Dispersive Approach to Current Algebra†

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The current-algebra scheme, not including equal-time commutators of two space components, is formulated entirely in terms of covariant dispersion relations (i.e., over the laboratory energy for fixed mass) where the number of subtractions is chosen according to the usual Regge picture. To do this, one expresses the equal-time commutators and retarded commutators directly in terms of the covariant dispersion integrals, using an identity between covariant and noncovariant dispersion integrals of Lorentz-invariant distributions which is proved here, by means of the Jost-Lehmann representation, from the locality of the currents. This last result is a generalization of previous works of Schroer and Stichel and of Le Bellac and the author. In the scheme discussed here, the equal-time commutators and retarded commutators are *rigorously* defined as a consequence of the *physical* high-energy behavior, contrary to the usual approach where they are only *formally* written as product of the commutator with $\delta(x_0)$ and $\theta(x_0)$, respectively. In fact, we show that the properties usually derived formally (as, e.g., by partial integration) are rigorously true in our scheme. The quantities which in the usual approach are not well defined, and/or are model-dependent—such as the Schwinger term, the seagull term, and the equal-time commutator of the current and the divergence—are given in this approach by the subtraction constants of the dispersion relations introduced. They are shown to exhibit the properties which usually are more or less assumed, or only obtained in models; e.g., we prove that the divergence of the seagull term is equal to the Schwinger term. Only matrix elements averaged over spin and/or relative momenta, with the vacuum matrix element subtracted, are considered. No explicit value of the equal-time commutator is assumed at the beginning, so as to clearly show the respective roles of the internal symmetry group and of the analyticity and high-energy behavior. This paper also completes a previous work of the author on the infinite-momentum limit based on the same approach.

INTRODUCTION

At the present time, it is quite clear that the introduction of current algebra is a very successful idea.[1] However, it is still true that a lot remains to be done in order to put the scheme on a rigorous basis. In fact one usually writes the equal-time commutators (E.T.C.) and retarded commutators (R.C.) as ordinary products of the commutator by $\theta(x_0)$ and $\delta(x_0)$, respectively. As a consequence, the computations are purely formal. In fact, as products of distributions cannot be given a general meaning, one deals with rather undefined objects. As an example, it is usually claimed that current algebra first leads to low-energy theorems which, making the extra assumption that certain dispersion integrals converge, can be transformed into sum rule. It seems to us that this point of view is misleading since the very existence of the E.T.C. and R.C. considered to derive the low-energy theorems already implies assumptions on the behavior at infinity in momentum space of the commutator. The reason is that this behavior, by Fourier transform, is connected to the regularity of the commutator near the origin which, in turn, is the key to define its product by $\delta(x_0)$ or $\theta(x_0)$.

This appears, for instance, in a paper of Schoer and Stichel[2] where it is shown that, essentially, if the E.T.C. of the two charges can be defined, then the Adler-Weisberger[1] sum rule holds and is convergent. An analogous result was obtained by Le Bellac and the author[3] in the case of the Fubini[4]–Dashen–Gell-Mann[5] sum rule.

Another difficulty which was pointed out recently by the author[6] is in the infinite-momentum limit method used by Dashen and Gell-Mann.[5] It was shown that taking the limit inside the noncovariant integral considered in (5) was certainly not permitted. Here also, to reformulate the method, as done recently by the author,[7] one makes an essential use of the behavior at infinity in momentum space.

The basic problem is that one would need to know this behavior for, say $q_0\rightarrow\infty$ in a given Lorentz frame (q being the variable of the Fourier transform of the commutator) for which nothing is known. In fact, the main difficulty of current-algebra calculations is in going from integrals of the type $\int dq_0$, computed for fixed $\mathbf{q}$, to integrals performed for fixed q^2. This problem has been studied in Ref. 2 for the particular case involved in the Adler-Weisberger sum rule, using the vanishing of the commutator for spacelike separations. The idea is to introduce the Jost-Lehmann-Dyson representation which is a consequence of this property and is well suited for the problem since it explicitly exhibits the dependence of the commutator in the four components of q. In Ref. 3 the same method was also used in a

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[1] S. L. Adler and R. F. Dashen, *Current Algebra and Applications to Particle Physics* (W. A. Benjamin, Inc., New York, 1968).
[2] B. Schroer and P. Stichel, Commun. Math. Phys. 3, 258 (1966).
[3] J. L. Gervais and M. Le Bellac, Nuovo Cimento 7, 8224 (1967).
[4] S. Fubini, Nuovo Cimento 43A, 475 (1966).
[5] R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles of High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966).
[6] J.-L. Gervais, Phys. Rev. Letters 19, 50 (1967).
[7] J.-L. Gervais, Phys. Rev. 169, 1365 (1968).

particular case to obtain rigorously the Fubini–Dashen–Gell-Mann sum rule.

In the present paper, we will first show that the method of Refs. 2 and 3 is indeed general and gives a connection between covariant and noncovariant dis-dispersion relations for Lorentz-invariant amplitudes if only a certain inequality holds among the invariants.

This will make it possible to relate the current algebra scheme, which does not involve the E.T.C. of two space components, to the behavior at infinity of the commutator for fixed mass (i.e., q^2) which is what high-energy physics actually predicts. Let us remark that, up to now, the studies of current algebra based on the Jost-Lehmann-Dyson representation,[2,3,8] were somewhat unphysical, since they were based on assumptions like the existence of E.T.C., or the behavior at infinity of Jost-Lehmann-Dyson weight functions, for which one has no physical insight. Our point of view will be essentially opposite to the one used in Refs. 2, 3, and 8. We will start from an assumed high-energy behavior of the commutator for fixed q^2, which one knows more or less physically, e.g., from Regge theory,[9] and then build up completely the current-algebra scheme from this hypothesis by computing the E.T.C. and R.C. as a sum of covariant dispersion integrals and by showing that they satisfy the properties usually derived by formal computations.[1]

Recently, Mandelstam[10] has put forward the idea of obtaining current-algebra results from analyticity and Regge poles alone. Our work is in the same direction and we will not, before the end, assume any particular value of the E.T.C. so as to show clearly the role of the internal symmetry group.

Preliminary results obtained by the same method on the infinite-momentum limit and existence of the antisymmetric part of the E.T.C. have already been given by the author in a recent paper.[7] We will reproduce them briefly here for the sake of completeness.

Our last remark will be that here, as in the infinite-momentum limit discussed in Ref. 7, our method actually works only in a certain continuous subset of Lorentz frames which we will describe. However we feel that the results are sufficiently general since the physical results are ultimately Lorentz-covariant.

In Sec. I we express the covariant dispersion relations in terms of the retarded Jost-Lehmann representation. Section II is devoted to do the same for the noncovariant dispersion integral and, comparing with the result of Sec. I, derive the connection between covariant and noncovariant integrals of invariant distributions. In Sec. III we consider the matrix elements of the commutator of two currents averaged over spin and relative momenta, and discuss what relation between covariant and noncovariant integrals one may expect to hold on the basis of Regge behavior, free-field models, and the results of Sec. II. The purpose of Sec. IV is to compute the E.T.C. involving at least one time component in terms of covariant dispersion integrals, and to show that they automatically have the form usually assumed[1]—the operator Schwinger terms and the E.T.C. of the divergence with the time component of the current being determined by the subtractions constants introduced in the dispersion relations discussed in Sec. III. The antisymmetric part of the E.T.C. is directly given by the covariant integral of the Fubini–Dashen–Gell-Mann sum rule. This integral is shown to be constant for negative q^2, a result usually deduced only from the current-algebra hypothesis. Taking the derivative with respect to q^2 at $q^2=0$, we then obtain the Cabibbo-Radicati[11] sum rule which thus follows only from our high-energy hypothesis and Sec. III. In Sec. V, we express the retarded commutators also as a sum of dispersion integrals, and determine the covariant amplitudes. The corresponding "seagull term" is computed, again in terms of the subtraction constants of Sec. III. We prove that the formal partial integration formulas obtained by taking the divergences are correct and that, as is known in electrodynamics, the divergence of the seagull term is equal to the Schwinger term. The aim of Sec. VI is to show directly that the low-energy theorems are satisfied for the symmetric and for the antisymmetric part of the amplitude defined in Sec. V. In Sec. VII we discuss the infinite-momentum limit of the R.C., and show that it is given by the Bjorken limit[12] even for finite q_0. Our conclusion and general remarks are the subject of Sec. VIII, while complementary results are discussed in Appendices A–C.

The reader who is unwilling to spend time on the Jost-Lehmann representation may very well begin reading at Sec. III and convince himself that our method is correct by looking at the free-field examples studied in Appendix C.

I. CONNECTION BETWEEN COVARIANT DISPERSION INTEGRAL AND RETARDED JOST-LEHMANN REPRESENTATION

In this section we consider a distribution f of the form

$$f(q,p,\Delta)=\int e^{iq\cdot x}\langle p_1|[A(\tfrac{1}{2}x),B(-\tfrac{1}{2}x)]|p_2\rangle d^4x, \quad (1.1)$$

where

$$p=\tfrac{1}{2}(p_1+p_2), \quad \Delta=p_1-p_2.$$

A and B are assumed to satisfy

$$[A(\tfrac{1}{2}x),B(-\tfrac{1}{2}x)]=0 \quad \text{if} \quad x^2<0, \qquad (1.2)$$

and to be Lorentz-invariant. In this section, as in Sec. II, we treat the case where $p_1\neq p_2$ to show the generality

[8] J. W. Meyer and H. Suura, Phys. Rev. **160**, 1366 (1967).
[9] I am indebted to S. L. Adler, who emphasized this point of view to me.
[10] S. Mandelstam, Berkeley Report (unpublished).
[11] N. Cabibbo and L. A. Radicati, Phys. Letters **19**, 697 (1966).
[12] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

of the method, though we will use only the result for $\Delta=0$ in this paper.

We restrict ourselves to the case where

$$p_1^2=p_2^2=m^2; \quad (1.3)$$

it then follows that

$$p\cdot\Delta=0. \quad (1.4)$$

For the discussion of Refs. 13 and 14 (see also Ref. 3), one needs to introduce the lowest intermediate states appearing in (1.1), i.e., the lowest masses μ_1 and μ_2 such that

$$\begin{aligned}&\langle p_1|A|\chi_1\rangle\neq 0,\quad \langle\chi_1|B|p_2\rangle\neq 0 \quad\text{if}\quad p_{\chi_1}{}^2\geqslant\mu_1{}^2,\\ &\langle p_1|B|\chi_2\rangle\neq 0,\quad \langle\chi_2|A|p_2\rangle\neq 0 \quad\text{if}\quad p_{\chi_2}{}^2\geqslant\mu_2{}^2;\end{aligned} \quad (1.5)$$

we will assume that

$$\mu_1=\mu_2=\mu.$$

This will be the case in current algebra if one uses the Hermitian components of the currents.[15] Then, as shown in Ref. 13, f is given by the Jost-Lehmann representation, which can be written in a covariant form as

$$f(q,\cdots)=\int_{(\Sigma)}d\sigma_\mu ds\left[\frac{\partial}{\partial u_\mu}\Phi^1(u,s,\cdots)+q_\mu\Phi^2(u,s,\cdots)\right]\times\epsilon(q_0-u_0)\delta[(q-u)^2-s], \quad (1.6)$$

where the integration in the four-vector u is in the plane Σ orthogonal to p, s being a Lorentz scalar. The support of Φ^1 and Φ^2 is given by the set of points $\{u,s\}$ such that

$$\{(p\pm u)\in V^+,\ \sqrt{s}\geqslant\sup[0,\mu-\sqrt{(p+u)^2},\mu-\sqrt{(p-u)^2}]\}.$$

They are invariant distributions of the vectors u, p, and Δ uniquely determined by f. The precise meaning of (1.6) is that, given a test function $\varphi(q)\in\mathfrak{D}$, one has, applying the distribution f to it,

$$f_\varphi=\int_{(\Sigma)}d\sigma^\mu ds\times\left(\frac{\partial}{\partial u^\mu}\Phi^1(u,s)\left\{\int d^4q\ \epsilon(q_0-u_0)\delta[(q-u)^2-s]\varphi(q)\right\}+\Phi^2(u,s)\left\{\int d^4q\ q_\mu\epsilon(q_0-u_0)\delta[(q-u)^2-s]\varphi(q)\right\}\right). \quad (1.7)$$

