Continuum Contributions to Vector-Meson Decays. II. Broken U_3 Symmetry

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We have constructed a model for estimating the continuum contributions to the propagators of the vector mesons ρ, ω, and ϕ. With this model, we compute vector-meson decay rates predicted by a model of octet-broken U_3 symmetry based on spectral-function sum rules for the propagators. The model is consistent with experiment, but present experimental accuracy is not sufficient to distinguish clearly between the pole-dominance form of the model and the model with continuum included.

I. INTRODUCTION

In most calculations involving single-particle states, it has been customary to make the narrow-resonance (or pole-dominance) approximation to the particle propagator, including in the inverse propagator a simple imaginary part when required to take into account the finite width of the particle. In the present work, we construct a more detailed model of the continuum contributions to the propagators of the $Y=0$ members of the vector-meson nonet. In this model the continuum (in the inverse propagator) is represented by a dispersion integral, with absorptive part given by the product of a known kinematical factor and a smooth cutoff function. In addition, we make a simplifying assumption about the contribution of the three-pion continuum to the 2×2 matrix inverse propagator which describes ω and ϕ; this assumption is motivated by the observed suppression of the decay $\phi\to\rho+\pi$.

With this model, we compute corrections to the pole-dominance approximation for the decays

(1) neutral vector meson $\to$ lepton pairs,

(2) $\phi\to K+\bar{K}$,

(3) $\omega,\phi\to\pi^0+\gamma$,

(4) $\pi^0\to\gamma+\gamma$.

As it turns out, the effect of the continuum are not dramatic (of order 25% at most), but large enough that pole dominance can give a distorted view. Especially in view of the colliding-beam experiments and the possibility of the precise measurements of some of the abovementioned decay rates, we think that these effects are important to know.

We are concerned particularly with the predictions of broken U_3 and SU_3 symmetry, which are obtained from our model by imposing two conditions: (i) The spectral functions of the propagators are required to satisfy octet-broken U_3 sum rules of the type discussed recently by Das, Mathur, and Okubo[1] and by Oakes and Sakurai,[2] and (ii) vertex functions which appear are taken to have minimal dependence on momentum, and the associated coupling constants are required to satisfy exact SU_3 symmetry.

We find that the symmetry model thus constructed leads to predictions consistent with experiment, provided that (i) we choose the symmetry-breaking condition which corresponds to a Gell-Mann–Okubo-type relation between the inverse masses (squared) of the vector nonet, rather than between the masses (squared), and (ii) the ρ-meson width is between 110 and 130 MeV. We also find that the continuum effects are not large enough to alter the conclusions of Sakurai[2] that Weinberg's second sum rule[3] (with no symmetry breaking) for the SU_3 spectral functions cannot bemaintained.

In Sec. II, we discuss the spectral-function sum rules for the propagators of a set of vector fields $V_\mu{}^a$ ($a=0,1,\cdots,8$) which are assumed to be proportional (with constant of proportionality independent of a) either to a set of currents which satisfy the U_3 algebra of currents or to a set of fields which satisfy the U_3 algebra of fields. We characterize broken U_3 symmetry in terms of these sum rules.

We then introduce our model for the vector-meson propagators; in Sec. III, we give our ρ-meson propagator, and in Sec. IV, we give the 2×2 matrix propagator which describes ω and ϕ. In Sec. V, the numerical results of the model are presented. In Appendix A, we collect together the formulas for the decay rates predicted by our model; in Appendix B, we discuss the pole-dominance limit.

II. PROPAGATIONS AND SPECTRAL-FUNCTION SUM RULES

Consider the propagators

$$\Delta_{\mu\nu}{}^{ab}(q)=F^{ab}(q^2)g_{\mu\nu}-G^{ab}(q^2)q_\mu q_\nu \qquad (2.1)$$

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[1] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967).
[2] J. J. Sakurai, Phys. Rev. Letters **19**, 803 (1967); R. J. Oakes and J. J. Sakurai, *ibid.* **19**, 1266 (1967).
[3] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

for a set of vector fields $V_\mu{}^a$ ($a=0, 1, \cdots, 8$) which are supposed to be proportional (with constant of proportionality independent of a) either to a set of currents which satisfy the U_3 current algebra or to a set of fields which satisfy the U_3 algebra of fields.

$F^{ab}(s)$ and $G^{ab}(s)$ have the standard spectral representations

$$F^{ab}(s)=\frac{1}{\pi}\int\frac{\sigma_1{}^{ab}(\kappa)}{s-\kappa}d\kappa\,, \tag{2.2}$$

$$G^{ab}(s)=\frac{1}{\pi}\int\frac{1}{s-\kappa}\left(\frac{\sigma_1{}^{ab}(\kappa)}{\kappa}+\sigma_0{}^{ab}(\kappa)\right)d\kappa\,, \tag{2.3}$$

where the spin-zero spectral function $\sigma_0{}^{ab}(\kappa)$ is not present for the fields which are coupled to the conserved currents.

