Even though $g_{\phi\rho\pi}^2/g_{\omega\rho\pi}^2 \cong 10^{-3}$, the $g_{\phi\rho\pi}$ term is not entirely negligible here, because of the interference term, and because $|\beta_N/\alpha_N|$ is somewhat less than 1 for favored values of the mixing angles.

APPENDIX B: POLE-DOMINANCE LIMIT

Coleman and Schnitzer[5] write

$$D(s)=s(1-\epsilon)-M_0^2 \tag{B1}$$

for the inverse propagator of V_μ^3, and

$$D^{-1}(s)=s\begin{pmatrix}1 & \beta \\ \beta & 1+\epsilon\end{pmatrix}-\begin{pmatrix}M_s^2 & 0 \\ 0 & M_0^2\end{pmatrix} \tag{B2}$$

for the inverse 2×2 matrix propagator of V_μ^8 and V_μ^0.

Our version of U_3 symmetry requires

$$M_s^2=M_0^2\equiv M^2, \tag{B3}$$

whereupon we obtain the numerical values

$$M=0.8351 \text{ BeV}, \tag{B4}$$

$$\epsilon=-0.1912, \tag{B5}$$

$$\beta=0.212. \tag{B6}$$

The parameters of interest in Secs. III and IV are then given by

$$Z_\rho^{-1}=1-\epsilon, \tag{B7}$$

$$\alpha_N=\{2[M^2-(1+\epsilon)m_\omega^2]/3\Delta\}^{1/2}, \tag{B8a}$$

$$\alpha_Y=-[4(M^2-m_\omega^2)/3\Delta]^{1/2}, \tag{B8b}$$

$$\beta_N=\{2[(1+\epsilon)m_\phi^2-M^2]/3\Delta\}^{1/2}, \tag{B8c}$$

$$\beta_Y=[4(m_\phi^2-M^2)/3\Delta]^{1/2}, \tag{B8d}$$

where

$$\Delta=(1+\epsilon-\beta^2)(m_\phi^2-m_\omega^2). \tag{B9}$$

Also,

$$\tan^2\theta_Y=\frac{m_\phi^4}{m_\omega^4}\frac{M^2-m_\omega^2}{M_\phi^2-M^2}. \tag{B10}$$

PHYSICAL REVIEW VOLUME 177, NUMBER 5 25 JANUARY 1969

Unitarity, Current Algebra, and s-Wave K^+p Scattering

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(Received 14 August 1968)

Two procedures for unitarizing current-algebra results are studied and applied to the elastic K^+p scattering amplitude. We note that if the current-algebra amplitude in the off-mass-shell limit is used as input, the unitarity correction leads to good agreement with experiments on phase shifts and other low-energy parameters.

I. INTRODUCTION

It is well known that the usual current-algebra calculations of low-energy parameters for scattering processes performed either in the soft-meson limit ($q_\mu\to 0$) or in the off-shell limit ($q_\mu^2\to 0$) need extrapolation in the masses of external mesons together with unitarity corrections. The unitarity corrections (corrections arising because of the extrapolation in the s variable) to current-algebra results have been studied by several authors. Akiba and Kang[1] extrapolated the π-π amplitude using elastic unitarity. They used the dispersion-relation technique and assumed that absorptive parts in a fixed-t dispersion relation are given by the Chew-Mandelstam or by the nonrelativistic effective-range approximation, and estimated the corrections due to unitarity from the symmetry point to the physical threshold. Bhargava, Biswas, Gupta, and Datta[2] (hereafter referred to as BBGD) unitarized the results of soft-pion current-algebra calculations for low-energy s-wave π-N scattering, using the N/D formalism. They replaced the N function in the N/D formalism by a single pole on the negative real axis, and determined the strength of the pole by current algebra in the soft-pion limit. Later, Datta, Gupta, and Varma[3] used off-shell ($q_\pi^2\to 0$) current algebra to obtain a π-π amplitude which is explicitly crossing-symmetric, and unitarized this amplitude to obtain s-wave scattering lengths and phase shifts. They also used the N/D formalism.