Letting

$$\nu=p\cdot q,$$

our purpose is now to relate covariant dispersion integrals of the type

$$I(\nu)=\int\frac{d\nu'}{\nu'-\nu}f(\nu',q^2,q\cdot\Delta,\cdots) \quad (1.8)$$

to the retarded Jost-Lehmann representation corresponding to (1.6). In (1.8) one integrates for fixed values of the other invariants q^2, $q\cdot\Delta$; it corresponds to a standard dispersion relation in the laboratory energy. For the moment we do not consider the problem of subtraction; we will discuss it later. In other to apply (1.7), we replace (1.8) by

$$I=\lim_{\epsilon\to0}\int\frac{d\omega}{\omega-E}\varphi_\epsilon(\omega)f(\omega,\cdots)=\lim_{\epsilon\to0}I_\epsilon,$$

where

$$\varphi_\epsilon\in\mathfrak{D},\quad \lim_{\epsilon\to0}\varphi_\epsilon(\omega)=1,$$

and where

$$\nu'=m\omega,\quad \nu=mE.$$

Also we choose ν so that $I(\nu)$ is real, i.e., outside the singularities. The general result can be obtained by analytic continuation. According to (1.7) one has, in the Breit frame (i.e., where $\mathbf{p}=0$),

$$I_\epsilon=\int d^3\mathbf{u}\ ds\left[\left(\frac{\partial}{\partial u_0}\Phi^1\right)G_1(u,s)+\Phi^2G_32(u,s)\right], \quad (1.9)$$

with

$$G_1(u,s)=\int d\omega\ \epsilon(\omega)\delta[\omega^2-(\mathbf{q}-\mathbf{u})^2-s]\frac{\varphi_\epsilon(\omega)}{\omega-E}, \quad (1.10)$$

$$G_2(u,s)=\int d\omega\ \omega\epsilon(\omega)\delta[\omega^2-(\mathbf{q}-\mathbf{u})^2-s]\frac{\varphi_\epsilon(\omega)}{\omega-E}.$$

In (1.10), as explained above, one integrates for fixed q^2, $q\cdot\Delta$. We restrict ourselves to the case where

$$q^2=-\zeta^2,\quad \Delta^2=-4\eta^2,\quad q\cdot\Delta=-2c\eta\zeta, \quad (1.11)$$

ζ and η being real. We shall take in our calculation q and Δ to be real so that one has

$$|c|\leqslant1.$$

To compute (1.10), we choose, in the spatial part of the Breit frame, Δ as third axis and $\mathbf{q}$ in the plane orthogonal to the second axis. Accordingly, the coordinates of $\mathbf{q}$ will be written

$$q_1=(\omega^2+\xi^2)^{1/2},\quad q_2=0,\quad q_3=c\zeta, \quad (1.12)$$

where

$$\xi^2=\zeta^2(1-c^2)\geqslant0,$$

and one has

$$\omega^2-(\mathbf{q}-\mathbf{u})^2-s=2[u_1(\omega^2+\xi^2)^{1/2}-y], \quad (1.13)$$

where

$$y=\tfrac{1}{2}(s+\zeta^2+\mathbf{u}^2)-c\zeta u_3.$$

The integrals (1.10) are evaluated from the general

[13] R. Jost and H. Lehmann, Nuovo Cimento 5, 1598 (1957).
[14] F. J. Dyson, Phys. Rev. 110, 1960 (1958).
[15] In the more general case, one can still carry out the proof by introducing the Dyson representation (Ref. 14). However, in this representation the weight function is no longer unique.

formula[16]

$$\delta(g(x))=\sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}, \tag{1.14}$$

where $\{x_i\}$ is the set of points where $g(x)$ vanishes. Formula (1.14) holds if the equation $g(x)=0$ has no double solution. One thus gets

$$G_1=\frac{y}{2u_1{}^2\omega_r}\left(\frac{\varphi_\epsilon(\omega_r)}{\omega_r-E}+\frac{\varphi_\epsilon(-\omega_r)}{\omega_r+E}\right),$$
$$G_2=\frac{y}{2u_1{}^2}\left(\frac{\varphi_\epsilon(\omega_r)}{\omega_r-E}-\frac{\varphi_\epsilon(-\omega_r)}{\omega_r+E}\right), \tag{1.15}$$

were ω_r is the positive solution of

$$\omega^2-(\mathbf{q}-\mathbf{u})^2-s=0,$$

i.e., according to (1.13),

$$\omega_r=[(y/u_1)^2-\xi^2]^{1/2}. \tag{1.16}$$

If $\omega_r\neq 0$, (1.14) will hold since there will be no double roots. From (1.13), (1.16) one sees that ω_r will vanish in the domain of integration of (1.9) if and only if there exists admissible hyperboloids going through the points $\{q_0=0, \mathbf{q}^2=\zeta^2\}$. This is impossible if those points are outside the region

$$|q_0|\geqslant|(\eta^2+m^2)^{1/2}-(\mathbf{q}^2+\mu^2)^{1/2}|,$$

which is the support of $f(q)$. The boundary of this region intersects the plane $q_0=0$ for

$$\mathbf{q}^2=\eta^2+m^2-\mu^2.$$

Accordingly, if

$$\zeta^2>\eta^2+m^2-\mu^2, \tag{1.17}$$

then ω_r cannot vanish, and one has

$$y^2>u_1{}^2\xi^2,$$

so that ω_r is automatically real.

We will assume that (1.17) holds so that (1.15) makes sense. Choosing φ_ϵ to be a symmetric function, one can then rewrite (1.15) as

$$G_1=\tfrac{1}{2}\varphi_\epsilon(\omega_r)\left(\frac{1}{y-u_1(E^2+\xi^2)^{1/2}}+\frac{1}{y+u_1(E^2+\xi^2)^{1/2}}\right)$$
$$G_2=\tfrac{1}{2}\varphi_\epsilon(\omega_r)\omega_r\left(\frac{1}{y-u_1(E^2+\xi^2)^{1/2}}+\frac{1}{y+u_1(E^2+\xi^2)^{1/2}}\right). \tag{1.18}$$

Now we use the fact that Φ^1 and Φ^2, being Lorentz-invariant, must be even distributions of u_1, so that in (1.18), the two terms give the same contribution to I_ϵ. On the other hand, one can write an equation similar to (1.13), i.e.,

$$r_0{}^2-(\mathbf{r}-\mathbf{u})^2-s=2[u_1(E^2+\xi^2)^{1/2}-y], \tag{1.19}$$

if one introduces a new Lorentz vector r such that

$$r^2=-\zeta^2,\quad r\cdot p=\nu,\quad r\cdot\Delta=-2c\zeta\eta. \tag{1.20}$$

Expressing the result in a covariant manner, we have finally proved that

$$I_\epsilon=2\int\frac{d\sigma^\mu ds}{s-(r-u)^2}\left(\frac{\partial}{\partial u^\mu}\Phi^1+r_\mu\Phi^2\right)\varphi_\epsilon(\omega_r). \tag{1.21}$$

If, e.g., the dispersion relation for f needs one subtraction, one will replace (1.21) by

$$(\nu-\nu_0)\int\frac{d\nu'}{(\nu'-\nu_0)(\nu'-\nu)}f(\nu')$$
$$=2\int d\sigma^\mu ds\left[\frac{1}{s-(r-u)^2}\left(\frac{\partial}{\partial u^\mu}\Phi^1+r_\mu\Phi^2\right)-\frac{1}{s-(v-u)^2}\left(\frac{\partial}{\partial u^\mu}\Phi^1+v_\mu\Phi^2\right)\right], \tag{1.22}$$

where v is such that

$$v\cdot p=\nu_0,\quad v^2=-\zeta^2,\quad v\cdot\tfrac{1}{2}\Delta=-c\eta\zeta.$$

Finally, one can continue (1.21) for complex r and ν, keeping ζ, η, and c fixed and real. It is easy to verify that both members of (1.21) will automatically have the same analyticity domain. Letting $\epsilon\to 0$, one sees that one will have

$$\int\frac{d\nu'}{\nu'-\nu}f(\nu')=2\int\frac{d\sigma^\mu ds}{s-(r-u)^2}\left(\frac{\partial}{\partial u^\mu}\Phi^1+r_\mu\Phi^2\right), \tag{1.23}$$

if one can replace φ_ϵ by 1 in (1.21) when taking the limit. Choosing the test function

$$\varphi_\epsilon=1\quad\text{if}\quad|\omega|<1/\epsilon,$$
$$\varphi_\epsilon=0\quad\text{if}\quad|\omega|>1/\epsilon+\delta,$$

one sees that the only troublesome points of (1.21) are those where ω_r becomes infinite. From (1.16), this happens if $u_1\to 0$ or if $s\to\infty$. In fact, $\partial_\mu\phi^1$ and ϕ^2 are temperate distributions in u and s, so that we can write

$$\Phi^1=\sum_k \mathcal{P}_k{}^1\left(\frac{\partial}{\partial u^\mu},\frac{\partial}{\partial s}\right)F_k{}^1(u,s),$$
$$\frac{\partial}{\partial u^\mu}\Phi^2=\sum_k \mathcal{P}_k{}^2\left(\frac{\partial}{\partial u^\mu},\frac{\partial}{\partial s}\right)F_{k,\mu}{}^2(u,s),$$

where the F's are continuous functions and the $\mathcal{P}$'s are polynomials. A sufficient condition for (1.23) to hold is

[16] See, e.g., I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. 1.

that near $u_1=0$, and $s\to\infty$ all the F's tend to zero except those for which $\mathcal{O}^1=\mathcal{O}^2=1$, namely, that Φ^1, $\partial_\mu\Phi^2$ behave like functions near those points. We pospone the discussion of this question until Sec. III.

If (1.23) holds, there is a one-to-one correspondence between the number of subtractions needed in the dispersion relation and in the Jost-Lehmann representation, since, if one member of the equality converges, it gives a meaning to the other.

II. CONNECTION BETWEEN COVARIANT AND NONCOVARIANT DISPERSION INTEGRALS

As we already indicated in the introduction and in Ref. 7, we want to relate the dispersion integrals studied in Sec. I, to noncovariant integrals over the time component of q in a given frame of reference $\mathfrak{F}$. Accordingly, f being defined as in (1.1), we now consider, in $\mathfrak{F}$,

$$J=\int \frac{dq_0}{q_0-\kappa}f(q_0,\mathbf{q})\,, \tag{2.1}$$

which is performed for fixed $\mathbf{q}$. As in Sec. I, we compute it as

$$J=\lim_{\epsilon\to 0} J_\epsilon=\lim_{\epsilon\to 0}\int \frac{dq_0}{q_0-\kappa}\varphi_\epsilon(q_0)f(q_0,\mathbf{q})\,, \tag{2.2}$$

where

$$\lim_{\epsilon\to 0}\varphi_\epsilon=1,\quad \varphi_\epsilon\in\mathfrak{D}.$$

As in Sec. I, we use the Jost-Lehmann representation (1.6) for f, so that we will have to compute, in $\mathfrak{F}$,

$$\begin{aligned}H_1&=\int dq_0\varphi_\epsilon(q_0)\frac{1}{q_0-\kappa}\epsilon(q_0-u_0)\delta[(q-u)^2-s]\,,\\ H_2&=\int dq_0 q_0\varphi_\epsilon(q_0)\frac{1}{q_0-\kappa}\epsilon(q_0-u_0)\delta[(q-u)^2-s]\,.\end{aligned} \tag{2.3}$$

Since one integrates for fixed $\mathbf{q}$, the calculation is very easy; one gets

$$H_1=\frac{1}{2\rho}\left(\frac{\varphi_\epsilon(u_0+\rho)}{u_0+\rho-\kappa}-\frac{\varphi_\epsilon(u_0-\rho)}{u_0-\rho-\kappa}\right),$$

$$H_2=\frac{1}{2\rho}\left(\frac{(u_0+\rho)\varphi_\epsilon(u_0+\rho)}{u_0+\rho-\kappa}-\frac{(u_0-\rho)\varphi_\epsilon(u_0-\rho)}{u_0-\rho-\kappa}\right),$$

where ρ stands for

$$\rho=[(\mathbf{q}-\mathbf{u})^2+s]^{1/2}.$$

As in Sec. I, letting $\epsilon\to 0$, one may replace φ_ϵ by 1, in general, except if $u_0\pm\rho$ becomes infinite. This happens if $s\to\infty$. Under the same regularity assumption as in Sec. I, one thus finally obtains

$$J=2\int\frac{d\sigma_\mu ds}{s-(r'-u)^2}\left(\frac{\partial}{\partial u_\mu}\Phi^1+\Phi^2\right) \tag{2.4}$$

if r' is defined in $\mathfrak{F}$ by

$$r_0'=\kappa,\quad \mathbf{r}'=\mathbf{q}.$$

In formula (2.4) again, as in (1.23), the retarded Jost-Lehmann representation appears. It has now to be computed for

$$r'^2=\kappa^2-\mathbf{q}^2,\quad r'\cdot p=\kappa p_0-\mathbf{p}\cdot\mathbf{q},\quad r'\cdot\Delta=\kappa\Delta_0-\mathbf{q}\cdot\Delta \tag{2.5}$$

instead of (1.20). However, the retarded Jost-Lehmann distribution is invariant. Since we consider it in a region where it is analytic, it is known[17] that it can be written as a function of invariants only. Accordingly, it follows from (1.23) and (2.4) that

$$\int\frac{d\nu'}{\nu'-\nu}f(\nu',\zeta,\eta,c)=\int\frac{dq_0}{q_0-\kappa}f(q_0,\mathbf{q},p,\Delta) \tag{2.6}$$

if, according to (1.20) and (2.5),

$$\nu=\kappa p_0-\mathbf{q}\cdot\mathbf{p},\quad \zeta^2=\mathbf{q}^2-\kappa^2,$$
$$\Delta^2=-4\eta^2,\quad 2c\zeta\eta=(\mathbf{q}\cdot\mathbf{\Delta}-\kappa\Delta_0)^2$$

and if (1.17) is satisfied.