Weinberg's first sum rule[3] requires

$$\int\left(\frac{\sigma_1{}^{ab}(\kappa)}{\kappa}+\sigma_0{}^{ab}(\kappa)\right)d\kappa=C\delta^{ab}. \tag{2.4}$$

In the present context, the propagators that we construct are normalized according to Eq. (2.4) to ensure that these propagators indeed describe fields which are proportional to currents (or fields) satisfying the U_3 algebra. In particular, we *define* the normalization of the field $V_\mu{}^0$ by the condition (2.4) and a baryon current

$$N_\mu\sim(\tfrac{2}{3})^{1/2}V_\mu{}^0. \tag{2.5}$$

U_3 symmetry is implemented by imposing relations between the coupling constants of $V_\mu{}^0$ thus normalized and those of the $V_\mu{}^a$ ($a=1, \cdots, 8$).

We assume that SU_3 is exact for $s\to\infty$, in particular, that[4]

$$\lim_{s\to\infty} sF^{08}(s)=0. \tag{2.6}$$

We should also like to require that the breaking of U_3 be pure octet. This requirement can be imposed in one of two (inequivalent) forms[1]:

(a) Octet breaking for $q^2\to\infty$: The spectral functions satisfy the sum rules

$$\int\rho^{ab}(\kappa)d\kappa=S\delta^{ab}+S'c^{ab}\,, \tag{2.7a}$$

where $c^{ab}=d^{ab8}$ ($a,b=1, \cdots, 8$) and 0 otherwise.

(b) Octet breaking for $q^2\to 0$: The spectral functions satisfy the sum rules

$$\int\frac{\rho^{ab}(\kappa)}{\kappa^2}d\kappa=A\delta^{ab}+A'c^{ab}+B(\delta^{a0}\delta^{b8}+\delta^{a8}\delta^{b0}). \tag{2.7b}$$

An octet-singlet mixing term, proportional to $(\delta^{a0}\delta^{b8}+\delta^{a8}\delta^{b0})$, is not present in (2.7a) due to the assumption (2.6).

That the underlying symmetry is supposed to be U_3 rather than SU_3 is indicated by the absence of a term proportional to $\delta^{a0}\delta^{b0}$ in the sum rules. We do not always insist that the octet-broken U_3 sum rules be satisfied, but we find that (2.7b), in particular, is not inconsistent with experiment. Octet breaking of SU_3 alone we do not check, since we do not consider here the K^* (moreover, the sum rules for nonconserved currents have additional, unknown contributions from spin-zero states).

In the narrow-resonance approximation, (2.7b) corresponds to octet breaking of U_3 in the current-mixing model of Coleman and Schnitzer.[5] Note, however, that (2.6) cannot be satisfied in this approximation, since it would lead to no mixing at all.[6]

III. ρ-MESON PROPAGATOR

For the inverse propagator of $V_\mu{}^3$ we write

$$[F^{33}(s)]^{-1}\equiv D(s)=s[1+\pi(s)]-M^2\,, \tag{3.1}$$

with

$$\pi(s)=\frac{s}{\pi}\int_{4m_\pi{}^2}^{\infty}\frac{\rho(s')}{s'^2(s'-s)}ds'\,, \tag{3.2}$$

and choose the phenomenological form[7]

$$\rho(s)=\lambda\frac{(s-4m_\pi{}^2)^{3/2}}{s^{1/2}}\left(\frac{\alpha^2}{\nu+\alpha^2}\right)^2, \tag{3.3}$$

where $\nu=s/4m_\pi{}^2-1$. In S-matrix language, $D(s)$ is a phenomenological D function for p-wave π-π scattering, in which the left-hand cut of the partial-wave amplitude has been approximated by a double pole, and the influence on D of inelastic cuts has been approximated by a linear function of s.[8]

For fixed cutoff parameter α^2, the constants λ and M^2 are adjusted to fit the mass and width of the ρ, defined by

$$\mathrm{Re}D(m_\rho{}^2)=0\,, \tag{3.4}$$

$$\Gamma_\rho=\frac{\mathrm{Im}D(m_\rho{}^2)}{m_\rho\,\mathrm{Re}D'(m_\rho{}^2)}. \tag{3.5}$$

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[4] It can, of course, always be arranged for the 2×2 matrix $\lim_{s\to\infty} sF^{ab}(s)$ ($a,b=0,8$) to be diagonal along with $F^{ab}(0)$; denote the fields for which this is the case by $\tilde{V}_\mu{}^0$ and $\tilde{V}_\mu{}^8$. The physical content of the assumption is that (i) $\tilde{V}_\mu{}^0$ and $\tilde{V}_\mu{}^8$ are the singlet and octet fields (proportional to the baryon and hypercharge currents, respectively) in the *physical* U_3 algebra, or (ii) $\tilde{V}_\mu{}^8$ couples directly to the isoscalar electromagnetic current and $\tilde{V}_\mu{}^0$ does not. We find below that this assumption is consistent with experiment; an alternative, in which $d[F^{ab}(s)]/ds$ is diagonal at $s=0$ and a mixing term is allowed in Eq. (2.7a), is not.

[5] S. Coleman and H. Schnitzer, Phys. Rev. **134**, B863 (1964); N. Kroll, T. D. Lee, and B. Zumino, *ibid.* **157**, 1376 (1967). The U_3 version is described in Appendix B.

[6] Thus our model can be regarded as an attempt to understand ω-ϕ mixing entirely in terms of low-mass intermediate states.

