Spectral Function of the Photon Propagator—Mass Spectrum and Timelike Form Factors of Particles

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We consider contributions to the spectral function of the photon propagator from the states which contain a particle (charge e and mass m_J) of arbitrary spin J, and its antiparticle. We express the contributions in terms of timelike form factors $\mathfrak{F}_J(a)$ (where $a=-P^2>0$, and P is the momentum of the photon) with the normalization $\mathfrak{F}_J(0)=(2J+1)e^2$. The unitarity limit of the spectral function can be transformed into the asymptotically bounded condition $\mathfrak{F}_J(a)\lesssim O(a)$. The experimental information about the anomalous magnetic moment of the muon gives a restriction on the sum of all the contributions. Using the restriction, we examine various mass spectra of charged particles and obtain simple results. For example: If there is an infinitely rising mass spectrum $m_J=f(J)$, then the asymptotic form of the mass formula must be bounded by condition $m_J>O(J)$ (case I or II) or $m_J>O(J^{1/(2l+1)})$ (case III), where l is a parameter in the form factors, assumed to be $\mathfrak{F}_J(a)=(2J+1)e^2(a/4m_J^2-1)|1-a/\mu^2|^{-2l}$ for $J=0, 1, 2, \cdots$, and $\mathfrak{F}_J(a)=(2J+1)e^2|1-a/\mu^2|^{-2l}$ for $J=\frac{1}{2}, \frac{3}{2}, \cdots$. For the purpose of experimental observations of the timelike form factors and the spectral function of the photon, the colliding-beam experiments e^++e^- (or $\mu^++\mu^-$) $\rightarrow \lambda+\bar{\lambda}$ (where λ is a particle of arbitrary spin) are discussed in some detail.

I. INTRODUCTION

MANY particles and resonances[1] have been found since big accelerators became available about ten years ago. Most of them can be classified into rotational series in which the mass of member particle increases with its spin. On the other hand, several models[2–7] have been proposed in order to explain or interpret the existence of these many particles. Some of them predict a series of particles with infinitely rising mass and spin levels as well as their spacelike form factors. We can, of course, never imagine that the infinitely rising series should be realized to its asymptotic limit, but we are very much interested in how and in what energy regions the present mass-spin spectra would come to an end.

Although we have no definite and powerful theory on the strong interactions of particles, we know, for their electromagnetic interactions at least, the universality which means that every particle of charge Q interacts with a photon through its vector current with the coupling constant Q at zero momentum transfer. If we want to inspect the over-all features of the particle spectrum from the viewpoint of the electromagnetic interaction, we should remember the universality. Above all, we should examine the spectral function of the photon to which all particles contribute more or less through their electromagnetic interactions with the photon. The anomalous magnetic moment of the muon, for example, must be affected by all the particles through their modification of the photon propagator. Therefore, the experimental data on the muon magnetic moment can give us valuable information about the unknown spectrum of particles. In the usual way, introducing the timelike form factors of particles, we treat the effects of the strong interaction on the electromagnetic current. Thus we can transform the information into some possible combinations of spectra and form factors of particles, which is a main purpose of this paper.

In Sec. II we discuss the matrix elements of the electromagnetic current between the vacuum and the particle-antiparticle states, introducing timelike form factors whose properties are investigated briefly. The contributions to the spectral function of the photon from the particle-antiparticle states are expressed in terms of the timelike form factors in Sec. III. In Sec. IV we show how the contributions can be observed by the anomalous magnetic moment of the muon, and from the experimental information we obtain the upper limit for the sum of all the contributions. In Sec. V we examine various mass and spin spectra of particles by means of the results given in Sec. IV. We obtain some simple restrictions on the relation between the spectrum and the timelike form factors. An assumption on the form of the spectrum gives upper bounds of the timelike form factors. For the purpose of experimental observation of the timelike form factors and the spectral function of the photon, the colliding-beam experiments e^++e^- (or $\mu^++\mu^-$) $\rightarrow\lambda+\bar{\lambda}$ (where λ is a particle of arbitrary spin) are discussed in some detail in the Appendix.