If no subtraction is needed, (2.6) converges as it stands; if one subtraction is needed; one subtracts both members, obtaining, e.g.,

$$\nu\int\frac{d\nu'}{(\nu'-\nu)\nu'}f=\kappa\int\frac{dq_0}{(q_0-\kappa)q_0}f;$$

writing formally

$$\int\frac{dq_0}{q_0-\kappa}f=\sigma+\kappa\int\frac{dq_0}{(q_0-\kappa)q_0}f,$$

one will obtain

$$\int\frac{dq_0}{q_0-\kappa}f=\sigma+\nu\int\frac{d\nu'}{(\nu'-\nu)\nu'}f, \tag{2.7}$$

which we will take as a definition of the left integral. As expected, in this case, the bad behavior at infinity of f results in arbitrariness in the definition of the corresponding retarded function (2.7) since σ is *a priori* arbitrary.

It is to be remarked that the assumptions from which (2.6) was deduced are certainly not very general. For instance, if (2.6) holds, one can consider the distribution $(q\cdot p)f$ which clearly has the same support properties in q and x spaces as f itself. However, it follows from (2.6) that

$$\int\frac{d\nu'}{\nu'-\nu}(\nu' f)=\int\frac{dq_0}{q_0-\kappa}(q_0p_0-\mathbf{q}\cdot\mathbf{p})f+\rho, \tag{2.8}$$

[17] D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 29, 12 (1955).

with

$$\rho=\int d\nu\, f-p_0\int dq_0 f\,,$$

so that for νf, (2.6) is not true. Thus one expects that, in general, extra terms appear in (2.6). We will discuss them in Sec. III.

III. RETARDED FUNCTIONS IN THE PHYSICAL CASE

As in Refs. 3 and 7, we consider[18]

$$t_{\mu\lambda}{}^{\alpha\beta}(q)=\frac{1}{i}\int d^4x\, e^{iq\cdot x}[\langle p_1|[J_\mu{}^\alpha(\tfrac{1}{2}x),J_\lambda{}^\beta(-\tfrac{1}{2}x)]|p_2\rangle -\langle p_1|p_2\rangle\langle 0|[J_\mu{}^\alpha(\tfrac{1}{2}x),J_\lambda{}^\beta(-\tfrac{1}{2}x)]|0\rangle]\,, \quad (3.1)$$

where α and β are SU_3 indices. To simplify, we consider only the case where

$$p_1=p_2=p\,,$$

though a general treatment is possible from the results of Secs. I and II. If $|p\rangle$ is a state of nonzero spin, and/or a multiparticle state, we average (3.1) over spin, and/or relative momenta so that one has (we suppress α and β in the following except if they are explicitly needed)

$$t_{\mu\lambda}=ap_\mu p_\lambda+b(q_\mu q_\lambda+q_\lambda p_\mu)+cq_\mu q_\lambda+dg_{\mu\lambda}\,, \quad (3.2)$$

where a, b, c, and d are Lorentz invariants.[19] As shown in Appendix A, they are Fourier transforms in the variable q of local distributions in x so that the results of Sec. II can be applied. As already indicated, we will reduce the E.T.C. and R.C. associated to $t_{\mu\lambda}$ to integrals of the form (2.2), so that (2.6), subtracted or not, may be applied. To do so, we will have to introduce the commutators involving divergences, namely,[20]

$$t_\mu=q^\lambda t_{\mu\lambda}=lp_\mu+hq_\mu\,,$$

where

$$l=a(p\cdot q)+bq^2\,,$$
$$h=b(p\cdot q)+cq^2+d\,, \quad (3.3)$$

and

$$w=q^\mu q^\lambda t_{\mu\lambda}=l(p\cdot q)+hq^2\,. \quad (3.4)$$

In Appendix C, we discuss different free-field models for spins 0 and $\frac{1}{2}$. In those examples, one sees that an equality of the type (2.6) is satisfied without extra terms for all $ab\cdots w$, except for d, which satisfies (2.8) instead, since it corresponds exactly to the example discussed at the end of Sec. II. In general, for dimensionality, one expects that d will contain, compared to a, b, and c, extra factors of the dimension of mass squared which will prevent (2.6) from holding. The simplest factors are q^2 or $q\cdot p$. In those two cases, as in the free-field model, the extra term which appears is symmetric in α and β [see (3.6).]

We will make the simplest possible dynamical assumption which agrees with the free-field models discussed in Appendix C, namely, that, apart from possible subtractions, (2.6) holds for a, b, c, l, h, and the antisymmetric part of d, while for the antisymmetric part of d, (2.8) holds instead.

We will discuss this hypothesis in the conclusion. Let us remark, for the moment, that from (3.3) and (3.4), one may think, *a priori*, that the factors q^2 and $q\cdot p$ which appear in the definition of l, h and w invalidate our hypothesis for these functions. As we have discussed, multiplying by q^2 or $q\cdot p$ separately certainly does. However, in the free-field case, the particular combinations of a, b, c, and d which appear in (3.3) and (3.4) do satisfy (2.6), since the extra terms cancel. This is likely to happen in general since, first of all, if J_μ is conserved $w=l=h=0$ so that (2.6) certainly holds; on the other hand, in Lagrangian field theory, $\partial^\mu J_\mu$ is not generally a more singular operator than J_μ, so that it is reasonable to make the same kind of hypothesis for l, h, w, as for a, b, c; finally for the axial current letting $m_\pi{}^2=0$, one has $\partial^\mu J_\mu=0$ and (2.6) holds for l, h, w; in the real world, *taking (2.6) for l, h, w is thus in the spirit of the partially conserved axial-vector current (PCAC) hypothesis.*

An interesting remark is that, again in Lagrangian field theory, $p^\mu\partial_\mu J^\lambda$ is a more singular operator than J^λ since it contains higher powers of the field. Though we cannot, for the moment, exhibit a clear cut connection, this is certainly related to the fact that multiplying by $(q\cdot p)$ a distribution satisfying (2.6) introduces extra terms. In Ref. 3 we also found the same type of phenomenon in the free-field case by exhibiting explicitly the weight functions of the Jost-Lehmann-Dyson representation: while they were all of the same type for a, b, c, h, l, w (i.e., with compact support), the weight function corresponding to, e.g., $(q\cdot p)a$ was shown to tend to a constant at infinity, making doubtful the derivation of sum rules from time derivatives computed in the Breit frame.

Now we must decide upon the number of subtractions to be performed in (2.6) or (2.8). As already remarked, it is given by the behavior of a, b, $\cdots$, w for large value

[18] The fact that one must subtract the vacuum expectation value to give a meaning to the E.T.C. was emphasized in Ref. 2. It is necessary here also; otherwise the spectrum properties necessary to use the Jost-Lehmann representation would not be satisfied. Thus our treatment of the vacuum expectation value given in Ref. 3 is not entirely correct; however, this does not invalidate the results of this paper.

[19] If J_μ is conserved, a, b, c, and d are not independent since in this case there are only two independent amplitudes. We keep them anyhow, since there exists no choice of independent amplitude which allows us to use the Jost-Lehmann representation completely (see Ref. 8). When necessary we will take into account the fact that a, b, c, and d are not independent if $\partial_\mu J^\mu=0$ (see Sec. VI).

[20] This is the same technique as in Ref. 7. In Ref. 1, Adler and Dashen also introduce l, h, and w to discuss the $p\to\infty$ limit. However, they have no general argument on the interchange of the limit $p\to\infty$ with the integration as we gave in Ref. 7. In Ref. 6 we showed that this is a delicate question, so that the analysis of Ref. 1 is incomplete.

of ν and fixed q^2. We take it to be of the Regge type.[21] In fact, the exchange of a pole at $l=\delta$ leads to

$$a\simeq\nu^{\delta-2}\,,$$
$$b,\, l\simeq\nu^{\delta-1}\,, \tag{3.5}$$
$$c,\, d,\, h,\, w\simeq\nu^{\delta}\,,\quad \text{as}\quad \nu\to\infty\,.$$

Of course, we will use this hypothesis for negative q^2 where it is not well established. However, we only need to assume that the number of subtractions does not increase from $q^2>0$ to $q^2\leqslant 0$.

Since δ is not the same for the part of the amplitude which is SU_3-symmetric ($\delta\leqslant 1$) as for the part which is antisymmetric ($\delta<1$), we separate them out by a superscript $+$ or $-$. We will finally introduce no subtraction for $a^\pm$, b^-, l^-, one subtraction for b^+, l^+, c^-, d^-, h^-, w^-, and two subtractions for c^+, d^+, h^+, w^+.

It is very convenient to perform those subtractions at $\nu=0$, since, as discussed, e.g., in Ref. 3, $a,\cdots,w$ have well-defined crossing properties in $\nu\to-\nu$, e.g.,

$$(a^\pm,b^\pm,c^\pm,d^\pm,l^\pm,h^\pm,w^\pm)\to(\mp a^\pm,\,\pm b^\pm,\,\mp c^\pm,\,\mp d^\pm,\,\pm l^\pm,\,\mp h^\pm,\,\mp w^\pm)\,, \tag{3.6}$$

so that many of the subtraction constants will actually be zero. This is strictly possible only if $\mu>m$, or, if $\mu=m$, only if $q^2<0$. The reason is that if $m=\mu$, the one-particle pole is at $\nu=q^2$. In this case, we will assume first that $q^2<0$ and take the limit $q^2\to 0$. The fact that we derive the low-energy theorems shows that this procedure is correct.