[7] For $\alpha^2\to\infty$, this reduces to the propagator considered by B. W. Lee and M. T. Vaughn, Phys. Rev. Letters **4**, 578 (1960); see also W. R. Frazer and J. Fulco, *ibid.* **2**, 365 (1959); **117**, 1603 (1960); **117**, 1609 (1960).

[8] The K-$\bar{K}$ cut in particular is small, and its influence is well approximated in this way for the energies under consideration.

TABLE I. Computed values of dimensionless coupling constants and branching ratios for several values of α^2 and Γ_ρ. Here $\Gamma(\omega\to 3\pi)$ is taken to be 10 MeV, $\Gamma(\phi\to 3\pi)$ to be 0.4 MeV.

Γ_ρ (MeV)	α^2	Z_ρ	M (MeV)	a_{11}	$g_{\omega\rho\pi}^2/4\pi$	$\Gamma(\omega\to\pi^0\delta)/\Gamma(\omega\to 3\pi)$	$g_{\phi\rho\pi}^2/4\pi$	$\Gamma(\phi\to\pi^0\gamma)/\Gamma(\phi\to 3\pi)$
100	5.0	1.220	702.7	0.1338	0.3363	0.1767	0.00042	0.0126
	10.0	1.176	722.4	0.0665	0.3528	0.1854		
	20.0	1.142	735.5	0.0422	0.3644	0.1915		
	50.0	1.106	746.3	0.0306	0.3710	0.1949		
	100.0	1.086	751.3	0.0272	0.3723	0.1956		
	200.0	1.073	754.6	0.0257	0.3725	0.1957		
	500.0	1.062	757.1	0.0247	0.3723	0.1956		
	(Pole model)	1.0	765.0	0.0248				
120	5.0	1.264	692.3	0.1611	0.2780	0.1217	0.00039	0.0099
	10.0	1.212	715.2	0.0795	0.2935	0.1285		
	20.0	1.170	730.3	0.0503	0.3045	0.1333		
	50.0	1.127	742.9	0.0365	0.3108	0.1361		
	100.0	1.104	748.8	0.0325	0.3120	0.1366		
	200.0	1.088	752.6	0.0306	0.3121	0.1367		
	500.0	1.075	755.6	0.0295	0.3119	0.1366		
	(Pole model)	1.0	765.0	0.0297				

The renormalized inverse propagator $D_\rho(s)$ of the ρ is characterized by the requirement

$$\mathrm{Re}D'(m_\rho^2)=1. \tag{3.6}$$

Evidently,

$$D_\rho(s)=Z_\rho D(s), \tag{3.7}$$

where

$$Z_\rho^{-1}=\mathrm{Re}D'(m_\rho^2). \tag{3.8}$$

The p-wave π-π scattering phase shift is given by

$$\tan\delta(s)=-\mathrm{Im}D(s)/\mathrm{Re}D(s), \tag{3.9}$$

from which we can compute the p-wave scattering length

$$a_{11}=\lim_{s\to 4m_\pi^2}\left[\frac{4m_\pi^2 s^{1/2}}{(s-4m_\pi^2)^{3/2}}\tan\delta(s)\right]. \tag{3.10}$$

We give in Table I the p-wave scattering length computed for various values of α^2 and Γ_ρ, together with the scattering length

$$a_{11}=\frac{4m_\rho^2 m_\pi^2\Gamma_\rho}{(m_\rho^2-4m_\pi^2)^{5/2}}, \tag{3.11}$$

deduced by extrapolating a p-wave Breit-Wigner formula to threshold. We remark that hard-pion current-algebra calculations[9] (which are based on pole dominance) give $a_{11}\sim 0.030-0.035$ for $\Gamma_\rho\sim 115$–120 MeV. We also give the values of M and Z_ρ computed for these values of α^2.

We use the modified ρ-meson propagator to extract numerical values for the ω-ρ-π and ϕ-ρ-π coupling constants, and predict the branching ratios for $\omega\to\pi^0+\gamma$ and $\phi\to\pi^0+\gamma$ according to the model of Gell-Mann, Sharp, and Wagner.[10] We write the V-ρ-π vertex as

$$(g_{V\rho\pi}/m_\pi)\epsilon_{\kappa\lambda\mu\nu}p_\kappa{}^V p_\lambda{}^\pi, \tag{3.12}$$

where $V=\omega,\phi$. Then the decay rate for $V\to 3\pi$ is given by

$$\Gamma(V\to 3\pi)=(f_\rho^2/4\pi)(g_{V\rho\pi}^2/4\pi)\rho_{3\pi}(m_V^2)/24\pi m_V^2 m_\pi, \tag{3.13}$$

where $\rho_{3\pi}(s)$ is the 3π phase-space integral computed using the inverse ρ-meson propagator $Z_\rho D(s)$ and including interference between ρ bands, and $f_\rho^2/4\pi$ is determined from the ρ width (we neglect momentum dependence at the ρ-π-π vertex beyond that contained in the propagator). The decay rate for $V\to\pi^0+\gamma$ is given by

$$\Gamma(V\to\pi^0\gamma)=\alpha(f_\rho^2/4\pi)^{-1}(g_{V\rho\pi}^2/4\pi)(m_V^2-m_\pi^2)^3/16m_\pi^2 m_V^3, \tag{3.14}$$

where $\alpha\cong 1/137$.