II. TIMELIKE FORM FACTORS

We consider the transition matrix elements of the electromagnetic current j_μ between the vacuum and

[1] A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, Paul Söding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. **40**, 77 (1968).
[2] Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 368 (1966); Phys. Rev. **160**, 1171 (1967).
[3] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).
[4] A. O. Barut and H. Kleinert, Phys. Rev. **161**, 1464 (1967); **156**, 1546 (1967).
[5] H. Bebié and H. Leutwyler, Phys. Rev. Letters **19**, 618 (1967).
[6] M. Gell-Mann, D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Interscience Publishers, Inc., New York, 1968), p. 479.
[7] S. J. Chang, J. G. Kuriyan, and L. O'Raifeartaigh, Phys. Rev. **169**, 1275 (1968).

the states which contain a particle (charge e and mass m_J) of arbitrary spin J, and its antiparticle:

$$\langle 0|j_\mu|p;\bar{p}\rangle, \tag{2.1}$$

where p and $\bar{p}$ are the momenta of the particle and the antiparticle, respectively ($p^2=\bar{p}^2=-m_J^2$). The momentum of the photon is

$$P=p+\bar{p}. \tag{2.2}$$

For our purpose, it is sufficient to consider only the timelike region of the momentum, i.e.,

$$P^2=(p+\bar{p})^2<0. \tag{2.3}$$

The transition matrix elements of the initial (momentum p) and final (momentum p') single-particle states $\langle p'|j_\mu|p\rangle$ have been studied first by Yennie[8] and more generally by Durand, DeCelles, and Marr.[9] We have only to transform carefully their results in the spacelike region into those in the timelike region. Although we shall soon find that this transformation is merely a replacement of the momentum p' by $-\bar{p}$ formally, we should note that there is no relationship between spacelike and timelike form factors introduced in the following manner.

At first we follow the method given by Yennie.[8] We choose the special frame in which $\mathbf{P}=0$ (i.e., $\bar{\mathbf{p}}=-\mathbf{p}$), and set the z axis along $\mathbf{p}$. In this frame the transition matrix elements are

$$\langle 0|j_\mu|m,\mathbf{p};\bar{m},-\mathbf{p}\rangle, \tag{2.4}$$

where m and $\bar{m}$ are the spin quantum numbers of the particle and the antiparticle, respectively, along the z axis [$m,\bar{m}=-J,-(J-1),\cdots,J$]. Charge conservation requires

$$P_\mu\langle 0|j_\mu|m,\mathbf{p};\bar{m},-\mathbf{p}\rangle=0, \tag{2.5}$$

which implies[10]

$$\langle 0|j_0|m,\mathbf{p};\bar{m},-\mathbf{p}\rangle=0. \tag{2.6}$$

It is convenient to consider the matrix elements of j_z and $j_\pm=j_x\pm j_y$. According to the transformation properties under a rotation about the z axis, we introduce a set of timelike form factors $A^{(J)}(m;a)$ and $B_\pm^{(J)}(m;a)$ as follows:

$$\langle 0|j_z|m,\mathbf{p};\bar{m},-\mathbf{p}\rangle=A^{(J)}(m;a)\delta_{\bar{m},-m} \tag{2.7}$$

and

$$\langle 0|j_\pm|m,\mathbf{p};\bar{m},-\mathbf{p}\rangle=B_\pm^{(J)}(m;a)\delta_{\bar{m},-m\mp 1}, \tag{2.8}$$

where $a=-P^2$ (>0). Some useful properties can be obtained by making the following combinations of transformations: (i) Space inversion and rotation of π about the y axis give the following properties:

$$A^{(J)}(m;a)=A^{(J)}(-m;a) \tag{2.9}$$

and

$$B_\pm^{(J)}(m;a)=-B_\mp^{(J)}(-m;a). \tag{2.10}$$

(ii) Charge conjugation and rotation of π about the y axis give another property to $B_\pm^{(J)}(m;a)$ as follows:

$$B_\pm^{(J)}(m;a)=-B_\mp^{(J)}(m\pm 1;a). \tag{2.11}$$

No additional properties independent of Eqs. (2.9)–(2.11) can be obtained by any other transformations. Note that these timelike form factors can be complex whereas spacelike ones are always real. From Eq. (2.9) and the combination of Eqs. (2.10) and (2.11), it can easily be seen that the numbers of independent form factors are $J+1$ for $A^{(J)}(\mathrm{m};a)$ and J for $B_\pm^{(J)}(m;a)$ in the boson case [$J+\frac{1}{2}$ for $A^{(J)}(m;a)$ and $J+\frac{1}{2}$ for $B_\pm^{(J)}(m;a)$ in the fermion case].