Finally, we are ready to introduce the retarded distributions associated to a, b, $\cdots$, w. From now on we consider only the case where

$$\mathbf{q}\cdot\mathbf{p}=0\,,$$

which is the only one we will actually study. For a, b, and l, we let, e.g., for a,

$$\frac{1}{2\pi}\int\frac{dq_0}{q_0-\kappa}a(q_0,\mathbf{q})=A(\kappa p_0,\,-\zeta^2), \tag{3.7}$$

while for c, d, h, w we have, e.g., for d,

$$\frac{1}{2\pi}\int\frac{dq_0}{q_0-\kappa}d(q_0,\mathbf{q})=\sigma_d(\kappa,|\mathbf{q}|)+D(\kappa p_0,\,-\zeta^2)\,, \tag{3.8}$$

where

$$\zeta^2=\mathbf{q}^2-\kappa^2\geqslant 0\,,$$

and where σ_c, σ_d, σ_h, σ_w are subtraction "constants" which are necessarily symmetric in α and β from our previous considerations. They, in fact, depend on κ since, in general, they may depend on ζ^2. The capital letters represent the subtracted dispersion integrals over ν without the subtraction constants; i.e., for $a^\pm$, b^-, l^-, one has, e.g., for $a^\pm$,

$$A^\pm(\nu,q^2)=\frac{1}{2\pi}\int\frac{d\nu'}{\nu'-\nu}a^\pm(\nu',q^2)\,, \tag{3.9}$$

while for b^+, e^+, c^-, d^-, h^-, w^- one has, e.g., for b^+,

$$B^+(\nu,q^2)=\frac{\nu}{2\pi}\int\frac{d\nu'}{\nu'(\nu'-\nu)}b^+(\nu',q^2)\,, \tag{3.10}$$

and for c^+, d^+, h^+, w^+ one has, e.g.,

$$D^+(\nu,q^2)=\frac{\nu^2}{2\pi}\int\frac{d\nu'}{\nu'^2(\nu'-\nu)}d^+(\nu',q^2) \tag{3.11}$$

and, of course,

$$A=A^++A^-,\quad \text{etc}\cdots.$$

IV. EQUAL-TIME COMMUTATORS

We take the E.T.C. as defined, in a given Lorentz frame by

$$\mathcal{T}_{\mu\lambda}{}^{\alpha\beta}(\mathbf{q})=\frac{i}{2\pi}\lim_{\epsilon\to0}\int_{-\infty}^{+\infty}dq_0\varphi_\epsilon(q_0)t_{\mu\lambda}{}^{\alpha\beta}(q_0,\mathbf{q})\,, \tag{4.1}$$

where $t_{\mu\lambda}$ is given by (3.1), and

$$\lim_{\epsilon\to0}\varphi_\epsilon(q_0)=1\,. \tag{4.2}$$

In general, the limit (4.1) may not exist. However, we will prove that, if the assumptions of Sec. III are satisfied, then (4.1) does make sense for $\mu=\lambda=0$, and $\mu=0$, $\lambda=1$, 2, 3, provided that one has

$$\mathbf{q}\cdot\mathbf{p}=0.$$

Thus, $\mathbf{q}_B$ being given in the Breit frame, we will define $\mathcal{T}$ only in the set of Lorentz frames deduced from the Breit frame by a pure Lorentz transformation in a direction orthogonal to $\mathbf{q}_B$ multiplied by a pure rotation.

A. Equal-Time Commutator of the Time Components

This was already studied in Ref. 7 but we consider it again briefly for the sake of completeness. Inserting (3.1) into (4.1) taken for $\mu=\lambda=0$, one gets (the limit $\epsilon\to0$ is not explicitly mentioned)

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q})=\frac{i}{2\pi}\int dq_0[ap_0^2+2bp_0q_0+cq_0^2+d]^{\alpha\beta}. \tag{4.3}$$

Now we introduce l, h, and w, writing (3.3) and (3.4) under the form

$$b=(l-a\nu)/q^2,\quad c=(h-d-b\nu)/q^2,\quad h=(w-l\nu)/q^2. \tag{4.4}$$

We will thus divide by q^2 inside the integral. It vanishes for

$$q_0=\pm|\mathbf{q}|=\pm\lambda;$$

[21] As will be clear in the following, we do not need behavior of exactly the Regge type. We use it just to simplify the writing. Possible fixed Regge poles do not matter here, since we consider only the imaginary parts in our hypothesis.

accordingly, we first replace $\int dq_0$ in (4.3) by

$$\int dq_0 = \lim_{\eta\to 0,\ \eta'\to 0}\left(\int_{-\infty}^{-\lambda-\eta} dq_0 + \int_{-\lambda+\eta}^{\lambda-\eta'} dq_0 + \int_{\lambda+\eta'}^{\infty} dq_0\right),$$

so that in the final formula the principal value is to be taken at $q_0=\pm\lambda$. This will ensure that the E.T.C. is purely imaginary. Inserting (4.4) into (4.3), one gets

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q}) = \frac{i}{2\pi}p_0\int \frac{dq_0}{q_0^2-\lambda^2}\left(-p_0\lambda^2 a^- - \lambda^2 q_0 b^- + \frac{1}{p_0}[\lambda^2(h^- - d^-)+w^-]\right)^{\alpha\beta}. \quad (4.5)$$

According to (3.6), only the parts of a, b, $\cdots$, w which are antisymmetric in β and α contribute. Now we use (3.7) and (3.8). For this we have to let $\kappa=\lambda$. The dispersion integrals are thus computed at zero mass and condition (1.17) holds only if $\mu>m$. Assuming for the moment that this is true, one gets

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q}) = ip_0 \,\mathrm{Re}\left(-\lambda p_0 A^-(\lambda p_0,0) - \lambda^2 B^-(\lambda p_0,0) + \frac{\lambda}{p_0}[H^-(\lambda p_0,0) - D^-(\lambda p_0,0)] + \frac{1}{\lambda p_0}W^-(\lambda p_0,0)\right)^{\alpha\beta}. \quad (4.6)$$

In (4.6) $\mathrm{Re}(\cdots)$ means of course the real part of $(\cdots)$; it comes from the principal value.

As in Ref. 7, (4.6) can be transformed into

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q}) = \frac{i}{2\pi}p_0\left(\int d\nu\, a^- + \int \frac{d\nu}{\nu-\lambda p_0}(l^- - \nu a^-) - \lambda^2\int \frac{d\nu}{\nu(\nu-\lambda p_0)}(\nu b^- - h^- - d^-) + \int \frac{d\nu}{\nu(\nu-\lambda p_0)}(w^- - \nu l^-)\right)^{\alpha\beta}. \quad (4.7)$$

But, according to (3.3) and (3.4), the last three integrals vanish since one integrates for zero mass.[22] One then gets

$$\mathcal{T}_{00}{}^{\alpha\beta} = \frac{i}{2\pi}p_0\int d\nu\, a^-(\nu,0). \quad (4.8)$$

This shows that the E.T.C. is well defined, as a consequence of our hypothesis, and is equal to the time component of a vector. As discussed in (7), the expression (4.8) is directly equal to the integral of the Fubini–Dashen–Gell-Mann sum rule. In the case where $m=\mu$, we replace (4.5) by

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q}) = \lim_{\kappa\to\lambda,\ \kappa^2\leqslant\lambda^2} i\frac{p_0}{2\pi}\int \frac{dq_0}{q_0^2-\kappa^2}\left(-p_0\kappa^2 a^- + \kappa^2 q_0 b^- + \frac{1}{p_0}[\kappa^2(h^- - d^-)+w^-]\right)^{\alpha\beta}. \quad (4.9)$$

This leads to a formula similar to (4.6) with λ replaced by κ, and one finally gets

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q}) = i\frac{p_0}{2\pi}\int d\nu\, a^-(\nu,0) + \lim_{\kappa\to\lambda,\ \kappa^2\leqslant\lambda^2}(\kappa^2-\lambda^2) \times\left[\int \frac{d\nu}{\nu-\kappa p_0}\left(b^-(\nu,\kappa^2-\lambda^2) + \frac{c^-(\nu,\kappa^2-\lambda^2)}{\nu} + \frac{h^-(\nu,\kappa^2-\lambda^2)}{\nu}\right)\right]. \quad (4.10)$$

In the limit, only the one-particle intermediate state contribution to c^- and h^- may lead to a finite extra term in this equation. However, the corresponding term vanishes by symmetry (see Appendix B) and (4.8) is also obtained for $m=\mu$.

One usually deduces from the current algebra hypothesis that the integral of the Fubini–Dashen–Gell-Mann sum rule, namely, $\int d\nu\, a^-(\nu,q^2)$, does not depend on q^2. We will now show that this fact can be directly proved, using our method,[23] in the region $q^2\leqslant 0$. For this we transform (4.3) by using, instead of (4.4), the relation

$$b=\frac{-a\nu+l+b\zeta^2}{q_0^2-\kappa^2},\quad c=\frac{h+c\zeta^2-d-b\nu}{q_0^2-\kappa^2},$$

$$h=\frac{w+h\zeta^2-l\nu}{q_0^2-\kappa^2},$$

where

$$\kappa^2=\mathbf{q}^2-\zeta^2$$

and ζ^2 is *a priori* a parameter such that $\zeta^2<\mathbf{q}^2$. Equation (4.5) now becomes (there is, here also, a principal value)

$$\mathcal{T}_{00}{}^{\alpha\beta}(\mathbf{q}) = \frac{ip_0}{2\pi}\int \frac{dq_0}{q_0^2-\kappa^2}\left(-p_0\kappa^2 a^- - \kappa^2 q_0 b^- + \frac{1}{p_0}[\kappa^2(h^- + \zeta^2 c^- - d^-) + w^- + \zeta^2 h^-]\right)^{\alpha\beta}. \quad (4.11)$$

This last equation can be transformed again as (4.6) to

[22] Of course, this is correct only if b and c do not contain $1/q^2$ factors. That it is so can be checked directly, e.g., in the case of Compton scattering. The optical theorem then gives, as $q^2\to 0$ for ν fixed in the continuum (see, e.g., Ref. 24): $a\simeq(q^2/\nu)\sigma_T$, $d\simeq\nu\sigma_T$, from which it follows that, if $q^2=0$, then $a=0$, $\nu b=d$, which is what we use since here $l=h=w=0$.

[23] This is similar to results obtained in Ref. 3.

yield

$$\mathcal{T}_{00}{}^{\alpha\beta}=\frac{i}{2\pi}p_0\int d\nu\; a^-(\nu,\,-\zeta^2)\,,$$

which shows that

$$\frac{d}{dq^2}\left(\int d\nu\; a^-(\nu,q^2)\right)=0\quad\text{if}\quad q^2\leqslant 0.\quad\text{Q.E.D.}$$

In the case of vector current, upon extracting the one-particle contribution one gets according to Appendix B

$$\left(2\frac{d}{dq^2}M^-(q^2)+\frac{d}{dq^2}\int_{\text{continuum}}d\nu\; a^-(\nu,q^2)\right)_{q^2=0}=0\,. \quad (4.12)$$

This can be recognized as the Cabibbo-Radicati sum rule (see, e.g., Gourdin[24]) which is thus only a consequence of our hypothesis of Sec. III and not of any particular value of the E.T.C.[23]

B. E.T.C. of a Space Component with the Time Component

Inserting (3.1) into (4.1), one now gets

$$\mathcal{T}_{0k}{}^{\alpha\beta}=\frac{i}{2\pi}p_k\int dq_0(ap_0+bq_0)^{\alpha\beta}+\frac{i}{2\pi}q_k\int dq_0(bq_0+cq_0)^{\alpha\beta}.$$

Assuming also $\mathbf{p}\cdot\mathbf{q}=0$, it is easy to see, from (3.6), that the coefficient of p_k is antisymmetric in α,β while the coefficient of q_k is symmetric. The first term is treated in the same way as $\mathcal{T}_{00}{}^{\alpha\beta}$, obtaining

$$p_k\int dq_0(ap_0+bq_0)=p_k\int d\nu\; a(\nu,0)\,.$$

Using the same method, the second term is written as

$$\mathcal{T}_{0k}{}^{+}=i\frac{q_k}{2\pi}\int dq_0(bp_0+cq_0)$$
$$=i\frac{q_k}{2\pi}\int\frac{dq_0}{q_0{}^2-\lambda^2}[-\lambda^2p_0b^++q_0(h^+-d^+)].$$

As in Sec. III A, we assume first $\mu>m$ and make use of (3.7) and (3.8) to get

$$\mathcal{T}_{0k}{}^{+}=\frac{i}{2\pi}q_k\bigg(\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)+(\lambda p_0)^2\int\frac{d\nu}{\nu^2(\nu-\lambda p_0)}(h^+-d^+-\nu b^+)\bigg),\quad (4.13)$$

which, according to (3.3) and (3.4), leads to

$$\mathcal{T}_{0k}{}^{+}=\frac{i}{2\pi}q_k[\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)].\quad (4.14)$$

We finally obtain the E.T.C. as

$$\mathcal{T}_{0k}{}^{\alpha\beta}=\frac{i}{2\pi}\bigg(p_k\int d\nu\; a^-+q_k[\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)]\bigg)^{\alpha\beta}.\quad (4.15)$$

One sees that we obtain the form usually assumed in current algebra calculations; namely $\mathcal{T}_{00}$ and the antisymmetric part of $\mathcal{T}_{0k}$ form a Lorentz vector $p_\mu\int d\nu a$. The symmetric term of $\mathcal{T}_{0k}$ corresponds to the operator Schwinger term which we have shown to be necessarily symmetric in α,β.

If $m=\mu$, we use the same method as in Sec. IV A. $\mathcal{T}_{0k}{}^{-}$ is not changed since the same asymmetric argument applies. On the contrary, one gets another contribution to $\mathcal{T}_{0k}{}^{+}$, which becomes

$$\mathcal{T}_{0k}{}^{+}=\frac{i}{2\pi}[\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)+\mathcal{B}]\,,\quad (4.16)$$

where $\mathcal{B}$ is given in Appendix B [formula (B7) or (B8)]. In the case of $m=\mu$, the extra term is crucial in deriving the low-energy theorems.