Numerical results for the dimensionless coupling constants $g_{V\rho\pi}^2/4\pi$ and for the branching ratios $\Gamma(V\to\pi^0\gamma)/\Gamma(V\to 3\pi)$ are given in Table I for several values of α^2 and Γ_ρ; in this table, $\Gamma(\omega\to 3\pi)$ is taken to be 10 MeV and $\Gamma(\phi\to 3\pi)$ is taken to be 0.4 MeV. It is evident that, within the framework of the Gell-Mann–Sharp–Wagner model and the current experimental value $(10.5\pm 1.0)\%$ [11] for the ratio of $\omega\to\pi^0\gamma$ to $\omega\to 3\pi$, a ρ width somewhat greater than 120 MeV is preferred.

The modified ρ propagator given here is also relevant to the pion electromagnetic form factor and the ρ decay into lepton pairs; we have discussed this elsewhere.[12]

IV. ω-ϕ PROPAGATOR

For the inverse propagator of fields proportional to the hypercharge and baryon currents Y_μ and N_μ,

[9] R. Arnowitt, M. Friedman, P. Nath, and R. Suitor, Phys. Rev. Letters **20**, 475 (1968); Phys. Rev. **175**, 1802 (1968); **175**, 1820 (1968); this work contains references to earlier calculations.

[10] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[11] A. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[12] M. T. Vaughn and K. C. Wali, Phys. Rev. Letters **21**, 938 (1968); see also G. J. Gounaris and J. J. Sakurai, *ibid.* **21**, 244 (1968).

respectively, we write

$$\mathbf{D}(s)=\mathbf{S}D_0(s)\mathbf{S}^T =\begin{pmatrix} F^{NN}(s) & F^{NY}(s) \\ F^{YN}(s) & F^{YY}(s)\end{pmatrix}^{-1}, \qquad (4.1)$$

where $\mathbf{S}$ is a renormalization matrix to be discussed below. $\mathbf{D}_0(s)$ has the form

$$\mathbf{D}_0(s)=s[1+\boldsymbol{\pi}(s)]-\mathbf{M}_0^2, \qquad (4.2)$$

where

$$\mathbf{M}_0^2=\begin{pmatrix} M_{0N}^2 & 0 \\ 0 & M_{0Y}^2\end{pmatrix} \qquad (4.3)$$

and $\boldsymbol{\pi}(s)$ is given by a dispersion integral

$$\boldsymbol{\pi}(s)=\frac{1}{\pi}\int\frac{\boldsymbol{\varrho}(s')}{s'(s'-s)}ds'. \qquad (4.4)$$

In this dispersion integral, we wish to include phenomenologically the 3π cut (which is dominated by $\rho+\pi$ for the $I=0$ states under consideration) and the K-$\bar{K}$ cut (which, according to SU_3, enters only in the Y-Y element of $\mathbf{D}_0$). Hence we write

$$\rho_{NN}(s)=\lambda_N^2\rho_{3\pi}(s)\left(\frac{s_0}{s+s_0}\right)^n, \qquad (4.5a)$$

$$\rho_{YN}(s)=\lambda_Y\lambda_N\rho_{3\pi}(s)\left(\frac{s_0}{s+s_0}\right)^n=\rho_{NY}(s), \qquad (4.5b)$$

$$\rho_{YY}(s)=\lambda_Y^2\rho_{3\pi}(s)\left(\frac{s_0}{s+s_0}\right)^n +\xi\frac{(s-4m_K^2)^{3/2}}{s^{1/2}}\left(\frac{\alpha_K^2}{\nu_K+\alpha_K^2}\right)^2, \qquad (4.5c)$$

with θ functions vanishing below the appropriate threshold implied.

In the K-$\bar{K}$ term in $\rho_{YY}(s)$,

$$\nu_K=s/4m_K^2-1,$$

and we choose $\alpha_K^2=m_\pi^2\alpha^2/m_K^2$, where α^2 is the cutoff parameter in the π-π integral of Sec. III. This choice of α_K^2 corresponds to choosing the same location for the approximate left-hand singularity in π-π and K-$\bar{K}$ scattering; since ξ and λ are varied independently, our results should not be very sensitive to this choice.

$\rho_{3\pi}(s)$ is the phase space for the decay of an $I=0$ vector meson of mass $s^{1/2}$ to three pions via the $\rho+\pi$ intermediate state, computed according to the model of Gell-Mann, Sharp, and Wagner,[10] using the ρ propagator of Eq. (3.1). The cutoff is now mandatory, since $\rho_{3\pi}(s)\sim s^{5/2}$ for $s\to\infty$; for numerical convenience, we generally choose $n=3$, which makes the integral $O(s^{-1})$ for $s\to\infty$, and allow the cutoff parameter s_0 to vary.[13]

The renormalization matrix $\mathbf{S}$ is to be chosen so that the propagators satisfy Weinberg's first sum rule, Eq. (2.4). If

$$V_\mu{}^3=aT_\mu{}^3, \qquad (4.6a)$$

then we wish to have

$$V_\mu{}^8=(\tfrac{3}{4})^{1/2}aY_\mu,\quad V_\mu{}^0=(\tfrac{3}{2})^{1/2}aN_\mu \qquad (4.6b)$$

($a=$const); it is thus appropriate to take

$$\mathbf{S}=\begin{pmatrix} (\frac{3}{2})^{1/2}M_{0N}/M & 0 \\ 0 & (\frac{3}{4})^{1/2}M_{0Y}/M\end{pmatrix}, \qquad (4.7)$$

with M from Eq. (3.1).