We may show the relations between these form factors and the familiar ones of lower-spin particles (mass m_J and charge e) defined by the covariant expression for a current as follows:

$$\langle 0|j_\mu|p;\bar{p}\rangle=\frac{e}{2m_0}(p-\bar{p})_\mu F(P^2)\quad \text{for } J=0 \tag{2.12}$$

and

$$\langle 0|j_\mu|p;\bar{p}\rangle=ie\bar{v}(\bar{p})\times\left[\gamma_\mu F_1(P^2)+\frac{\kappa}{2m_{1/2}}\sigma_{\mu\nu}P_\nu F_2(P^2)\right]u(p)\quad \text{for } J=\tfrac{1}{2}, \tag{2.13}$$

where κ is the anomalous magnetic moment of the particle in the unit $e/2m_{1/2}$. The relations are

$$A^{(0)}(0;a)=e(a/4m_0^2-1)^{1/2}F(-a),\quad \text{for } J=0 \tag{2.14}$$

$$A^{(1/2)}(\tfrac{1}{2};a)=A^{(1/2)}(-\tfrac{1}{2};a)=e\left[F_1(-a)+\kappa\frac{a}{4m_{1/2}^2}F_2(-a)\right], \tag{2.15}$$

and

$$B_+^{(1/2)}(-\tfrac{1}{2};a)=-B_-^{(1/2)}(\tfrac{1}{2};a)=2e\frac{|P|}{2m_{1/2}}[F_1(-a)+\kappa F_2(-a)]\quad \text{for } J=\tfrac{1}{2}. \tag{2.16}$$

In the case of higher spins we find that the form factors $A^{(J)}(m;a)$ and $B_\pm^{(J)}(m;a)$ contain various parts from the electric and magnetic multipole moments. It is important to note that by *formally* putting the momentum P equal to zero,[9,11]

$$A^{(J)}(m;0)=e \tag{2.17}$$

[8] D. R. Yennie, M. M. Lévy, and D. G. Ravenhall, Rev. Mod. Phys. **29**, 144 (1957).
[9] L. Durand, III, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).
[10] The vanishing of $\langle 0|j_0|\mathbf{p};-\mathbf{p}\rangle$ is in fact independent of current conservation, as can be proved in the same manner as in the paper by Durand *et al.* (see Ref. 9).
[11] As $P\to 0$, i.e., $a\to 0$, $(a/4m_0^2-1)^{1/2}$ in Eq. (2.14) should be replaced by $|a/4m_0^2-1|^{1/2}$. Then, $(a/4m_0^2-1)^{1/2}\to 1$ as $P\to 0$.

and

$$B_{\pm}^{(J)}(m;0)=0\,, \qquad (2.18)$$

which is one of the expressions of the universality and gives a guide to the normalization of the timelike form factors in their physical regions.

III. SPECTRAL FUNCTION OF THE PHOTON PROPAGATOR

We start from the definition of the spectral function of the photon propagator[12]:

$$\Pi(-P^2)=\frac{(2\pi)^3}{-3P^2}\sum_{p^{(z)}=P}\langle 0|j_\mu|z\rangle\langle z|j_\mu|0\rangle\,, \qquad (3.1)$$

where z denotes an arbitrary intermediate state with the total momentum $p^{(z)}$. As usual we can express the renormalized photon propagator in the Källén spectral representation[12] as follows:

$$D(q^2)=\frac{1}{q^2}+\int_0^\infty\frac{da}{a}\frac{\Pi(a)}{q^2+a}\,. \qquad (3.2)$$

In accordance with the purpose of this paper, we take only the particle (charge e, spin J, and mass m_J)-antiparticle state as one of many candidates for the intermediate states. The contribution from the pair state is

$$\Pi_J(-P^2)=\frac{(2\pi)^3}{-3P^2}\sum_{p+\bar{p}=P}\langle 0|j_\mu|p;\bar{p}\rangle\langle p;\bar{p}|j_\mu|0\rangle. \qquad (3.3)$$