Since the Schwinger term is given by subtraction constants, one understands clearly in our approach why it is model-dependent. In the real world where the integrals (3.7) and (3.8) do not converge without subtractions, the σ's are arbitrary. This illustrates the fact that the commutator being singular near the origin, the E.T.C. is not uniquely defined. In models, on the contrary, the convergence may be better so that the σ's may in certain cases be calculated. Equation (4.16) then gives a new method to determine the operator Schwinger terms. This is in particular the case in free-field models (see Appendix C). For instance, in the free-quark model of vector currents one finds that all the σ's are zero so that, in agreement with the usual method, the Schwinger terms are found to be c numbers.

In general, if the σ's are arbitrary one can choose them so that $\mathcal{T}_{0k}{}^{+}=0$, i.e., so that the Schwinger terms are c number. We will come back to this in Sec. VIII.

Finally, one usually expects $\mathcal{T}_{0k}{}^{+}$ to contain only a finite number of derivatives of $\delta_3(x)$. For this, $\sigma_h-\sigma_d$ has to be a polynomial in λ.[25]

C. Remarks on the E.T.C. of Space Components

To complete this study of the equal-time commutators, one may ask oneself if the same method can be used for the E.T.C. of two space components. In this case, the answer is no, the reason being that they correspond to bad charges in the classification of Fubini,

[24] M. Gourdin, Nuovo Cimento **18**, 145 (1967); **18**, 195 (1967).

[25] In fact, because of Lorentz invariance, only integer powers of ∇^2 appear.

Segrè, and Walecka.[26] For the other E.T.C., we have been able to use our method because, by introducing the divergences, one ends up with expression like (4.5) where the power in q_0 is larger in the denominator than in the numerator so that it can be reduced to dispersion relations. As discussed in Ref. 7 and in Appendix C, this fact suppresses the Z graphs in the $p\to\infty$ limit (see also Ref. 1). This was precisely the criterion discussed in Ref. 26 to distinguish between good and bad charges. Time components give good charges while space components lead to bad charges. And, as discussed in Ref. 26, the one-particle intermediate state gives the correct answer as $p\to\infty$ only for good-good and good-bad commutation relations. Thus our result agrees with those of Ref. 26.

D. E.T.C. of the Current and its Divergence

It is studied in the same way as $\mathcal{T}_{\mu\lambda}$; if $m\neq\mu$, one gets

$$\mathcal{T}_0{}^{\alpha\beta}=\frac{i}{2\pi}\int dq_0(lp_0+hq_0)=\frac{i}{2\pi}\sigma_w{}^{\alpha\beta}(\lambda,\lambda)\,,\qquad(4.17)$$

so that $\mathcal{T}_0{}^{\alpha\beta}$ is symmetric as expected. If $m=\mu$, one must add to σ_w the contribution $\mathcal{B}'$ of the one-particle intermediate state as given in Appendix B.

In our approach this commutator is thus on the same footing as the operator Schwinger term found in Sec. IV B, being also given by a subtraction constant. One thus understands why it is model-dependent. It is arbitrary if the integral which gives w does not converge without the subtraction as happens in the real world. In models, if the convergence is better one can compute it, and one gets the same answer as (e.g., in the free-quark model) when one uses the field equations.

As in Sec. IV C, our method does not determine $\mathcal{T}_k{}^{\alpha\beta}$.

V. RETARDED COMMUTATORS

In general, the retarded commutator is computed from $t_{\mu\lambda}(q)$ as

$$R_{\mu\lambda}{}^{\alpha\beta}(q_0,\mathbf{q})=\frac{1}{2\pi}\int\frac{dq_0'}{q_0'-q_0}t_{\mu\lambda}{}^{\alpha\beta}(q_0',\mathbf{q})\,.\qquad(5.1)$$

As for $\mathcal{T}_{\mu\lambda}$, we will consider $R_{\mu\lambda}$ only in those Lorentz frames where

$$\mathbf{q}\cdot\mathbf{p}=0.$$

Also we introduce the retarded commutators involving divergences, namely,

$$R_\mu{}^{\alpha\beta}(q_0,\mathbf{q})=\frac{1}{2\pi}\int\frac{dq_0'}{q_0'-q_0}t_\mu{}^{\alpha\beta}(q_0',\mathbf{q})\,,\qquad(5.2)$$

$$R^{\alpha\beta}(q_0,\mathbf{q})=\frac{1}{2\pi}\int\frac{dq_0'}{q_0'-q_0}w^{\alpha\beta}(q_0',\mathbf{q})\,.\qquad(5.3)$$

[26] S. Fubini, G. Segrè, and J. Walecka, Ann. Phys. (N. Y.) **39**, 381 (1966).

A. Expression of the Retarded Commutators in Terms of Dispersion Integrals

Consider first R_{00}. As in Sec. IV A, we introduce (3.2) into (5.1) and use (4.4) to get

$$R_{00}=\frac{1}{2\pi}p_0\int\frac{dq_0'}{(q_0'-q_0)(q_0'^2-\lambda^2)}\bigg[-p_0\lambda^2a-\lambda^2q_0'b+\frac{1}{p_0}[\lambda^2(h-d)+w]\bigg]\,,\qquad(5.4)$$

where again the principal value has to be taken at $q_0=\pm\lambda$. One now decomposes the factor $[(q_0'-q_0)(q_0{}^2-\lambda^2)]^{-1}$ so that (3.7) and (3.8) apply. Using (3.3) and (3.4) to simplify the result, one finally gets, if $\mu>m$,

$$R_{00}=Ap_0{}^2+2Bp_0q_0+Cq_0{}^2+D+\frac{1}{q_0{}^2-\mathbf{q}^2}\{\lambda^2[\sigma_h(q_0,\lambda)-\sigma_d(q_0,\lambda)]+\sigma_w(q_0,\lambda)-\lambda^2[\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)]-\sigma_w(\lambda,\lambda)\}\,.\qquad(5.5)$$

Similarly, for R_{0k} we write

$$R_{0k}=\frac{p_k}{2\pi}\int\frac{dq_0'}{q_0'-q_0}(ap_0+bq_0')+\frac{q_k}{2\pi}\int\frac{dq_0'}{q_0'-q_0}(bp_0+cq_0')\,,\qquad(5.6)$$

which, if $\mu>m$, leads to

$$R_{0k}=Ap_0p_k+B(p_0q_k+q_0p_k)+Cq_0q_k+q_0\sigma_c\,,\qquad(5.7)$$

or to

$$R_{0k}=Ap_0p_k+B(p_0q_k+q_0p_k)+Cq_0q_k+\frac{q_0}{q_0{}^2-\lambda^2}[\sigma_h(q_0,\lambda)-\sigma_d(q_0,\lambda)-\sigma_h(\lambda,\lambda)+\sigma_d(\lambda,\lambda)]\,.\qquad(5.8)$$

(5.7) is obtained by expressing the symmetric part of the coefficient of q_k directly in term of B and C, while (5.8) follows if one introduces h there first according to (3.4). We thus have necessarily

$$[\sigma_h(q_0,\lambda)-\sigma_d(q_0,\lambda)]-[\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)]=(q_0{}^2-\lambda^2)\sigma_c(q_0,\lambda)\,.\qquad(5.9)$$

For R_{jk}, there is nothing to compute; one has immediately

$$R_{jk}=Ap_jp_k+B(p_jq_k+q_jp_k)+Cq_jq_k+Dg_{jk}+q_jq_k\sigma_c(q_0,\lambda)+g_{jk}\sigma_d(q_0,\lambda)\,.\qquad(5.10)$$

In the same way one treats R_μ, obtaining, if $\mu>m$,

$$R_\mu=Lp_\mu+Hq_\mu+\sigma_h(q_0,\lambda)q_\mu\,,\qquad(5.11)$$

or, for R_0, as in (5.8),

$$R_0=Lp_0+Hq_0+[g_0/(g_0^2-\lambda^2)]\times[\sigma_w(q_0,\lambda)-\sigma_w(\lambda,\lambda)], \quad (5.12)$$

from which it follows that

$$\sigma_w(q_0,\lambda)-\sigma_w(\lambda,\lambda)=(q_0^2-\lambda^2)\sigma_h(q_0,\lambda). \quad (5.13)$$

Finally, for completeness, let us recall that R is in fact already given by (3.8), namely,

$$R=W+\sigma_w(q_0,\lambda). \quad (5.14)$$

As in Sec. IV, if $m=\mu$ one must add to $\sigma_h(\lambda,\lambda)-\sigma_d(\lambda,\lambda)$ and to $\sigma_w(\lambda,\lambda)$ the quantities $\mathcal{B}$ and $\mathcal{B}'$, respectively, as given in Appendix B.

B. Covariant Amplitude

Until the end of Sec. V we consider only the case $\mu>m$ explicitly, since it is easy to modify the formulas, as was done in Sec. V A, to treat the case $\mu=m$.

First we use (5.9) and (5.13) to transform the expression (5.10) of R_{00} so that it contains only σ_c and σ_h. Collecting the result together with (5.7) and (5.10), one sees that $R_{\mu\nu}$ can be written as

$$R_{\mu\nu}=\tilde{R}_{\mu\nu}+(1-\delta_{\mu0})(1-\delta_{\nu0})g_{\mu\nu}[\sigma_d(\lambda,\lambda)-\sigma_h(\lambda,\lambda)], \quad (5.15)$$

where $\tilde{R}_{\mu\nu}$ is covariant, being given by

$$\tilde{R}_{\mu\nu}=Ap_\mu p_\nu+B(p_\mu q_\nu+q_\mu p_\nu)+Cq_\mu q_\nu+Dg_{\mu\nu}+(q_\mu q_\nu-q^2g_{\mu\nu})\sigma_c(q_0,\lambda)+g_{\mu\nu}\sigma_h(q_0,\lambda). \quad (5.16)$$

$\tilde{R}_{\mu\nu}$ will in general be identified with the physical amplitude. One sees that, as expected, the retarded product being not uniquely defined according to our hypothesis is also noncovariant. In electrodynamics $R_{\mu\nu}-\tilde{R}_{\mu\nu}$ corresponds to the so-called "seagull term" which, in this case is given by the E.T.C. of the time derivative of the electromagnetic field with the Lagrangian.[1] It is thus model-dependent. This, in our approach, comes from the fact that the seagull term is given by the arbitrary subtraction constants.

It has been proposed in general by Bjorken[12] that $R_{\mu\nu}-\tilde{R}_{\mu\nu}$ be given by the Schwinger term. According to (4.14) this is exactly what we obtain.

C. Integration by Parts

The expressions (3.9), (3.10), and (3.11), together with (5.11) and (5.16), allow us to compute easily the divergences of the retarded products. One finds, taking also (3.6) into account,

$$q^\mu\tilde{R}_{\mu\rho}=R_\mu+\frac{1}{2\pi}p_\rho\int d\nu\, a^-, \quad (5.17)$$

$$q^\mu R_\mu=R-\sigma_w(\lambda,\lambda). \quad (5.18)$$

Combining (5.17) and (5.15), one verifies that the standard results of the formal partial integration are correct, namely, that the difference between $q^\mu R_{\mu\rho}$ and R_ρ is given by the E.T.C. calculated in Sec. III. This was not obvious, *a priori*, since $R_{\mu\nu}$ and R_ν contain subtraction constants. It follows in fact from (5.9) and (5.13).

We have, in particular,

$$q^\mu(R_{\mu\nu}-\tilde{R}_{\mu\nu})=(1-\delta_{\nu0})q_\nu[\sigma_d(\lambda,\lambda)-\sigma_h(\lambda,\lambda)],$$

which shows that the divergence of the seagull term is equal to the Schwinger term. This is a standard result in the particular case of electrodynamics.[1] We obtain it in general in our scheme. This property is crucial in deriving the low-energy theorems.

VI. LOW-ENERGY THEOREMS

Low-energy theorems were discussed for the case of the electromagnetic currents by Low[27] and Gell-Mann and Goldberger[28] long before current algebra was introduced.[29,30] Bég[31] used the current-algebra hypothesis to generalize them to the SU_3-antisymmetric part of the amplitude.

