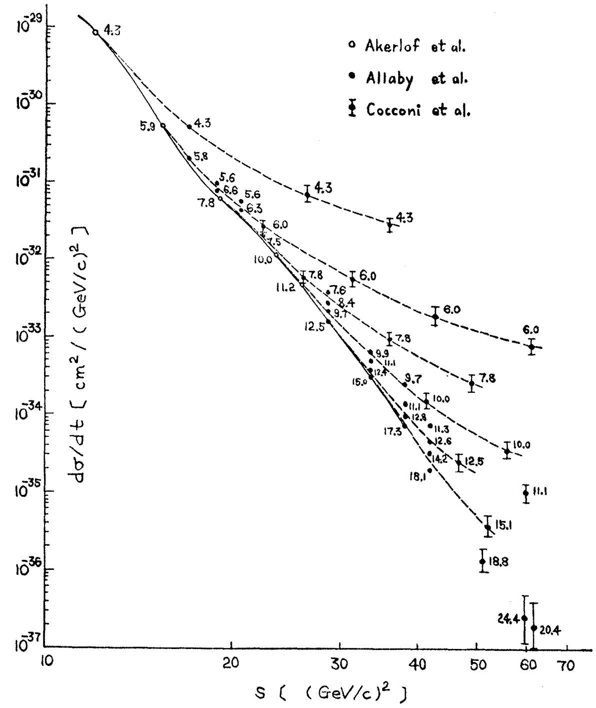

FIG. 2. The differential cross section of elastic pp scattering plotted against s, both in logarithmic scale. The experimental points are labeled by the corresponding value of t. Equal-t contours are shown by dashed lines. The data are taken from Refs. 17 and 18. Not all the data are shown. The full line results from a smooth fit to the 90° data of Ref. 17 (Akerlof *et al.* and Allaby *et al.*). The data of Akerlof *et al.* are dense on the curve between $|t|=4.3$ and 11.2.

$$d\sigma/dt = Dps^{-2}\sin^4\theta\exp(-p_\perp/T_0) \quad \text{(Etim-Greco[21])}, \quad (6.5)$$

and

$$d\sigma/dt = Es^{-2}\exp[-C_\gamma(\theta)s^\gamma \ln s] \quad \text{(Chiu-Tan[22])}, \quad (6.6)$$

where

$$p_\perp = \sin\theta,$$

β is the center-of-mass velocity of the protons, and $C_\gamma(\theta)$ is a known function of θ. The formula (6.6) is the lower bound of Chiu and Tan, and $\gamma=\frac{1}{2}$ corresponds to that of Cerulus and Martin.[23] In addition to these formulas there are ones connecting $d\sigma/dt$ with the fourth power of the electromagnetic form factor.[24]

In any case the angular distribution shows an exponential decrease with respect to $\sin\theta$ or $\sin^2\theta$, and the fixed-θ distribution decreases exponentially with respect to p or s. As we want to analyze large-angle pp scattering in the framework of the theory of the complex angular momentum, the behavior with fixed t is important. In Fig. 2 the experimental data[17,18] ($\log d\sigma/dt$ versus

[16] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).
[17] C. W. Akerlof *et al.*, Phys. Rev. Letters **17**, 1105 (1966); Phys. Rev. **159**, 1138 (1967); J. V. Allaby *et al.*, Phys. Letters **25B**, 156 (1967).
[18] G. Cocconi *et al.*, Phys. Rev. **138**, B165 (1965).
[19] J. Orear, Phys. Letters **13**, 190 (1964).
[20] A. D. Krisch, Phys. Rev. Letters **19**, 1149 (1967).
[21] E. Etim and M. Greco, Phys. Letters **26B**, 313 (1968).
[22] C. B. Chiu and C. I. Tan, Phys. Rev. **162**, 1701 (1967).
[23] F. Cerulus and A. Martin, Phys. Letters **8**, 80 (1964).
[24] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965); H. D. I. Abarbanel, S. D. Drell, and F. J. Gilman, Phys. Rev. Letters **20**, 280 (1968); T. T. Chou and C. N. Yang, Phys. Rev. **170**, 1591 (1968).

$\log s$) are plotted. The GeV will be used for the unit of s, t, and p. Data due to Cocconi *et al.*[18] at large s and $|t|$ indicate that the decrease with respect to s at fixed t slows down. More precisely, the data at $|t|=6.0$, for example, indicate that

$$A\propto s^{-3}\text{–}s^{-5} \quad \text{for} \quad s\approx 15\text{–}20\,,$$
$$A\propto s^{0}\text{–}s^{-1} \quad \text{for} \quad s\approx 50\text{–}60\,.$$

In this respect the formula (6.4) is unsatisfactory, because it behaves at fixed t as

$$d\sigma/dt\simeq\sum_{i=1}^{2} C_i \exp[-2(-t)^{1/2}s^{1/2}/g_i]$$

and goes below the data for $s\gtrsim 40$. Of course one may reproduce the data by adding more and more terms in the summation. In this connection we note that the formula (6.6) with $\gamma=1$ shows a power decrease with respect to s at fixed t.[25] While the formula (6.2) shows some systematic errors, the improved one (6.5) fits the data well for $\theta\approx 90°$ in the energy region both below and above the break.