Since the contribution from every state is positive definite, $\Pi(-P^2)$ is always larger than $\Pi_J(-P^2)$, i.e.,

$$\Pi(-P^2)\geqq\Pi_J(-P^2)\geqq 0. \qquad (3.4)$$

After transforming the summation $\sum_{p+\bar{p}=P}$ into the invariant integral

$$\int\frac{d^4p}{(2\pi)^3}\int\frac{d^4\bar{p}}{(2\pi)^3}\delta(p+\bar{p}-P)\delta(p^2+m_J^2)\delta(\bar{p}^2+m_J^2)\,,$$

we may choose the special frame mentioned in Sec. II in the intermediate state. Thus, we can express $\Pi_J(-P^2)$ in terms of the timelike form factors as follows:

$$\Pi_J(a)=\frac{\mathfrak{F}_J(a)}{12\pi^2}\frac{m_J^2}{a}\left(1-\frac{4m_J^2}{a}\right)^{1/2}\theta(a-4m_J^2)\,, \qquad (3.5)$$

where $\mathfrak{F}_J(a)$ is a reduced timelike form factor defined by

$$\mathfrak{F}_J(a)=\sum_m|A^{(J)}(m;a)|^2+\sum_m|B_+^{(J)}(m;a)|^2. \qquad (3.6)$$

[12] G. Källén, Helv. Phys. Acta 25, 417 (1952).

Here we have used the following relation:

$$\sum_{m,\bar{m}}\langle 0|j_\mu|m,\mathbf{p};\bar{m},-\mathbf{p}\rangle\langle m,\mathbf{p};\bar{m},-\mathbf{p}|j_\mu|0\rangle =\sum_m|A^{(J)}(m;a)|^2+\sum_m|B_+^{(J)}(m;a)|^2, \qquad (3.7)$$

which we can easily derive from Eqs. (2.7) and (2.8). Considering Eqs. (2.14)–(2.16), we may illustrate the simple results for lower spins as follows:

$$\mathfrak{F}_0(a)=e^2(a/4m_0^2-1)|F(-a)|^2, \qquad (3.8)$$

and then

$$\Pi_0(a)=\frac{\alpha}{12\pi}\left(1-\frac{4m_0^2}{a}\right)^{3/2}|F(-a)|^2\theta(a-4m_0^2) \quad \text{for } J=0; \qquad (3.9)$$

$$\mathfrak{F}_{1/2}(a)=2e^2\left[\left|F_1(-a)+\kappa\frac{a}{4m_{1/2}^2}F_2(-a)\right|^2 +\frac{a}{2m_{1/2}^2}|F_1(-a)+\kappa F_2(-a)|^2\right], \qquad (3.10)$$

and then

$$\Pi_{1/2}(a)=\frac{\alpha}{3\pi}\left(1-\frac{4m_{1/2}^2}{a}\right)^{1/2}\left[|F_1(-a)+\kappa F_2(-a)|^2 +\frac{2m_{1/2}^2}{a}\left|F_1(-a)+\kappa\frac{a}{4m_{1/2}^2}F_2(-a)\right|^2\right] \quad \text{for } J=\tfrac{1}{2}. \qquad (3.11)$$

From Eqs. (2.17) and (2.18), we obtain the following normalization condition even for higher spins:

$$\mathfrak{F}_J(0)=(2J+1)e^2, \qquad (3.12)$$

when we *formally* put the momentum P equal to zero. Furthermore, the unitarity limit of the spectral function, $\Pi(a)\to\text{const}$ $(a\to\infty)$,[13,14] places the following restriction on the asymptotic behavior of the timelike form factors:

$$\mathfrak{F}_J(a)\lesssim O(a) \qquad (3.13)$$

or

$$|A^{(J)}(m;a)|\lesssim O(a^{1/2})$$

and

$$|B_\pm^{(J)}(m;a)|\lesssim O(a^{1/2}). \qquad (3.14)$$

[13] V. N. Gribov, B. L. Ioffe, and I. Ya. Pomeranchuk, Phys. Letters **24B**, 554 (1967).