The usual proof of the low-energy theorem is based on the remark that, as $q\to0$, only the one-particle intermediate state contributes in the expression (5.1) of $R_{00}(q)$. This is a simple consequence of current conservation (in this section we assume that $\partial_\mu J^\mu=0$ and $m=\mu$). It allows us to determine the physical amplitude to lowest order in q_0.

In our approach, the dependence in q is such that as $q\to0$ we will end up with dispersion relations computed at zero mass. Here also, the one-particle intermediate state will play a special role since the corresponding pole is at the origin for $q^2=0$.

We will consider explicitly the case of Compton scattering of a spin $\frac{1}{2}$ target. In this case, according to Appendix B, when one separates out the one-particle intermediate state, Eq. (5.16) reads

$$\tilde{R}_{\mu\lambda}{}^\pm=-\left(\frac{1}{q^2+2q_0p_0}\pm\frac{1}{q^2-2q_0p_0}\right)[M^\pm(2p_\mu p_\lambda+\tfrac{1}{2}q_\mu q_\lambda)-q_\mu q_\lambda\tfrac{1}{2}N^\pm]+q_\mu q_\lambda\left(\frac{1\pm1}{2}\right)\frac{M^\pm-N^\pm}{q^2}$$
$$-\left(\frac{1}{q^2+2q_0p_0}\mp\frac{1}{q^2-2q_0p_0}\right)[(q_\mu p_\lambda+q_\lambda p_\mu)M^\pm-p_0q_0g_{\mu\lambda}N^\pm]+A'p_\mu p_\lambda+B'(p_\mu q_\lambda+q_\lambda p_\mu)+C'q_\mu q_\lambda+D'$$
$$+\left(\frac{1\pm1}{2}\right)(q_\mu q_\lambda-g_{\mu\lambda}q^2)\sigma_c(q_0,\lambda), \quad (6.1)$$

[27] F. E. Low, Phys. Rev. **96**, 1428 (1954).
[28] M. Gell-Mann and M. Goldberger, Phys. Rev. **96**, 1433 (1954).
[29] See also S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).
[30] For recent references on the subject see H. D. I. Abarbanel and M. L. Goldberger, Phys. Rev. **165**, 1594 (1968), and Ref. 1.
[31] M. A. B. Bég, Phys. Rev. **150**, 1276 (1966).

where A', B', C', and D' are given by (3.9)–(3.11) without the one-particle contribution; $M^{\pm}$, $N^{\pm}$ stand for

$$M^{\alpha\beta}=(M^{+}+M^{-})^{\alpha\beta}=\tfrac{1}{2}([\lambda^{\alpha},\lambda^{\beta}]_{+}+[\lambda^{\alpha},\lambda^{\beta}]_{-})(F_1^2-q^2F_2^2),$$
$$N^{\alpha\beta}=(N^{+}+N^{-})^{\alpha\beta}=\tfrac{1}{2}([\lambda^{\alpha},\lambda^{\beta}]_{+}+[\lambda^{\alpha},\lambda^{\beta}]_{-})(F_1+F_2)^2. \quad (6.2)$$

F_1 and F_2 being the standard Pauli and Dirac form factors. Now it is convenient to consider R^{+} and R^{-} separately.

A. Symmetric Part

The behavior of A', B', C', D' is easily obtained for $q\rightarrow 0$ since the one-particle intermediate state has been subtracted out. One gets, taking (3.6) into account,

$$2\pi A'^{+}\simeq\lim_{q\rightarrow 0}\int'\frac{d\nu}{\nu}a^{+}(\nu,q^2),$$
$$2\pi B'^{+}\simeq\lim_{q\rightarrow 0}\int'\frac{d\nu}{\nu^2}b^{+}(\nu,q^2),$$
$$2\pi C'^{+}\simeq q_0p_0\lim_{q\rightarrow 0}\int'\frac{d\nu}{\nu^3}c^{+}(\nu,q^2), \quad (6.3)$$
$$2\pi D'^{+}\simeq q_0p_0\lim_{q\rightarrow 0}\int'\frac{d\nu}{\nu^3}d^{+}(\nu,q^2).$$

(The prime on the integral sign means that the one-particle intermediate state has been dropped.) Of course, in this case a, b, c, and d are not independent. Using the optical theorem (see, e.g., Ref. 24), one has, as $q\rightarrow 0$,

$$a\simeq(1/\nu)q^2\sigma_T,\quad \nu b\simeq d\simeq\nu\sigma_T,$$

where σ_T is the total electroproduction cross section by transverse photons so that, for ν fixed in the continuum,

$$a=O(q_0^2),\quad b=O(1),\quad c=O(1),\quad d=O(1); \quad (6.4)$$

thus one gets, from (6.3), (6.4),

$$A'^{+}=O(q_0^2),\quad B'^{+}=O(q_0),\quad C'^{+}=O(q_0^2),\quad D'^{+}=O(q_0^2).$$

Thus, in (6.1), the multiparticle intermediate state contribution is of second order in q_0 at least. On the other hand, since $m=\mu$, (5.9) becomes

$$\sigma_c(q_0,\lambda)=(1/q^2)[\sigma_d(\lambda,\lambda)-\sigma_d(q_0,\lambda)+M^{+}-N^{+}];$$

thus, if $q\rightarrow 0$,

$$\sigma_c(q_0,\lambda)\simeq\frac{M^{+}-N^{+}}{q^2}+O(1). \quad (6.5)$$

Finally, one gets [here $M^{+}(0)=1$]

$$\tilde{R}_{\mu\nu}=p_\mu p_\nu\frac{q^2}{(q_0p_0)^2}-(p_\mu q_\nu+q_\mu p_\nu)\frac{1}{q_0p_0}+g_{\mu\nu}+O(q_\mu q_\nu). \quad (6.6)$$

The result corresponds to the Thomson formula; namely, using the Coulomb gauge in the rest frame ($\mathbf{p}=0$), denoting by $\boldsymbol{\varepsilon}$ the polarization of the photon ($\boldsymbol{\varepsilon}\cdot\mathbf{q}=0$), one has

$$\epsilon^{\mu}\epsilon^{\nu}\tilde{R}_{\mu\nu}=-\boldsymbol{\varepsilon}^2=-1, \quad (6.7)$$

and also

$$\tilde{R}_{00}=-\mathbf{q}^2/q_0^2. \quad (6.8)$$

Formulas (6.7) and (6.8) are the standard results averaged over spin (see, e.g., Ref. 27).

It is to be remarked that, in this approach, the one-particle intermediate state dominates as $q\rightarrow 0$ for every component of $\tilde{R}_{\mu\nu}$, while in the usual calculation that is true, in Eq. (5.1), only for $\tilde{R}_{00}$.

B. Antisymmetric Part

In Sec. IV, we have shown that $\int d\nu\, a(\nu,q^2)$ in fact does not depend on q^2 as a consequence of our hypothesis. We will first show here that, in the case considered in this section, one can actually compute $\int d\nu\, a$ by studying also its limit as $q\rightarrow 0$. In fact, from the optical theorem one has (see, e.g., Ref. 24)

$$\frac{1}{2\pi}\int d\nu\, a^{-}(\nu,q^2)=2M^{-}(q^2)+\frac{2q^2}{3\pi e^2}\int'\frac{d\nu}{(\nu^2+q^2m^2)^{1/2}}(\sigma_T+\sigma_L)^{-}, \quad (6.9)$$

where σ_T (σ_L) is the total electroproduction cross section with transverse (longitudinal) photons, and where we have extracted the one-particle intermediate state as computed in Appendix B. Letting $q\rightarrow 0$ on the right and since the left member does not depend on q, one gets

$$\frac{1}{2\pi}\int d\nu\, a^{-}(\nu,q^2)^{\alpha\beta}=2M^{-\alpha\beta}(0)=[\lambda^{\alpha},\lambda^{\beta}], \quad (6.10)$$

which is the Fubini–Dashen–Gell-Mann sum rule, and from (4.8) leads to

$$\mathcal{T}_{0\mu}{}^{\alpha\beta}=ip_\mu f^{\alpha\beta\gamma}\lambda^{\gamma}+(1-\delta_{\mu 0})(\text{sym. terms}) \quad (6.11)$$

in agreement with the usual hypothesis on the E.T.C.

In this case, *our approach is thus sufficient to determine the current-algebra scheme completely*. Now we go back to the retarded commutator. Let us first consider $\tilde{R}_{00}{}^{-}$. It is convenient to rewrite it as

$$\tilde{R}_{00}{}^{-}=\frac{1}{2\pi}\frac{p_0}{q_0}\left[-\int d\nu\, a^{-}+\lambda^2\int d\nu\frac{b^{-}}{\nu-\lambda p_0}\right]+\frac{1}{2\pi}q_0p_0\int\frac{d\nu}{\nu(\nu-\lambda p_0)}c^{-}. \quad (6.12)$$

From (6.4), as $q\rightarrow 0$ in the last integral, the contribution of the continuum is of higher order. For the second integral, one has

$$\int'\frac{d\nu}{\nu}b^{-}(\nu,0)=\left[\frac{1}{q^2}\int' d\nu\, a^{-}(\nu,q^2)\right]_{q^2=0}. \quad (6.13)$$

It follows from (6.9) that

$$\int' d\nu\; a(\nu,0)=0,$$

so that (6.13) can be rewritten as

$$\int' \frac{d\nu}{\nu}\, b^-(\nu,0)=\left[\frac{d}{dq^2}\int' d\nu\; a^-(\nu,q^2)\right]_{q^2=0},$$

which, from (4.12), leads to

$$\frac{1}{2\pi}\int' \frac{d\nu}{\nu}\, b^-(\nu,0)=2\left[\frac{dM^-}{dq^2}\right]_{q^2=0}.$$

Finally, letting $q^2=0$, one gets

$$\tilde{R}_{00}=\frac{p_0}{q_0}[-2M^-(0)+\lambda^2 M'(0)]-\frac{\lambda^2}{2q_0p_0}[M^-(0)-N^-(0)]+O(q_0^2).$$

This formula, in the rest frame ($\mathbf{p}=0$), reduces to

$$R_{00}^{-\alpha\beta}=\tfrac{1}{2}[\lambda^\alpha,\lambda^\beta]m\left[-\frac{2}{q_0}+4q_0\left(F_1'+\frac{F_2}{4m^2}\right)\right].\quad(6.14)$$

In the same frame, one easily studies the space components obtaining

$$R_{jk}^-=-\frac{1}{2q_0m}q_jq_k(M^--N^-)-q_0m\delta_{jk}\frac{1}{2\pi}\int\frac{d\nu}{\nu^2}d^-+O(q_0^2).\quad(6.15)$$

As previously, and using (6.4), one gets

$$\frac{1}{2\pi}\int'\frac{d\nu}{\nu^2}d^-=-2M'^-(0),$$

which finally leads to

$$R_{jk}^{-\alpha\beta}=\tfrac{1}{2}[\lambda^\alpha,\lambda^\beta]\left[\frac{1}{2q_0m}(F_2^2+2F_2)q_jq_k+4q_0m\left(F_1'-\frac{F_2^2}{8m^2}\right)\right]+O(q_0^2).\quad(6.16)$$

Formulas (6.15) and (6.16) coincide with the corresponding results given by Bég.[31]

It is interesting to remark that, for the antisymmetric part, the one-particle intermediate state does not dominate in the covariant integrals as $q\to 0$. This is due to the fact that, in this case, we made fewer subtractions than for the symmetric part. Nevertheless, one gets the low-energy theorems because the contribution of the continuum can be evaluated. This involves taking the derivative of $\int d\nu\, a$ at $q^2=0$. As we emphasized in Sec. IV, this leads to the Cabibbo-Radicati sum rule. On the other hand, this sum rule was precisely rededuced by Bég from using formulas (6.14) and (6.16) which he derived directly. One thus sees the complete equivalence of our result with the results of the conventional approach.

C. Remarks

It was, of course, predictable *a priori* that we would get the correct low-energy theorems since we know from Sec. V that the standard partial integration formulas hold in our approach while the usual assumption on the Schwinger term have also been verified. However, it was interesting to show explicitly how they were satisfied.

We will not discuss the case of axial-vector current which can be treated along the same line. We know already that the standard results are obtained. Let us remark, however, that if one takes $m_\pi^2=0$ and $\partial_\mu A^\mu=0$, then as in Sec. IV B, one can compute $\int d\nu\, a$ by letting $q\to 0$ and then obtain the standard $SU_2\times SU_2$ E.T.C.