Elastic K^+p scattering has been studied within the framework of current algebra by several authors.[4] Roy[4] obtained reasonably good results by including the contribution from the weak-amplitude term. The purpose of this note is to obtain s-wave K^+p phase shifts and other low-energy parameters by applying unitarity

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[1] K. Kang and T. Akiba, Phys. Rev. **164**, 1836 (1967).

[2] S. C. Bhargava, S. N. Biswas, K. C. Gupta, and K. Datta, Phys. Rev. Letters **20**, 558 (1968).

[3] K. Datta, K. C. Gupta, and V. S. Varma, Phys. Rev. **173**, 1549 (1968).

[4] Y. Tomozawa, Nuovo Cimento **46A**, 707 (1967); A. P. Balachandran, G. M. Gundzik, and F. Nicodemi, *ibid.* **44A**, 1257 (1966); P. Roy, Phys. Rev. **162**, 1644 (1967); **172**, 1849(E) (1968).

corrections to the results obtained from current algebra. We will use two methods to study these unitarity corrections in the context of the N/D formalism. First, we use the method proposed by BBGD, where the N function is replaced by a single pole, and determine the strength of the pole from current algebra in the soft-kaon limit; and secondly, the method of Datta *et al.*[3] will be used. Here the input will be the amplitude obtained from current algebra by using PDDAC[5] (pole dominance of the divergence of the axial-vector current) and by including first-order corrections in kaon four-momenta q_K. The magnitudes of the corrections arising from extrapolation in the masses of the external kaons are not clearly known. The problem may be studied using the method of Fubini and Furlan[6]; however, we assume smooth extrapolation in the kaon mass and study corrections arising from unitarity alone to move the threshold to $s=(m_N+m_K)^2$.

Our numerical results allow us to conclude that the unitarizing process in which the N function is replaced by the amplitude from current algebra in the off-shell limit gives significantly better results for low-energy parameters and phase shifts in the low-energy region for K^+p scattering. We note that the good agreement for phase shifts farther away from the threshold gives us a better test of various procedures for unitarizing the results of current algebra, rather than a comparison of low-energy parameters only.

II. s-WAVE K^+p SCATTERING PROCESS

We assume that the off-mass-shell amplitude is an analytic function of s in the complex s plane[7] for each fixed value of q_K^2, and that it has cuts analogous to those imposed on the physical amplitude by unitarity and crossing. Thus for s-wave scattering in the $I=1$ channel, we define

$$g_0(s)=f_0(s)/\rho(s)\,, \tag{2.1}$$

$$f_0(s)=e^{i\delta_0(s)}\sin\delta_0(s)/q\,, \tag{2.2}$$

where $\rho(s)$ is the appropriate phase-space factor, taken as $1/\sqrt{s}$. We then write the amplitude $g_0(s)$ in the form

$$g_0(s)=N(s)/D(s)\,, \tag{2.3}$$

where $N(s)$ has the cuts arising from cross-channel exchanges and $D(s)$ has the unitarity cut. Assuming once-subtracted dispersion relations for the D function, we can write

$$g_0(s)=N(s)\bigg/\left(1+\frac{(s-s_0)}{\pi}\times\int_{s^{\text{th}}}^{\infty}ds'\frac{\operatorname{Im}D(s')}{(s'-s)(s'-s_0)}\right), \tag{2.4}$$

where s_0 is the subtraction point and $s^{\text{th}}=(m_N+m_K)^2$ is the point at which the unitarity cut begins. From elastic unitarity we have

$$\operatorname{Im}D(s)=-\frac{q}{\sqrt{s}}N(s)\,, \tag{2.5}$$

where q is the c.m. momentum in the direct channel, given by

$$q^2=\frac{[s-(m_N+m_K)^2][s-(m_N-m_K)^2]}{4s}\,. \tag{2.6}$$

Thus, from (2.4)–(2.6),

$$g_0(s)=N(s)\bigg/\left(1-\frac{(s-s_0)}{2\pi}\int_{(m_N+m_K)^2}^{\infty}ds'\frac{N(s')\{[s'-(m_N+m_K)^2][s'-(m_N-m_K)^2]\}^{1/2}}{s'(s'-s)(s'-s_0)}\right). \tag{2.7}$$