[14] From Eqs. (3.4), (3.5), and (3.18), we can easily find that this asymptotic condition is also required by the renormalizability of hadronic electromagnetic interactions (i.e., $Z_3^{-1}\lesssim$ logarithmic divergence). A more conservative requirement, the finiteness of the contribution to the muon magnetic moment, leads to a weaker asymptotic condition, $\mathfrak{F}_J(a)<O(a^2)$ [see Eqs. (3.5), (4.8), and (4.9)], while the hypothesis proposed by Kroll, Lee, and Zumino (Ref. 16) leads to a stronger one, $\mathfrak{F}_J(a)<O(1)$ [see Eqs. (3.4), (3.5), and (3.20)].

For lower spins, from Eqs. (2.14)–(2.16), this equation means

$$|F(-a)|\lesssim O(1)\,,\quad \text{for}\quad J=0 \tag{3.15}$$

$$|F_1(-a)|\lesssim O(1)\,, \tag{3.16}$$

and

$$|F_2(-a)|\lesssim O(a^{-1/2})\,,\quad \text{for}\quad J=\tfrac{1}{2} \tag{3.17}$$

which seem to be weak restrictions. For higher spins, however, the inequalities (3.14) may become much stronger constraints.[15]

In the latter part of this section, we want to note the relation between the asymptotic behavior of the spectral function (or the form factor) and the charge renormalization constant. It is well known that the charge renormalization constant Z_3 $(=e^2/e_0^2)$ can be presented in terms of the spectral function of the photon $\Pi(a)$, by[12]

$$Z_3^{-1}=1+\int_0^\infty da\frac{\Pi(a)}{a}\,. \tag{3.18}$$

The finiteness or infiniteness of Z_3^{-1}, therefore, depends on whether $\Pi(a)$ behaves asymptotically as $a^{-\eta}$ $(\eta>0)$ or a^{ξ} $(\xi\geq 0)$. Considering Eq. (3.5) and the positive definiteness of every contribution to the spectral function from the intermediate states shown in Eq. (3.4), the finiteness of Z_3^{-1} necessarily requires that the asymptotic behavior of $\mathfrak{F}_J(a)$ be as $a^{1-\eta}$, i.e.,

$$\mathfrak{F}_J(a)<O(a)\,. \tag{3.19}$$

It is remarkable that the hypothesis recently proposed by Kroll, Lee, and Zumino[16] leads to the finiteness of hadronic contributions to the charge renormalization constant. The hypothesis is that the entire hadronic electromagnetic current operator is identical with a linear combination of the known neutral vector-meson fields. Their result is derived from the situation that $\Pi(a)<O(a^{-1})$ as $a\to\infty$ because, according to their hypothesis, the photon spectral function is proportional to the spectral function of the neutral vector meson, $\sigma(a)$,[17] divided by a^2, i.e.,

$$\Pi(a)\propto\sigma(a)/a^2\,, \tag{3.20}$$

and $\sigma(a)<O(a)$ [which is required by the sum rule $\int da\,\sigma(a)/a^2=$ (mass of the meson)$^{-2}$].

IV. MUON MAGNETIC MOMENT

The value of the anomalous magnetic moment of the muon recently given by the experiment being performed at CERN[18] has become more precise than before and is expected to be much more so in the near future. The latest experimental value[18] is

$$\tfrac{1}{2}(g_\mu-2)_{\text{expt}}=(11666\pm 5)\times 10^{-7}\,, \tag{4.1}$$

while the theoretical value to order α^2 is[19–22]

$$\tfrac{1}{2}(g_\mu-2)_{\text{theor}}=\alpha/2\pi+0.766(\alpha/\pi)^2=11655\times 10^{-7}\,, \tag{4.2}$$

which is the result of quantum electrodynamics in a narrow sense, including only the interaction of the electron (e), the muon (μ), and the photon (γ). The difference between these two values is more than twice the experimental accuracy, i.e.,

$$\Delta\mu\equiv\tfrac{1}{2}(g_\mu-2)_{\text{expt}}-\tfrac{1}{2}(g_\mu-2)_{\text{theor}}=(11\pm 5)\times 10^{-7}\,. \tag{4.3}$$