VII. INFINITE-MOMENTUM LIMIT

As already discussed, e.g., in Ref. 7, in this case one lets $p\to\infty$ in a fixed direction such that $\mathbf{q}\cdot\mathbf{p}=0$. A formula of the type (2.6) is then very useful if one takes $\kappa^2-\mathbf{q}^2=$constant as $p\to\infty$ since then it contains the dependence in p explicitly so that the $p\to\infty$ limit is reduced to the high-energy limit of ordinary dispersion integrals.

In Ref. 7, we used this method to re-establish the proof of the Fubini–Dashen–Gell-Mann sum rule from an infinite-momentum limit. We will briefly apply the same idea to $R_{\mu\nu}^-$. In the pure Lorentz transformation along $\mathbf{p}$, the Lorentz vector $(0,\mathbf{q})$ is invariant since $\mathbf{p}\cdot\mathbf{q}=0$. Accordingly, if we keep q_0 fixed, $(q_0^2-\mathbf{q}^2)$ will be independent of $|\mathbf{p}|$ and the method will apply.

It thus follows from (3.5) that (see, e.g., Ref. 32)

$$A^-\simeq-\frac{1}{q_0p_0}\int d\nu\; a^-,$$

$$B^-\simeq(q_0p_0)^{\delta-1},\quad C^-,D^-\simeq(q_0p_0)^\delta,$$

with $0<\delta<1$ as $p_0\to\infty$. One thus gets

$$R_{\mu\nu}^-\simeq-p_\mu p_\nu\frac{1}{p_0q_0}\int d\nu\; a^-.\quad(7.1)$$

It has been proposed by Bjorken[12] that as $q_0\to\infty$, $R_{\mu\nu}$ behaves like the E.T.C. divided by q_0. According to (4.8), one sees that, after taking the limit, the same type of result is obtained also for finite q_0.

[32] J.-L. Gervais and F. J. Yndurain, Phys. Rev. **167**, 1289 (1968); **169**, 1187 (1968).

An equation of the type (7.1) is of great importance to determine, in particular the convergence of electromagnetic mass differences,[33] or β-decay radiative corrections.[34] Our result suggests that a $p\to\infty$ limit may be appropriate to study them. However, the condition $\mathbf{p}\cdot\mathbf{q}=0$ is *a priori* difficult to take into account, since in this type of problem, one integrates over q in the whole space. We may devote a further publication to this problem.

VIII. CONCLUSION

In conclusion, we present some comments on the results obtained in this paper as well as in Ref. 7.

We assume that $\mathbf{p}\cdot\mathbf{q}=0$. As already discussed p and q being given, there exists a continuous set of Lorentz frames where $\mathbf{p}\cdot\mathbf{q}=0$. Presently we do not know how to treat the general case. In fact this condition was essential to obtain expressions of the E.T.C. and R.C. where the power of q_0 was higher in the denominator than in the numerator so that our method could be used. This is unimportant in the results discussed here since the dependence of $\mathbf{q}$ ultimately disappears.

We considered only the case of diagonal matrix elements averaged over spin and/or relative momenta. This limitation is not fundamental since, in particular, we obtain the relation between covariant and noncovariant integrals also in the general case for which one simply would have more invariants. However, the calculation becomes rather tedious. It is possible that a more powerful method can be used based on group theory, since in the case we considered, the introduction of matrix elements involving divergences is equivalent to decomposing the tensor representation of the Lorentz group, by which the commutator transforms, into irreducible representations. If spin is taken into account, the corresponding representation is more complicated, but the method is perhaps to decompose it again in irreducible representations. We may study this question in another paper.

We cannot take commutators of two space components into account. As we already discussed, this limitation is natural and fundamental. It agrees with the general feeling one now has in current algebra, namely, that the results based on commutation of two space components are less reliable than the others.

Our formulas are the usual ones, except for an unknown parameter $\int d\nu\, a$. It can be directly computed if the current is conserved and is then just equal to the corresponding commutator of the infinitesimal generator of the internal symmetry group. This result is not surprising since, in this case the integrated charges form a representation of the corresponding Lie algebra. In general, one must assume the value of $\int d\nu\, a$, and this is the rigorous formulation of the current-algebra hypothesis in our approach.

From the mathematical point of view, as we already emphasized, the introduction of the E.T.C. and R.C. from the commutators amounts to a definition of products by $\delta(x_0)$ and $\theta(x_0)$, respectively. Our results show the familiar pattern. Namely, the behavior at infinity in momentum space of the commutator determines its singularity near the origin in position space. If it is too singular, then the products becomes ambiguous and noncovariant.

For the antisymmetric part, the physical high-energy behavior is just sufficient to make the E.T.C. and R.C. well defined and covariant, since the subtraction constants which enter are then symmetric from crossing symmetry. This explicitly shows how much the "low-energy" results depend on the assumed high-energy behavior. If more subtractions were needed they would produce, e.g., gradient terms in the E.T.C. of two time components, thus spoiling the low-energy theorems. On the contrary, the symmetric part decreases less rapidly at infinity in momentum space; as a result the corresponding E.T.C. and R.C. are noncovariant and ambiguous.

That the experimental high-energy behavior is exactly suited to getting the current-algebra results is a remarkable fact which shows the consistency between the present high-energy and low-energy schemes.

It is an interesting property of our approach that it introduces naturally, by means of subtraction constants, the quantities which usually are ambiguous and/or model-dependent, namely, the Schwinger and seagull terms and the E.T.C. of $\partial_\mu J^\mu$ with J_0. Thus in our formalism those terms are *a priori* completely arbitrary. However, let us remark that, first of all, in perturbation theory, it has been emphasized by, e.g., Bogoliubov and Shirkov[35] that renormalization amounts to the definition of products of distributions which are *a priori* ambiguous, being divergent in momentum space. Thus, in our approach, to choose a particular determination of the σ's can be interpreted as a sort of renormalization which takes the physical high-energy behavior into account. One may in particular define the E.T.C. by letting

$$\sigma_h(\lambda,\lambda)=\sigma_d(\lambda,\lambda).$$

Then there will be no operator Schwinger term, in agreement with the recently proposed algebra of fields.[36]

In the case of $\sigma_w(\lambda,\lambda)$, it seems to be very small if $|p\rangle$ is not the one-π state. For instance, letting $\sigma_w(\lambda,\lambda)=0$ in the calculation of the scattering length of Tomozawa[37]

[33] J. Das *et al.*, Phys. Rev. Letters **18**, 759 (1967).
[34] E. Abers *et al.*, Phys. Rev. Letters **18**, 676 (1967); N. Cabibbo *et al.*, Phys. Letters **25B**, 31 (1967); **25B**, 132 (1967); K. Johnson *et al.*, Phys. Rev. Letters **18**, 1224 (1967).
[35] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley-Interscience Publishers, Inc., New York, 1959).
[36] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[37] Y. Tomozawa, Nuovo Cimento **46A**, 707 (1966).

and Weinberg[38] leads to

$$2a_{3/2}+a_{1/2}=0, \qquad (8.1)$$

while experimentally this number is $0.005m_\pi^{-1}$.

For the retarded commutators involving the divergence of the axial current, the σ's should be chosen so that the amplitude is as smooth as possible from[39] $q^2=0$ to $q^2=m_\pi^2$, in agreement with the spirit of PCAC. From (5.14) one has up to a factor,

$$\langle\pi^i N|\pi^j N\rangle=(q^2-m_\pi^2)^2\sigma_w{}^{ij}+(q^2-m_\pi^2)^2W^{ij}; \qquad (8.2)$$

also, it follows from (5.11) that

$$\langle\pi^i N|A_\mu|N\rangle=(q^2-m_\pi^2)\sigma_h q_\mu +(q^2-m_\pi^2)(Lp_\mu+Hq_\mu). \qquad (8.3)$$

As expected, the on-mass-shell matrix elements do not depend on the subtraction constants, since the σ's are regular at $q^2=m_\pi^2$. However, the soft off-mass-shell matrix elements will be those for which

$$\sigma_w(q_0,\lambda)=\sigma_h(q_0,\lambda)=0, \qquad (8.4)$$

which also agrees with (8.1).[40]

In the case of a conserved current, condition (8.4) is of course satisfied; thus the axial current is put on the same footing as a conserved current by the smoothness property. This agrees with the approach used, e.g., by Weinberg[41] in which one lets $m_\pi^2=0$ and uses a conserved axial-vector current.

Our last comment will be about our hypothesis of Sec. III on the possible extra terms in the relation between covariant and noncovariant dispersion relations. They can actually be checked on field models since both members of (3.7) and (3.8) can be computed. Let us remark that one should not use a nonlocal approximation since locality was the key to our results of Secs. I and II. Thus many models, like, e.g., ladder-graph approximations, are inappropriate. We have shown that, given the physical high-energy behavior, the current algebra results follow, up to the value of the E.T.C., from our hypothesis of Sec. III. An interesting question is whether the converse is true; namely, if extra terms appear in (3.7) and (3.8), will they invalidate the results? The answer is that certainly no antisymmetric extra terms can be introduced without losing the connection with the usual formulas of current algebra, since, e.g., one obtains antisymmetric gradient terms in the E.T.C. (an example of such a case is given in Appendix C). There can be, in principle, symmetric extra terms which would not change the essential features of the results. However, this more general approach is unappealing since one no longer has the link between high-energy physics and current algebra. One must essentially postulate that the Schwinger terms are symmetric while, in this paper, we proved this result by using essentially crossing symmetry and the high-energy physical picture.

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APPENDIX A: LOCALITY OF THE INVARIANTS

Here we show that the invariant distributions a, b, c, and d can be determined from $t_{\mu\lambda}$ according to (3.2) so that they vanish outside the lightcone in position space. By inverse Fourier transformation, (3.2) becomes

$$\hat{t}_{\mu\lambda}(x)=\hat{a}p_\mu p_\lambda+\left(ip_\mu\frac{\partial}{\partial x^\lambda}+ip_\lambda\frac{\partial}{\partial x^\mu}\right)\hat{b} -\frac{\partial^2}{\partial x^\mu\partial x^\lambda}\hat{c}+g_{\mu\nu}\hat{d}. \qquad (A1)$$

$\hat{t}_{\mu\lambda}(x)$ vanishes for $x^2<0$. To carry out the proof, it is convenient to choose the rest frame where $\mathbf{p}=0$. In this case, (A1) takes a very simple form for the space components:

$$\hat{t}_{jk}(x)=-\frac{\partial^2}{\partial x^j\partial x^k}\hat{c}-\hat{d}\delta_{jk}. \qquad (A2)$$

On the other hand, $\hat{t}_{\mu\lambda}$ can also be expanded in x space on a set of invariants by writing

$$\hat{t}_{\mu\lambda}=a'p_\mu p_\lambda+b'(p_\mu x_\lambda+x_\mu p_\lambda)+c'x_\mu x_\lambda+d'g_{\mu\lambda}. \qquad (A3)$$

Since $\hat{c}$ is a function only of $\mathbf{x}^2$ and x^0, one has for $j\neq k$

$$\hat{t}_{jk}=-4x_jx_k\frac{\partial^2\hat{c}}{\partial(\mathbf{x}^2)^2}=x_jx_kc'(x^0,\mathbf{x}^2).$$

Solving this differential equation with the boundary condition

$$\lim_{|\mathbf{x}|\to\infty}\hat{c}=\lim_{|\mathbf{x}|\to\infty}\frac{\partial\hat{c}}{\partial(\mathbf{x}^2)}=0,$$

one gets

$$\hat{c}=-\tfrac{1}{4}\int_{(\mathbf{x})^2}^{\infty}du\int_u^\infty dv\, c'(x^0,v),$$

which shows that $\hat{c}$ vanishes for $x^2<0$, since c' does. Knowing this result and going back to (A2), now with $j=k$ one verifies immediately that d is also zero for $x^2<0$. On the other hand, for $\mu=0$, $\nu=k$, (A3) becomes

$$\hat{t}_{0k}=ip_0\frac{\partial}{\partial x^k}\hat{b}+\frac{\partial^2}{\partial x^0\partial x^k}\hat{c}.$$

[38] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).
[39] Our analysis does not directly apply since here $q^2\geqslant 0$; however, one can extend our result by analytic continuation in q^2.
[40] This is essentially equivalent to a discussion given in Ref. 38.
[41] See, e.g., S. Weinberg, Phys. Rev. Letters 16, 879 (1966).