For fixed cutoff parameters s_3 and α^2, the parameters M_{0Y}, M_{0N}, λ_Y, λ_N, and ξ are determined from the mass and width of the ω and the mass and partial widths, for decay to $K+\bar{K}$ and $\rho+\pi$, of the ϕ. The relevant equations are

$$\det[\mathrm{Re}\mathbf{D}(m_\omega^2)]=0=\det[\mathrm{Re}\mathbf{D}(m_\phi^2)], \qquad (4.8)$$

$$\Gamma^\omega=\frac{\mathrm{Im}[\det\mathbf{D}(m_\omega^2)]}{m_\omega\det'[\mathrm{Re}\mathbf{D}(m_\omega^2)]}, \qquad (4.9a)$$

$$\Gamma_{K\bar{K}}{}^\phi=\frac{\mathrm{Im}_K[\det\mathbf{D}(m_\phi^2)]}{m_\phi\det'[\mathrm{Re}\mathbf{D}(m_\phi^2)]}, \qquad (4.9b)$$

$$\Gamma_{3\pi}{}^\phi=\frac{\mathrm{Im}_{3\pi}[\det\mathbf{D}(m_\phi^2)]}{m_\phi\det'[\mathrm{Re}\mathbf{D}(m_\phi^2)]}, \qquad (4.9c)$$

where det′ means derivative of the determinant, and $\mathrm{Im}_{3\pi}$ and Im_K refer to the imaginary parts due to the 3π and K-$\bar{K}$ cuts, respectively.

To express the physical ω and ϕ fields in terms of the hypercharge and baryon currents, we write[14]

$$aY_\mu=\alpha_Y\omega_\mu+\beta_Y\phi_\mu, \qquad (4.10a)$$

$$aN_\mu=\alpha_N\omega_\mu+\beta_N\phi_\mu \qquad (4.10b)$$

[a is from Eq. (4.6)], and introduce the "residue matrices" $\mathbf{R}^\omega$ and $\mathbf{R}^\phi$ of the propagator according to

$$\mathbf{R}^\omega=\lim_{q^2\to m_\omega^2}(q^2-m_\omega^2)[\mathrm{Re}\mathbf{D}(q^2)]^{-1}, \qquad (4.11a)$$

$$\mathbf{R}^\phi=\lim_{q^2\to m_\phi^2}(q^2-m_\phi^2)[\mathrm{Re}\mathbf{D}(q^2)]^{-1}. \qquad (4.11b)$$

$\mathbf{R}^\omega$ and $\mathbf{R}^\phi$ are expressed in terms of the coefficients

[13] That the cutoff parameter is the same in all the matrix elements is a simplifying assumption; it leads to a simple qualitative understanding of the absence of the decay $\phi\to\rho+\pi$.

[14] As in Kroll, Lee, and Zumino (Ref. 5).

TABLE II. Computed values of decay rates of ω and φ, branching ratios (B_ω,B_ϕ), and mixing angles (θ_Y,θ_N) in various broken-symmetry models.

Model	$\Gamma(\phi\to K\bar{K})$ (MeV)	$\Gamma(\omega\to e^+e^-)$ (keV)	B_ω ($\times10^5$)	$\Gamma(\phi\to e^+e^-)$ (keV)	B_ϕ ($\times10^4$)	θ_Y (deg)	θ_N (deg)	$\Gamma(\pi^0\to 2\delta)$ [a] (eV)	$\Gamma(\pi^0\to 2\delta)$ [b] (eV)
Pole dominance; octet-broken SU_3 (Coleman-Schnitzer)	4.98	0.40	3.3	1.15	2.31	34.0	21.7	6.5	7.18 5.82
Pole dominance; octet-broken U_3	4.36	0.56	4.65	1.03	2.36	40.2	26.4	6.28	6.70 5.66
Octet-broken U_3; $\alpha^2=50$, $s_0=0.230$	3.17	0.31	2.55	0.83	2.61	34.9	15.2	4.77	5.52 4.08
Octet-broken U_3; $\alpha^2=100$, $s_0=0.224$	2.91	0.34	2.87	0.75	2.58	37.7	15.8	5.47	6.28 4.70
Octet-broken U_3[c]; $\alpha^2=100$, $s_0=0.253$	0.47	1.74	14.4	0.12	2.55	76.8	67.6	28.2	28.7 27.8

[a] Neglecting $\phi\to\rho+\pi$.
[b] With $\Gamma(\phi\to\rho+\pi)=0.4$ MeV; two values are included due to the sign ambiguity in $f_{\phi\rho\pi}/f_{\omega\rho\pi}$.
[c] Using sum rule (2.7a); the two preceding rows use the sum rule (2.7b).

appearing in (4.10) as

$$\mathbf{R}^\omega=\begin{pmatrix}\alpha_N^2 & \alpha_Y\alpha_N\\ \alpha_Y\alpha_N & \alpha_Y^2\end{pmatrix}, \tag{4.12a}$$

$$\mathbf{R}^\phi=\begin{pmatrix}\beta_N^2 & \beta_Y\beta_N\\ \beta_Y\beta_N & \beta_Y^2\end{pmatrix}, \tag{4.12b}$$

whence we can compute α_Y, α_N, β_Y, and β_N from the residue matrices.