In order to answer whether or not we should consider the difference $\Delta\mu$ as an evidence of the breakdown of quantum electrodynamics, we have to solve two problems. One is to calculate the sixth-order radiative corrections to the magnetic moment of the muon, and the other is to evaluate all the possible contributions of all particles other than e, μ, and γ. The former problem has been studied by some authors[23–25] but has not yet been completed. All of their results already reported are of the order 10^{-8}, which cannot explain the difference $\Delta\mu$ at all. On the latter problem, much work[26–34] has been done. Even if we are concerned only with particles existing at present other than e, μ, and γ, we must consider all the hadrons. All of them may contribute to $\frac{1}{2}(g_\mu-2)$ through their effects on the photon propagator (Fig. 1). Among these, the simplest and most important contribution is that of the $\pi^+-\pi^-$ intermediate state that was calculated by several authors.[26–28]

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[15] For $J=1$, for example, the result will be $|F_1(-a)|\lesssim O(a^{-1/2})$, $|F_2(-a)|\lesssim O(a^{-1})$, and $|F_3(-a)|<O(a^{-1})$, where $F_3(-a)$ is a timelike form factor of the anomalous electric quadrupole moment.

[16] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[17] This spectral function is different from that in Ref. 16 by the factor a.

[18] F. J. M. Farley *et al.*, in *Proceedings of the 1967 International Symposium on Electron and Photon Interactions at High Energies* (Stanford Linear Accelerator Center, Stanford, Calif., 1967). The most recent value is $\frac{1}{2}(g_\mu-2)=(11661.4\pm 3.1)\times 10^{-7}$. Then $\Delta\mu=(6.2\pm 3.1)\times 10^{-7}$. F. J. M. Farley *et al.*, Phys. Letters **28B**, 287 (1968).

[19] J. Schwinger, Phys. Rev. **73**, 416 (1948).

[20] H. Suura and E. H. Wichmann, Phys. Rev. **105**, 1930 (1957).

[21] A. Peterman, Phys. Rev. **105**, 1931 (1957).

[22] H. H. Elend, Phys. Letters **20**, 682 (1966); **21**, 720(E) (1966).

[23] S. D. Drell and J. S. Trefil (to be published). See S. D. Drell, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 85.

[24] T. Kinoshita, Nuovo Cimento **51B**, 141 (1967); lectures given at the Summer School of Theoretical Physics, Cargèse, 1967 (unpublished).

[25] H. Terazawa, Progr. Theoret. Phys. (Kyoto) **38**, 863 (1967).

[26] C. Bouchiat and L. Michel, J. Phys. Radium **22**, 121 (1961).

[27] L. Durand, III, Phys. Rev. **128**, 441 (1962).

[28] T. Kinoshita and R. J. Oakes, Phys. Letters **25B**, 143 (1967).

[29] H. Terazawa, Progr. Theoret. Phys. (Kyoto) **39**, 1326 (1968).

[30] F. J. M. Farley, Proc. Roy. Soc. (London) **285**, 248 (1965); S. J. Brodsky and E. de Rafael, Phys. Rev. **168**, 1620 (1968).

[31] G. Segrè, Phys. Letters **7**, 357 (1963); H. Pietschann, Z. Physik **178**, 409 (1964); R. A. Shaffer, Phys. Rev. **135B**, 187 (1964); S. J. Brodsky and J. D. Sullivan, *ibid.* **156**, 1644 (1968).

[32] N. Byers and F. Zachariasen, Nuovo Cimento **30**, 1572 (1963); R. D. Amado and L. Holloway, *ibid.* **30**, 1572 (1963).

[33] S. Nakamura, K. Itami, and H. Ugai, Progr. Theoret. Phys. (Kyoto) **34**, 256 (1965); S. Nakamura, H. Matsumoto, N. Nakazawa, and H. Ugai (to be published).

[34] H. Terazawa, Progr. Theoret. Phys. (Kyoto) **37**, 204 (1967).

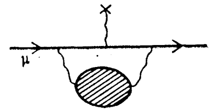


FIG. 1. Diagram for the contribution to $\frac{1}{2}(g_\mu-2)$ from the modification of the photon propagator.

Their result is[35]

$$\Delta\tfrac{1}{2}(g_\mu-2)_{\pi^+\pi^-}=6.1\times10^{-8}. \qquad (4.4)$$

The sum of all the hadronic contributions has been estimated by the author[29] using the hypothesis proposed by Kroll, Lee, and Zumino[16] stated in Sec. III. Its upper limit is obtained as

$$\Delta\tfrac{1}{2}(g_\mu-2)_{\text{all hadrons}}<2.2\times10^{-7}, \qquad (4.5)$$

which is subject to the reliability of the hypothesis.