To estimate the magnitude of the unitarity corrections, we approximate the N function in two ways:

(i) We follow BBGD and approximate the cuts in the N function by a single pole on the negative real s axis:

$$N(s)=\frac{R(q_K^2)}{(s+m_0^2)}\,. \tag{2.8}$$

To find R, we use current algebra in the soft-kaon limit. First, our assumption of smooth continuation in q_K^2 gives us

$$R(q_K^2)\approx R(0)\,.$$

Thus, if we choose $s_0=m_N^2$ as our subtraction point, then the current-algebra amplitude which is evaluated at the

[5] We assume a smooth extrapolation in the mass variable; in that case the off-mass-shell invariant amplitude becomes the physical scattering amplitude, and thus represents the usual Born terms that are fed in as the complete N function in the determinantal approximation (Ref. 9) for the partial-wave amplitude.

[6] S. Fubini and G. Furlan, MIT Report, 1968 (unpublished).

[7] We are aware of the presence of kinematic singularities in this plane. However, the present calculations are done in the spirit of our earlier work, Bhargava *et al.*, Ref. 1; see also V. Singh and B. M. Udgaonkar, Phys. Rev. **128**, 1820 (1962); S. K. Bose and S. N. Biswas, *ibid.* **134**, B635 (1964); D. P. Roy, *ibid.* **136**, B804 (1964).

point $s=m_N^2$ will give us

$$a_c=\operatorname{Re}f_0(s)\,|_{s=m_N^2,\,q_K^2=0}$$
$$=\operatorname{Re}\frac{g_0(s)}{\sqrt{s}}\bigg|_{s=m_N^2,\,q_K^2=0}$$
$$=\frac{R/(m_N^2+m_0^2)}{(m_N^2)^{1/2}}.$$

Thus

$$R=R(0)=a_c m_N(m_N^2+m_0^2)\,, \tag{2.9}$$

where a_c is the current-algebra scattering length. We follow Roy[4] and take

$$a_c^{I=1}=-\frac{1}{2\pi}\frac{m_N}{(m_N+m_K)m_K f_K^2}. \tag{2.10}$$

Finally,

$$\begin{aligned} f_0(s)&=g_0(s)/\sqrt{s}\\ &=a_c m_N[(m_N^2+m_0^2)/(\sqrt{s})(s+m_0^2)]\\ &\bigg/\left(1-\frac{(s-m_N^2)a_c m_N(m_N^2+m_0^2)}{2\pi}\int_{(m_N+m_K)^2}^{\infty}ds'\frac{\{[s'-(m_N+m_K)^2][s'-(m_N-m_K)^2]\}^{1/2}}{s'(s'-m_N^2)(s'-s)(s'+m_0^2)}\right). \end{aligned} \tag{2.11}$$

The pole position m_0^2 is a parameter which we will vary between m_N^2 and $15m_N^2$.

(ii) Following Roy,[4] we evaluate the off-shell current-algebra amplitude by taking into account the first-order corrections in the kaon four-momenta. Thus, we include the contribution from the so-called "weak-amplitude" term.[8] This amplitude has the characteristics of the input forces which would arise in the dynamical theory from crossed-channel exchanges. In the determinantal approximation[9] this input gives the Born term for the N function in a unitarized N/D formalism. Hence we replace the N function by

$$\begin{aligned} N(s)&=g_0(s)\,|_{\mathrm{CA}}\\ &=(\sqrt{s})f_0(s)\,|_{\mathrm{CA}}. \end{aligned} \tag{2.12}$$