We can also determine the mixing angles θ_Y and θ_N introduced by Kroll, Lee, and Zumino[5]; we have

$$\frac{m_\omega^4}{m_\phi^4}\tan^2\theta_Y=\frac{R_{YY}^\omega}{R_{YY}^\phi}, \tag{4.13a}$$

$$\frac{m_\phi^4}{m_\omega^4}\tan^2\theta_N=\frac{R_{NN}^\phi}{R_{NN}^\omega}. \tag{4.13b}$$

Even in our model, $\tan^2\theta_Y$ is determined from the ratio of decay rates of ω and ϕ into lepton pairs (we assume that the coupling of $V_\mu{}^8$ to the photon does not change from $m_\omega{}^2$ to $m_\phi{}^2$); however, the relation

$$\tan\theta_Y=(m_\phi{}^2/m_\omega{}^2)\tan\theta_N \tag{4.14}$$

is not valid, since the inverse propagator is not well approximated by a linear function of s on $m_\omega{}^2<s<m_\phi{}^2$.

V. NUMERICAL RESULTS AND DISCUSSION

With the aid of the formulas developed in Appendix A, we can compute rates for various vector-meson decays and, following the model due to Gell-Mann, Sharp, and Wagner,[10] the decay $\pi^0\to2\gamma$. In this computation, we take all vertex functions to have minimal momentum dependence, and assume SU_3 relations between the V^3-π-π and V^8-K-$\bar{K}$ coupling constants.

When we adjust the cutoff parameter in the three-pion integral to satisfy one or the other of the octet-broken U_3 sum rules, Eq. (2.7), we obtain typical results as shown in Table II, along with the corresponding results in the pole-dominance approximation. We remark that the sensitivity of the predictions of our version of octet-broken U_3 to the cutoff parameter α^2 in the ρ propagator is not large, provided that we require a reasonable p-wave π-π scattering length. The sensitivity of the over-all effects of the continuum to α^2 is even less, if we hold fixed the mixing angle θ_Y (abandoning the octet-broken U_3 sum rules).

The most striking result is that the predicted rate of $\phi\to K+\bar{K}$ is substantially reduced from the pole-dominance prediction to bring it into excellent agreement with the experimental result[11]

$$\Gamma_{\text{expt}}(\phi\to K\bar{K})=3.0\pm0.8\text{ MeV}.$$

The predicted rate of ϕ decay to lepton pairs is 20~25% lower than that predicted by pole dominance[15]; experimental results available at present favor larger values of the rate,[16] but are subject to large uncertainties. The predicted rate of ω decay to lepton pairs is rather sensitive to the mixing angle θ_Y, but is again ~20% lower than the rate predicted by pole dominance. Experimental determination of the rate will be complicated by ρ-ω interference[17]; we hope to discuss this problem elsewhere.

The prediction of the rate of $\pi^0\to2\gamma$ is complicated by the presence of the decay through virtual $\phi+\rho$. Even though, as indicated in Table I, $(f_{\phi\rho\pi}/f_{\omega\rho\pi})^2\sim10^{-3}$, interference between $\omega+\rho$ and $\phi+\rho$ states involves the ratio $(f_{\phi\rho\pi}\cos\theta_Y)/(f_{\omega\rho\pi}\sin\theta_Y)$, which is not entirely negligible, as can be seen from Table II. If the sign of this ratio could be determined independently, a more accurate measurement of the π^0 lifetime would be a

[15] Kroll, Lee, and Zumino (Ref. 5) give $\Gamma(\phi\to e^+e^-)=2.2$ keV. They used the experimental width of $\phi\to K\bar{K}$ rather than the SU_3 predicted rate, which explains the difference between their number and the entry in Table II.
[16] R. G. Astvacurov *et al.*, Phys. Letters **27B**, 45 (1968); D. M. Binnie *et al.*, *ibid.* **27B**, 106 (1968); D. Bollini *et al.*, CERN Report, 1968 (unpublished).
[17] R. G. Parsons and R. Weinstein, Phys. Rev. Letters **20**, 1314 (1968).

useful test of our model, since it is quite sensitive to the continuum contributions.

In these calculations, we have used a ρ width $\Gamma_\rho=120$ MeV; the dependence of the computed quantities on Γ_ρ and on θ_Y is given roughly by

(i) $\Gamma(\rho\to l^+l^-)\propto\Gamma_\rho^{-1}$,

(ii) $\Gamma(\omega\to\pi^0\gamma)/\Gamma(\omega\to3\pi)\propto\Gamma_\rho^{-2}$,

(iii) $\Gamma(\phi\to K\bar{K})\propto\Gamma_\rho\cos^2\theta_Y$,

(iv) $B_\omega\propto\Gamma_\rho^{-1}\sin^2\theta_Y$,

(v) $B_\phi\propto\Gamma_\rho^{-2}$,

(vi) $\Gamma(\pi^0\to2\gamma)\propto\Gamma_\rho^{-3}\sin^2\theta_Y$.