The indication of the data is, therefore, that the large-angle scattering is not dominated by a single singularity at fixed t for the present energies. In other words, we may be now observing, with increasing s, transition between singularities. For fixed t $(0\leq\theta\leq 90°)$ the 90° scattering is realized for the smallest possible physical s. Therefore, the 90° scattering may probe singularities located more to the left than those which the smaller-angle scattering may detect. As is indicated by the rate of decrease of the differential cross section around $\theta\approx 90°$ for fixed t, the 90° scattering $(5\lesssim|t|\lesssim 20)$ is dominated, at present, by singularities located near $j\simeq(-3)\text{–}(-6)$. The residues (or the discontinuities) of these singularities $[j\simeq(-3)\text{–}(-6)]$ must be larger than those of the singularities situated around $-3\lesssim j\lesssim 1$. In order to treat the smaller-angle scattering we need informations about singularities around $-3\lesssim j\lesssim 1$. At present energies, however, the asymptotic expressions given in Secs. 2 and 4 are not applicable and the dominant contributions come from nP and P'(or ω)$+nP$ with $n=1$–3, which introduces a large number of unknown parameters. In the present article, therefore, only the 90° scattering is analyzed. (See Fig. 1.)

The most powerful candidates for singularities near $j\simeq -3\text{–}-6$ $(5\lesssim|t|\lesssim 20)$ are the branch points of the type $(n-m)P'+m\omega$, whose contribution to the scattering amplitude is discussed in the preceding section. Other singularities, for example, the branch points generated by ρ and/or A_2 are neglected for simplicity. We will show below that the singularities represented simply by $j_{\text{eff}}(t)$ can, in fact, explain the rate of decrease with no *ad hoc* parameter.

Now, we examine the expression $C'(-2z_t)^{j_{\text{eff}}(t)}$ with $j_{\text{eff}}(t)$ given in (5.21). We neglect a possible oscillation of C' in the following analysis. For the 90° scattering

$$\begin{aligned}-z_t&=3-4/(p^2+2)\\&\simeq 2.2\text{–}2.8\,,\end{aligned}\tag{6.7}$$

and

$$\ln(-2z_t)\simeq 1.5\text{–}1.7\tag{6.8}$$

for

$$15\lesssim s\lesssim 60\,.$$

Therefore, we get

$$\begin{aligned}[A(s,t)]_{\theta=90°}&\simeq C'(-2z_t)^{j_{\text{eff}}(t)}\\&\simeq C''e^{-\ln(-2z_t)[8c(1-d)]^{1/2}p}\\&\equiv C''e^{-p/2T}\,,\end{aligned}\tag{6.9}$$

where

$$\begin{aligned}T&\simeq\{2\ln(-2z_t)[8c(1-d)]^{1/2}\}^{-1}\\&\simeq[4\ln(-2z_t)]^{-1}\\&\simeq 0.17\text{–}0.15\,,\end{aligned}\tag{6.10}$$

with $c\simeq 1$ and $d\simeq 0.5$ as suggested by various analyses.[26] The parameter T is experimentally [(6.2) and (6.5)]

$$T_0\simeq T_1\simeq 0.158\,.\tag{6.11}$$

Thus the agreement is surprisingly good.

Next we discuss the behavior with respect to t or θ, at fixed s, near $\theta\approx 90°$. The form $C'(-2z_t)^{j_{\text{eff}}(t)}$ with $j_{\text{eff}}(t)$ as given in (5.21) does not fit the angular distribution of the data so well. The behavior seems to be sensitive to the detailed positions of the singularities. For example, we will show that the form $(-2z_t)^{j_{\text{eff}}(t)}$ exhibits an exponential decrease with respect to $\sin\theta$ when we take for $j_{\text{eff}}(t)$ a slightly different form from that given in (5.21). Namely, we take the form

$$j_{\text{eff}}(t)=-a_1(-t)^{3/4}+a_2\,.\tag{6.12}$$

Then for $1\ll s$

$$\begin{aligned}A(s,t)&\simeq C'(-2z_t)^{j_{\text{eff}}(t)}\\&\simeq C'(-2z_t)^{a_2}e^{-a'\psi(z)}\,,\end{aligned}\tag{6.13}$$

where

$$a'=(2p^2)^{3/4}a_1\ln 6\,,\tag{6.14}$$

$$\psi(z)=\frac{(1-z)^{3/4}}{\ln 6}\ln\left(2\frac{3+z}{1-z}\right),\tag{6.15}$$

and

$$|\psi(z)/\sin\theta|=|\psi(z)/(1-z^2)^{1/2}|<1.006\tag{6.16}$$

for $0\leq z\leq 0.3$. We can certainly approximate the envelope of the branch-point trajectories by the form (6.12) in some finite interval of t. Thus the behavior with respect to t or θ seems to be sensitive to the detailed location of singularities. In addition, we must take into account the crossing symmetry $(t\leftrightarrow u)$ in a precise treatment. Here we do not go into such details.