$f_0(s)$ is found from current algebra (CA), and is the partial-wave projection of the full amplitude from current algebra using on the mass-shell kinematics. Thus we assume smooth extrapolation in the kaon masses for the invariant amplitudes. Again following Roy,[4] we obtain

$$\begin{aligned} f_0(s)=\frac{1}{4\pi f_K^2 m_K^2}&\left\{\frac{E+m_N}{2W}\left(\frac{\mu_p}{m_N}(\omega E+\tfrac{1}{2}q^2)-(W-m_N)(2+\mu_p)\right)-\frac{\mu_p q^2(E-m_N)}{12Wm_N}\right.\\ &+\left(\frac{g_A(\Lambda)^2}{M_\Lambda-m_N}+\frac{g_A(\Sigma)^2}{M_\Sigma-m_N}-\frac{g_A(Y_0^*)^2}{M_{Y_0^*}+m_N}\right)\left(\frac{E+m_N}{2W}[2\omega E-2m_N(W-m_N)]-\frac{q^2(E-m_N)}{3W}\right)\\ &+\frac{g_A(Y_1^*)^2}{M_{Y_1^*}-m_N}\left\{\frac{E+m_N}{2W}\left[q^2\left(1+\frac{2\omega E}{3M_{Y_1^*}^2}-\frac{2q^2}{3M_{Y_1^*}^2}\right)+\tfrac{2}{3}\omega E-\tfrac{1}{3}(W-m_N)\left(2m_N-\frac{q^2}{M_{Y_1^*}}\right)\right]\right.\\ &\left.\left.-\frac{q^2(E-m_N)}{6W}\left(\frac{W+m_N}{3M_{Y_1^*}}+\frac{2}{3M_{Y_1^*}^2}(q^2-\omega E)-\frac{1}{3}\right)\right\}\right\}. \end{aligned} \tag{2.13}$$

The notation is the same as Roy's.[4] In obtaining (2.13) we have taken into account the contributions of the Λ, Σ^0, $Y_0^*(1405)$, and $Y_1^*(1385)$ intermediate states only, neglecting their widths.

The integral for the D function diverges on account of high-spin exchanges included in the N function; we use a straight cutoff Λ_c, which we vary from $225m_\pi^2$ to $300m_\pi^2$. In this unitarizing procedure the cutoff Λ_c may

[8] We do not include a σ-term contribution. We follow P. Roy (Ref. 4) and ignore the contribution from this term by invoking Adler's consistency condition.

[9] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).

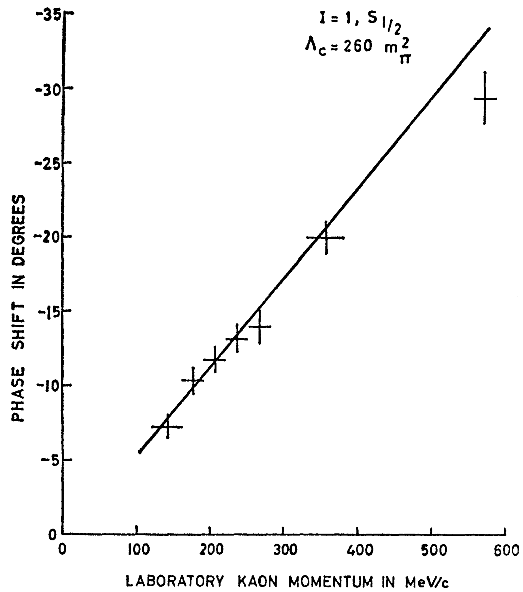

FIG. 1. s-wave K^+p phase shifts in the $I=1$ channel are plotted against the laboratory kaon momenta. The continuous line is our result, and the $+$'s are the experimental points.

be regarded as replacing the pole position which appeared in case (i).