Of these, (i) and (ii) depend only on the ρ propagator (and, of course, the ρ-π-π vertex), and thus are predicted more reliably than are (iii)–(vi).

As mentioned in Sec. III, the present experimental value of $\Gamma(\omega\to\pi^0\gamma)$ does not permit Γ_ρ to be much smaller than 120 MeV. Within that constraint, it is a success of our model to reduce the predicted rate of $\phi\to K\bar{K}$, and our model is consistent with other experimental results [although only marginally so with $\Gamma(\phi\to e^+e^-)$; more precise measurements are clearly needed here]. We must add, however, that the U_3 version of the Coleman-Schnitzer model[5] is already an improvement over the SU_3 version[18]; present experimental evidence does not necessarily require continuum contributions to maintain U_3 symmetry.

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APPENDIX A: DECAY RATES

We introduce the coupling constants f_3 and f_8 of the fields $V_\mu{}^3$ and $V_\mu{}^8$ to the corresponding conserved currents, and the coupling constants γ_3 and γ_8 of the fields to the photon. We assume the SU_3 relations

$$f_8=(\tfrac{3}{4})^{1/2}f_3\,, \tag{A1}$$

$$\gamma_8=(\tfrac{1}{3})^{1/2}\gamma_3\,, \tag{A2}$$

and suppose that the vertices are independent of momentum (the momentum dependence due to the lowest-mass intermediate states is at least approximately taken into account by using the modified propagators). We then obtain expressions for various decay rates in terms of these constants and the parameters introduced in Secs. III and IV.

$\rho\to\pi+\pi$

$$\Gamma(\rho\to2\pi)=\frac{f_\rho{}^2}{48\pi}\frac{(m_\rho{}^2-4m_\pi{}^2)^{3/2}}{m_\rho{}^2}\,, \tag{A3}$$

where

$$f_\rho=Z_\rho{}^{1/2}f_3\,. \tag{A4}$$

$V\to l^++l^-$

For vector-meson decay into a lepton pair, we have

$$\Gamma(V\to l^+l^-)=\tfrac{1}{3}\alpha\left(\frac{\gamma_V}{m_V{}^2}\right)^2\left(1+\frac{2m^2}{m_V{}^2}\right)\left(1-\frac{4m^2}{m_V{}^2}\right)^{1/2}m_V\,, \tag{A5}$$

where m_V is the vector-meson mass, m is the lepton mass, and

$$\gamma_\rho=-Z_\rho{}^{1/2}eM^2/f_3=Z_\rho{}^{1/2}\gamma_3\,, \tag{A6}$$

$$\gamma_\omega=\alpha_Y(\tfrac{3}{4})^{1/2}\gamma_8=\tfrac{1}{2}\alpha_Y\gamma_3\,, \tag{A7a}$$

$$\gamma_\phi=\beta_Y(\tfrac{3}{4})^{1/2}\gamma_8=\tfrac{1}{2}\beta_Y\gamma_3\,. \tag{A7b}$$

$\phi\to K+\bar{K}$

$$(\phi\to K\bar{K})=\frac{f_\phi{}^2}{48\pi}\left(\frac{(m_\phi{}^2-4m_+{}^2)^{3/2}}{m_\phi{}^2}+\frac{(m_\phi{}^2-4m_0{}^2)^{3/2}}{m_\phi{}^2}\right), \tag{A8}$$

where m_+ is the $K^\pm$ mass, m_0 is the K^0 mass, and

$$f_\phi=\tfrac{3}{4}\beta_Yf_3=\tfrac{3}{4}\beta_YZ_\rho{}^{-1/2}f_\rho\,. \tag{A9}$$

$\omega,\phi\to3\pi$

In the model of Gell-Mann, Sharp, and Wagner,[10] the decay $V(=\omega\text{ or }\phi)\to3\pi$ is supposed to proceed through the intermediate state of $\rho+\pi$. We write the V-ρ-π vertex as

$$M_{\mu\nu}=(g_{V\rho\pi}/m_\pi)\epsilon_{\kappa\lambda\mu\nu}(p^V)_\kappa(p^\pi)_\lambda\,, \tag{A10}$$

where $g_{V\rho\pi}$ is a dimensionless coupling constant. Then the decay rate for $V\to3\pi$ is given by

$$(V\to3\pi)=\frac{f_\rho{}^2}{4\pi}\frac{g_{V\rho\pi}{}^2}{4\pi}\frac{\rho_{3\pi}(m_V{}^2)}{24\pi m_Vm_\pi{}^2}\,. \tag{A11}$$

Here $\rho_{3\pi}(s)$ denotes the phase-space integral used also in Sec. IV; we do not give here an explicit formula, but remark that it is computed using the renormalized ρ propagator and including interference between the three ρ bands on the Dalitz plot. Numerical results are given in Table I.