Solving this equation in x_k with the boundary condition

$$\lim_{|\mathbf{x}|\to\infty} \hat{b}(x^0,\mathbf{q}^2)=0\,,$$

one obtains, after averaging over space and taking (A3) into account,

$$ip_0\hat{b}(x^0,\mathbf{x}^2)=\frac{\partial}{\partial x^0}\hat{c}-\tfrac{1}{6}\int_{\mathbf{x}^2}^{\infty} du[p_0b'(x^0,u)+x_0c'(x^0,u)]\,,$$

which shows that $\hat{b}$ also vanishes for $x^2<0$. Finally, from (A1) taken for $\mu=\nu=0$, it follows immediately that the same property holds for $\hat{a}$, Q.E.D.

APPENDIX B: ONE-PARTICLE INTERMEDIATE-STATE CONTRIBUTIONS

We recapitulate the relevant formulas and so doing indicate our conventions of normalization. In the spin-$\frac{1}{2}$ case, we write the vector current as

$$\langle p'|V_\mu{}^\alpha|p\rangle =\tfrac{1}{2}\bar{u}(p')\left[\gamma_\mu(F_1+F_2)^\alpha+\frac{(p+p')_\mu}{2m}F_2{}^\alpha\right]u(p)\,, \quad \text{(B1)}$$

where F_1 and F_2 are the standard Dirac and Pauli form factors, respectively. Our conventions are

$$\{a(p),a^*(p')\}=p_0\delta_3(\mathbf{p}-\mathbf{p}')\,,\qquad \bar{u}(p)u(p)=2m\,, \quad \text{(B2)}$$

and we omit everywhere an over-all factor of $(2\pi)^{-3}$. The one-particle intermediate-state contribution to $t_{\mu\lambda}$ is then easily computed; one gets, if $m=\mu$,

$$\begin{aligned} a^\pm&=4\pi M^\pm\varphi^\mp\,,\\ b^\pm&=2\pi M^\pm\varphi^\pm\,,\\ c^\pm&=\pi(M^\pm-N^\pm)\varphi^\mp\,,\\ d^\pm&=-2\pi(p\cdot q)N^\pm\varphi^\pm\,, \end{aligned} \quad \text{(B3)}$$

and, of course, $l=h=w=0$. In (B3) we have let

$$\varphi^\pm(q,p)=\epsilon(q_0+p_0)\delta[(q+p)^2-m^2] \pm\epsilon(p_0-q_0)\delta[(q-p)^2-m^2]\,, \quad \text{(B4)}$$

$$M^{\pm\alpha\beta}=\tfrac{1}{2}\left[[F_1{}^\alpha,F_1{}^\beta]_\pm-\frac{q^2}{4m^2}[F_2{}^\alpha,F_2{}^\beta]_\pm\right], \quad \text{(B5)}$$

$$N^{\pm\alpha\beta}=\tfrac{1}{2}[(F_1+F_2)^\alpha,(F_1+F_2)^\beta]_\pm\,. \quad \text{(B6)}$$

The term $\mathcal{B}$ which appears in (4.16) is easily computed as

$$\mathcal{B}=\lim_{q\to0} q^2q_0p_0\frac{1}{2\pi}\int\frac{d\nu\, c^+}{\nu(\nu-q_0p_0)}=M^+-N^+. \quad \text{(B7)}$$

In the case of the axial-vector current we let

$$\langle p'|A_\mu{}^\alpha|p\rangle=\tfrac{1}{2}\bar{u}(p')\{\gamma_\mu K_1{}^\alpha+[(p'-p)_\mu/2m]K_2{}^\alpha\}\gamma_5u(p)\,,$$

which leads to

$$\begin{aligned} a^\pm&=4\pi K_{11}{}^\pm\varphi^\mp\,,\\ b^\pm&=2\pi K_{11}{}^\pm\varphi^\pm\,,\\ c^\pm&=2\pi[K_{12}{}^\pm+K_{21}{}^\pm+(q^2/4m^2)K_{22}{}^\pm]\varphi^\mp\,,\\ d^\pm&=\pi K_{11}{}^\pm(q^2-4m^2)\varphi^\mp\,,\\ l^\pm&=0\,,\\ h^\pm&=-4\pi m^2K_{11}{}^\pm\varphi^\mp +q^2[K_{12}{}^\pm+K_{21}{}^\pm-(q^2/4m^2)K_{22}{}^\pm]\varphi^\mp\,, \end{aligned}$$

where

$$K_{ij}{}^{\pm\alpha\beta}=\tfrac{1}{2}[K_i{}^\alpha,K_j{}^\beta]_\pm\,.$$

$\mathcal{B}$ and $\mathcal{B}'$ are here also easily computed:

$$\mathcal{B}=2(K_{12}{}^++K_{21}{}^+),\quad \mathcal{B}'=-4m^2K_{11}{}^+. \quad \text{(B8)}$$

APPENDIX C: FREE-FIELD MODELS

In this appendix, we show that the results of Sec. III are true in various free-field models of the current for which current algebra is satisfied.

1. Free Spin-Zero Fields

In Ref. 3, we have considered the model where

$$J_\mu{}^i=i\sigma\frac{\overleftrightarrow{\partial}}{\partial x^\mu}\pi^i\,, \quad \text{(C1)}$$

where π and σ are free fields of isospin 1 and 0, respectively. It is easy to see that, in this model, taking the matrix element between two π states, one gets

$$\begin{aligned} &a^\pm=4\pi T^\pm\varphi^\mp, && b^\pm=2\pi T^\pm\varphi^\pm,\\ &c^\pm=\pi T^\pm\varphi^\pm, && d^\pm=0,\\ &l^\pm=2\pi(m_\sigma{}^2-m_\pi{}^2)T^\pm\varphi^\pm, && h^\pm=\pi(m_\sigma{}^2-m_\pi{}^2)T^\pm\varphi^\mp,\\ &w^\pm=\pi(m_\sigma{}^2-m_\pi{}^2)^2T^\pm\varphi^\mp, \end{aligned} \quad \text{(C2)}$$

where $T^\pm$ is given by the isotopic-spin factors.

Now one can verify directly from (B4) that for $\varphi^\pm$, the covariant and noncovariant dispersion integrals are equal; namely, if $\mathbf{p}\cdot\mathbf{q}=0$ and $q^2<0$, one has

$$\int\frac{dq_0'}{q_0'-q_0}\varphi^\pm(q_0,\mathbf{q},p)=\int\frac{d\nu}{\nu-q_0p_0}\varphi^\pm(\nu,q^2). \quad \text{(C3)}$$

Accordingly, one verifies that (3.7) and (3.8) hold in this model. As we indicated in Sec. III, when one computes l, w, and h from a, b, c, and d according to (3.3) and (3.4), the factor p_0q_0 and q^2 which appears in those formulas cancel so that l, w, and h also satisfy (3.7) or (3.8).

Since, of course, the integrals converge already before subtracting, one can actually compute the σ's from formulas of the type

$$\sigma_c(q_0,\lambda)=\frac{1}{2\pi}\int\frac{d\nu}{\nu}\,c^+(\nu,\,q_0{}^2-\lambda^2).$$

One gets

$$\sigma_c = T^+/(m_\sigma^2 - m_\pi^2 - q^2), \quad \sigma_d = 0,$$

$$\sigma_w = (m_\sigma^2 - m_\pi^2)\sigma_h = 2T^+(m_\sigma^2 - m_\pi^2)^2/(m_\sigma^2 - m_\pi^2 - q^2).$$

Thus (4.14) shows that there is an operator Schwinger term given by

$$\mathcal{T}_{ok}^+ = -(i/2\pi) q_k T^+,$$

while one has from (4.17)

$$\mathcal{T}_0 = (i/2\pi) T^+ (m_\sigma^2 - m_\pi^2).$$

Those formulas agree with what one obtains from the canonical commutation relations. Finally, our study of the $p \to \infty$ limit made in Ref. 6 has shown that

$$\lim_{p\to\infty} p_0 \int dq_0 a^-(q_0, \mathbf{q}) \neq \int d\nu \lim_{p\to\infty} a^-, \qquad \text{(C4)}$$

while in Ref. 7 we obtained

$$\lim_{p\to\infty} p_0 \int \frac{dq_0}{q_0^2 - \lambda^2} a^- = -\lambda^2 \int d\nu \lim_{p\to\infty} a^-. \qquad \text{(C5)}$$

One can see, in the model considered here, how this happens. One may separate a^- into two parts:

$$a^{(1)} = 4\pi T^- \{\theta(p_0 + q_0)\delta[(q+p)^2 - m_\sigma^2] + \theta(p_0 - q_0)\delta[(q-p)^2 - m_\sigma^2]\},$$

$$a^{(3)} = 4\pi T^- \{\theta(-p_0 - q_0)\delta[(q+p)^2 - m_\sigma^2] + \theta(-p_0 + q_0)\delta[(q-p)^2 - m_\sigma^2]\}.$$

$a^{(1)}$ comes from the one-particle intermediate state in the noncovariant expansion of the retarded product, i.e., "straight graphs," while $a^{(3)}$ corresponds to the three-particle intermediate states, namely, "Z graphs." As we already discussed, the left-hand side of (C4) is actually equal to zero since straight and Z graphs cancel, namely,

$$\int dq_0 a^{(1)} = -\int dq_0 a^{(3)}.$$

Nevertheless, for q_0 fixed and $p \to \infty$,

$$\theta(p_0 \pm q_0) \to 1, \quad \theta(-p_0 \pm q_0) \to 0.$$

Accordingly, the Z graphs tend to zero in the limit and the right-hand side of (C4) is not zero since, as expected, only straight graphs contribute. For fixed $\mathbf{q}$ a is a sum of δ function of q_0; in (C4) increasing p essentially translates $a^{(3)}$ without decreasing its contribution to the integral so that the interchange of limit and integral is not allowed. In (C5), on the contrary, the factor $(q_0^2 - \lambda^2)^{-1}$ suppresses the contribution of $a^{(3)}$ as $p \to \infty$ so that one can interchange the limit and the integration. This is similar to the discussion of Refs. 6 and 7.

2. Free-Spin-$\frac{1}{2}$ Fields and Vector Current

Here we take

$$V_\mu^\alpha = \bar{\psi}\gamma_\mu \lambda^\alpha \psi,$$

so that the result is obtained from (B3), letting

$$M^\pm \equiv N^\pm \equiv \tfrac{1}{2}[\lambda^\alpha, \lambda^\beta].$$

As in case (1), (3.7) and (3.8) follow from (B4) for a, b, c. However, d contains, as expected, an extra factor $q \cdot p$. Instead of (B4), we use the fact that

$$\int \frac{dq_0'}{q_0' - q_0} (q_0' p_0) \varphi^\pm = \int \frac{d\nu\, \nu}{\nu - q_0 p_0} \varphi^\pm(\nu, q^2) - \tfrac{1}{2}(1 \pm 1). \quad \text{(C6)}$$

This equality is a particular case of (2.8). It follows from (C6) that no extra term appears in d^- and that (3.8) is also satisfied for d.

It is easy in this model also to compute the σ's. One gets

$$\sigma_c = \sigma_d \equiv 0,$$

which shows that, as one knows already directly, the Schwinger term is a c number.

3. Free-Spin-$\frac{1}{2}$ Particle and Axial-Vector Current

As usual, we let

$$A_\mu^\alpha = \bar{\psi}\gamma_\mu \gamma_5 \lambda^\alpha \psi,$$

so that the result is obtained from (B8) by letting

$$K_{ij} = 0 \quad \text{if} \quad i \neq 1 \text{ or } j \neq 1.$$

Here also (3.7) and (3.8) are satisfied. Note that here, as in (1), no q^2 or $q \cdot p$ factor appears in l, w, and h. One finally gets

$$\sigma_h = \sigma_d = -4m^2/q^2 \qquad \text{(C7)}$$

and

$$\sigma_w = 0. \qquad \text{(C8)}$$

From (C7) one sees that the Schwinger term is a c number while (C8) shows that the E.T.C. of A_0 and $\partial_\mu A^\mu$ vanishes. This also agrees with the free-quark model where it is found to be proportional to $\bar{u}\gamma_5 u$ which vanishes in the forward direction.

4. Free-Spin-$\frac{1}{2}$ Particle with Anomalous Moment

In this model one would take (B3) with $F_2 \neq 0$. Then there are extra factors which prevent (3.7) and (3.8) from holding for the antisymmetric part. However, this model should not be taken into account, since it leads, e.g., to antisymmetric gradient terms in the E.T.C. of the time components which invalidate the usual current-algebra calculations.