III. RESULTS AND DISCUSSIONS

The scattering length a for the s wave in the $I=1$ channel and the effective range (r_0) are given by

$$\mathrm{Re}\,f_0^{-1}(s)=\mathrm{Re}\left[\left(\frac{g_0(s)}{\sqrt{s}}\right)^{-1}\right]$$
$$=q\cot\delta$$
$$=\frac{1}{a}+\tfrac{1}{2}r_0q^2. \tag{3.1}$$

Hence,

$$a=\mathrm{Re}\frac{g_0(s)}{\sqrt{s}}\bigg|_{q^2=0,\,s=(m_N+m_K)^2}$$

and

$$\tfrac{1}{2}r_0=\frac{\partial}{\partial q^2}\mathrm{Re}\left[\left(\frac{g_0(s)}{\sqrt{s}}\right)^{-1}\right]$$
$$=\left\{\frac{\partial}{\partial s}\mathrm{Re}\left[\left(\frac{g_0(s)}{\sqrt{s}}\right)^{-1}\right]\right\}\frac{\partial s}{\partial q^2}\bigg|_{s=(m_N+m_K)^2,\,q^2=0}. \tag{3.2}$$

TABLE I. s-wave, $I=1$ K^+p scattering length and effective range.

Parameters	N function replaced by a single pole $m_0^2=m_N^2$	$m_0^2=7m_N^2$	N function replaced by off-shell current-algebra amplitude	Experimental[a]
Scattering length a (F)	-0.13	-0.17	-0.28	-0.29 ± 0.02
Effective range r_0 (F)	-0.84	-0.61	0.57	$0.5\ \pm0.15$

[a] Reference 11.

For numerical evaluation, the values of the various coupling constants are taken from Roy (Ref. 4).[10] The values of a and r_0 are given in Table I, and those of the phase shifts in Fig. 1.

From the values of the scattering length and effective range as quoted in Table I, it can be seen that replacing the N function with a single pole is quite unsatisfactory in the present case. It results in a low value for the scattering length and a wrong sign for the effective range, the pole position being taken at $m_0^2=m_N^2$. The value of the scattering length may be improved by changing the pole position to $m_0^2=7m_N^2$, but the effective range again has the wrong sign. The phase shifts obtained also disagree with experiments. However, the numerical results are in good agreement with experiments[11] if we approximate the N function with the full off-mass-shell current-algebra amplitude. The phase shifts thus obtained are also in good agreement with experiments (Fig. 1). The best results are obtained for $\Lambda_c=260m_\pi^2$. Variation of Λ_c between $225m_\pi^2$ and $300m_\pi^2$ will change our results by 15%.

We have obtained the phase shifts up to kaon momenta of 500 MeV/c. For larger kaon momenta, a considerable discrepancy with experimental results appears, as in earlier N/D calculations of KN scattering.[12]

ACKNOWLEDGMENTS

The author is indebted to Professor S. N. Biswas, Dr. S. H. Patil, Dr. K. Datta, and Dr. K. C. Gupta for many helpful discussions and for numerous suggestions on the manuscript. He is also grateful to Professor R. C. Majumdar and Professor F. C. Auluck for their kind interest in this work.

[10] We take: $f_K=0.327$; $g_A(\Lambda)=0.68$; $g_A(\Sigma^0)=0.23$; $g_A(Y_0^*)=0.42$, and $g_A(Y_1^*)=0.67$. The values of $g_A(\Lambda)$ and $g_A(\Sigma^0)$ follow from Cabibbo theory. For $g_A(Y_0^*)$ and $g_A(Y_1^*)$ we use the widths of the decays $Y_0^*\to\Sigma^0\pi^0$ and $Y_1^*\to\Lambda\pi^0$, the Goldberger-Treiman relation, and $SU(3)$. For further details see Roy (Ref. 4).

[11] S. Goldhaber, W. Chinowsky, G. Goldhaber, W. Lee, T. Halloran, T. F. Stubbs, G. M. Pjerrou, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **9**, 135 (1962).

[12] S. C. Agarwal, Phys. Rev. **145**, 1196 (1966).