[25] C. B. Chiu, J. Harte, and C. I. Tan, Nuovo Cimento **53**, 174 (1968).

[26] See, for example, C. B. Chiu, S. Y. Chu, and L. L. Wang, Phys. Rev. **161**, 1563 (1967).

7. DISCUSSION

In the above analysis we have not taken into account the modification of the trajectory functions $\alpha(t)$ and $\beta(t)$ due to the presence of branch points. Namely, we have not considered the type of diagrams depicted in Fig. 3. The wavy line represents a Reggeon. According to Gribov and Migdal[27] the contribution of the diagram of Fig. 3 gives, for $|t|\ln s \gg 1$, the asymptotic behavior

$$A(s,t)=Kse^{-c'(-t\ln s)^{1/2}}(-t\ln s)^{1/4}. \tag{7.1}$$

It is interesting to note that this expression shows an exponential decrease with respect to $(-t\ln s)^{1/2}$ in the same way as the expression (2.30) does. The diagram of Fig. 3 may be regarded as that of the three-Reggeon exchange with a particular choice for Reggeon production amplitudes. Therefore, the inclusion of the type of diagrams of Fig. 3 into our analysis may lead to double counting. (In this respect we mention the case where the Pomeranchuk *pole* does not exist, but some other singularity passes through $j=1$ at zero energy. If this primary singularity other than a pole produces a scattering amplitude behaving effectively as $s^{\alpha(t)}$, then the discontinuity formula for the multiple exchange of this singularity is the same as that in the case of the Pomeranchuk pole.)

In Sec. 2 and Sec. 4 it has been shown that the totality of the contributions from the branch points of the type nP (P' or $\omega+nP$) gives an oscillatory behavior to the amplitude of elastic scattering of two identical particles. In cases of elastic scatterings of nonidentical particles and of inelastic scatterings the situation may be different. In these cases the Reggeon production amplitudes on both sides are different and their product may change sign as n varies in contrast to the case of the elastic scattering of two identical particles. Therefore the nature of oscillation may be different for scattering amplitudes of nonidentical particles. In the other respects from above, our argument in the preceding sections applies without change also to scatterings of nonidentical particles.

Next we comment on the works of Gervais and Yndurain.[28] One of their theorems states that the total contribution to the scattering amplitude from the branch points of the type nP or P'(or ω)$+nP$ should

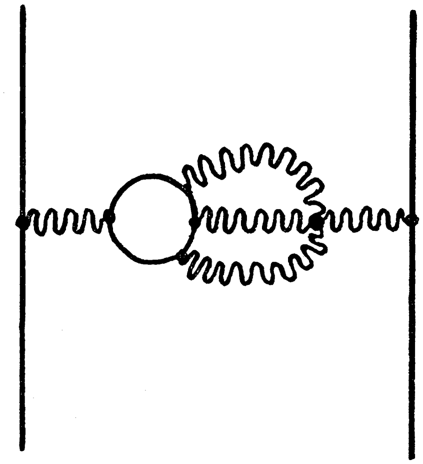

FIG. 3. An example of the diagrams that give rise to modification of the trajectory function. The wavy line represents a Reggeon.

satisfy the following condition:

$$\lim_{s\to\infty}\sup[\ln|A(s,t)|/\ln s]=\alpha(0) \quad\text{or}\quad \beta(0). \tag{7.2}$$

This condition is satisfied by our amplitudes (2.30) and (4.14). However, our result shows that the expansion

$$s^{j(0)}(\ln s)^{k(t)}(\ln\ln s)^{l(t)}\cdots$$

may be inappropriate. Incidentally, their criterion for nonoscillation may not be satisfied in general by the Mandelstam cut contribution.

Finally we discuss on the break[17] observed in the differential cross section of the 90° pp scattering. In the analysis of Sec. 6 we have been able to reproduce the gross feature of the data ($\theta\approx 90°$), but the question concerning the break has been left open. The break appears as a sharp break in the parametrization (6.4). On the other hand, the formula (6.5) fits the data ($\theta>90°$) in the energy region both below and above the break point, namely, there appears no break in the parametrization (6.5). In our plot of the data[17,18] in Fig. 2 ($\log d\sigma/dt$ versus $\log s$) the break may also be viewed as the first manifestation of a possible oscillation with a rather long period. By closely examining the phase function $f(n)$ introduced in (5.9) and estimating the integral (5.14), we might be able to reproduce this oscillatory behavior. However, this program is beyond the scope of the present article. In addition, it is not clear at present whether, at higher energies, the 90° scattering continues to be dominated by $j_{\rm eff}(t)$ or gradually becomes dominated by other singularities.

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[27] V. N. Gribov, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1966*, edited by C. R. Hagen *et al.* (Wiley-Interscience Publishers, Inc., New York, 1967).

[28] J. L. Gervais and F. J. Yndurain, Phys. Rev. Letters **20**, 27 (1968); Phys. Rev. **167**, 1289 (1968); **169**, 1187 (1968).