$\pi^0\to2\gamma$

$$\Gamma(\pi^0\to2\gamma)=\tfrac{1}{12}\alpha^2\left(\frac{f_\rho{}^2}{4\pi}\right)^{-1}\left(\frac{f_8{}^2}{4\pi}\right)^{-1}\frac{g_{Y\rho\pi}{}^2}{4\pi}m_\pi\,, \tag{A12}$$

where, we recall, $f_8{}^2=\tfrac{3}{4}Z_\rho{}^{-1}f_\rho{}^2$ according to SU_3, and

$$g_{Y\rho\pi}=\frac{\beta_Ng_{\omega\rho\pi}-\alpha_Ng_{\phi\rho\pi}}{\alpha_Y\beta_N-\alpha_N\beta_Y}\,. \tag{A13}$$

[18] This is an indication that the neglect of s-wave K-π states implicit in the SU_3 version is not quite justifiable.

Even though $g_{\phi\rho\pi}^2/g_{\omega\rho\pi}^2 \cong 10^{-3}$, the $g_{\phi\rho\pi}$ term is not entirely negligible here, because of the interference term, and because $|\beta_N/\alpha_N|$ is somewhat less than 1 for favored values of the mixing angles.

APPENDIX B: POLE-DOMINANCE LIMIT

Coleman and Schnitzer[5] write

$$D(s)=s(1-\epsilon)-M_0^2 \qquad \text{(B1)}$$

for the inverse propagator of V_μ^3, and

$$D^{-1}(s)=s\begin{pmatrix}1 & \beta \\ \beta & 1+\epsilon\end{pmatrix}-\begin{pmatrix}M_s^2 & 0 \\ 0 & M_0^2\end{pmatrix} \qquad \text{(B2)}$$

for the inverse 2×2 matrix propagator of V_μ^8 and V_μ^0.

Our version of U_3 symmetry requires

$$M_s^2=M_0^2\equiv M^2, \qquad \text{(B3)}$$

whereupon we obtain the numerical values

$$M=0.8351 \text{ BeV}, \qquad \text{(B4)}$$

$$\epsilon=-0.1912, \qquad \text{(B5)}$$

$$\beta=0.212. \qquad \text{(B6)}$$

The parameters of interest in Secs. III and IV are then given by

$$Z_\rho^{-1}=1-\epsilon, \qquad \text{(B7)}$$

$$\alpha_N=\{2[M^2-(1+\epsilon)m_\omega^2]/3\Delta\}^{1/2}, \qquad \text{(B8a)}$$

$$\alpha_Y=-[4(M^2-m_\omega^2)/3\Delta]^{1/2}, \qquad \text{(B8b)}$$

$$\beta_N=\{2[(1+\epsilon)m_\phi^2-M^2]/3\Delta\}^{1/2}, \qquad \text{(B8c)}$$

$$\beta_Y=[4(m_\phi^2-M^2)/3\Delta]^{1/2}, \qquad \text{(B8d)}$$

where

$$\Delta=(1+\epsilon-\beta^2)(m_\phi^2-m_\omega^2). \qquad \text{(B9)}$$

Also,

$$\tan^2\theta_Y=\frac{m_\phi^4}{m_\omega^4}\frac{M^2-m_\omega^2}{M_\phi^2-M^2}. \qquad \text{(B10)}$$

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Unitarity, Current Algebra, and s-Wave K^+p Scattering

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Two procedures for unitarizing current-algebra results are studied and applied to the elastic K^+p scattering amplitude. We note that if the current-algebra amplitude in the off-mass-shell limit is used as input, the unitarity correction leads to good agreement with experiments on phase shifts and other low-energy parameters.

I. INTRODUCTION

It is well known that the usual current-algebra calculations of low-energy parameters for scattering processes performed either in the soft-meson limit ($q_\mu \to 0$) or in the off-shell limit ($q_\mu^2 \to 0$) need extrapolation in the masses of external mesons together with unitarity corrections. The unitarity corrections (corrections arising because of the extrapolation in the s variable) to current-algebra results have been studied by several authors. Akiba and Kang[1] extrapolated the π-π amplitude using elastic unitarity. They used the dispersion-relation technique and assumed that absorptive parts in a fixed-t dispersion relation are given by the Chew-Mandelstam or by the nonrelativistic effective-range approximation, and estimated the corrections due to unitarity from the symmetry point to the physical threshold. Bhargava, Biswas, Gupta, and Datta[2] (hereafter referred to as BBGD) unitarized the results of soft-pion current-algebra calculations for low-energy s-wave π-N scattering, using the N/D formalism. They replaced the N function in the N/D formalism by a single pole on the negative real axis, and determined the strength of the pole by current algebra in the soft-pion limit. Later, Datta, Gupta, and Varma[3] used off-shell ($q_\pi^2 \to 0$) current algebra to obtain a π-π amplitude which is explicitly crossing-symmetric, and unitarized this amplitude to obtain s-wave scattering lengths and phase shifts. They also used the N/D formalism.

Elastic K^+p scattering has been studied within the framework of current algebra by several authors.[4] Roy[4] obtained reasonably good results by including the contribution from the weak-amplitude term. The purpose of this note is to obtain s-wave K^+p phase shifts and other low-energy parameters by applying unitarity

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