In this section we consider the contributions to $\frac{1}{2}(g_\mu-2)$ from the states which contain a particle (spin J and mass m_J) and its antiparticle, so that we may complete our purpose stated in Sec. I.

Generally, the contribution to $\frac{1}{2}(g_\mu-2)$ from the modification of the photon spectral function $\Delta\Pi(a)$ is given by

$$\Delta\tfrac{1}{2}(g_\mu-2)=\frac{\alpha}{\pi}\int_0^\infty \frac{da}{a}\Delta\Pi(a)g\left(\frac{a}{m_\mu{}^2}\right), \qquad (4.6)$$

where m_μ is the mass of the muon and

$$g(c)=\int_0^1 dx\frac{x^2(1-x)}{x^2+c(1-x)}. \qquad (4.7)$$

As far as hadrons other than π mesons are concerned, the thresholds for $\Delta\Pi(a)$ are much higher than $m_\mu{}^2$ (e.g., $4m_K{}^2/m_\mu{}^2\simeq89\gg1$). Therefore, Eq. (4.6), to a very good approximation, gives

$$\Delta\tfrac{1}{2}(g_\mu-2)=(\alpha/3\pi)(m_\mu{}^2/\Lambda^2), \qquad (4.8)$$

where

$$\frac{1}{\Lambda^2}\equiv\int_0^\infty da\frac{\Delta\Pi(a)}{a^2}. \qquad (4.9)$$

Here we note that the quantity Λ^2 play the same role as the phenomenological cutoff parameter[36] in quantum electrodynamics without considering its sign.[37]

At present we may propose an upper limit ϵ of the sum of all the contributions other than that from the $\pi^+-\pi^-$ state, using the experimental information shown in Eq. (4.3), i.e.,

$$\Delta\equiv\Delta\tfrac{1}{2}(g_\mu-2)_{\text{all hadrons}} -\Delta\tfrac{1}{2}(g_\mu-2)_{\pi^+\pi^-}<\epsilon\simeq2\times10^{-6}. \qquad (4.10)$$

Equations (4.8) and (4.10) show numerically that

$$1/\Lambda^2<1/(2.3\text{ GeV})^2, \qquad (4.11)$$

which gives an important criterion for the discussion in Sec. V.

From Eqs. (3.5) and (4.9), we express the contribution of the particle-antiparticle state in terms of the timelike form factors as follows:

$$\frac{1}{\Lambda_J{}^2}=\frac{1}{12\pi^2}\int_{4m_J{}^2}^\infty da\frac{m_J{}^2}{a^3}\left(1-\frac{4m_J{}^2}{a}\right)^{1/2}\mathfrak{F}_J(a). \qquad (4.12)$$

In order to calculate the further analytic form of $1/\Lambda_J{}^2$, we have to assume a proper analytic form of the timelike form factor $\mathfrak{F}_J(a)$. One of the simplest and most convenient candidates for $\mathfrak{F}_J(a)$ is

$$\mathfrak{F}_J(a)=(2J+1)e^2\left(\frac{a}{4m_J{}^2}-1\right)\Big/\left|1-\frac{a}{\mu_J{}^2}\right|^{2l_J}, \quad \text{for } J=0,1,2,\cdots$$

$$=(2J+1)e^2\Big/\left|1-\frac{a}{\mu_J{}^2}\right|^{2l_J}, \quad \text{for } J=\tfrac{1}{2},\tfrac{3}{2},\cdots \qquad (4.13)$$

whose normalization is taken to satisfy the condition given in Eq. (3.12). The reason for the different forms for bosons than for fermions is only that we wish to express the difference between the threshold behaviors in the S and P waves [see Eqs. (3.8) and (3.10)]. Here we have introduced two parameters, l_J and $\mu_J{}^2$. For $l_J=1$, the expression Eq. (4.13) is the same as that from the one-pole or one-resonance approximation.[38]

When we try to evaluate the integral in Eq. (4.12), we want to know the information about the behavior of the form factor $\mathfrak{F}_J(a)$ near the threshold rather than that in the asymptotic region. It is, therefore, nothing serious to have determined the asymptotic behavior of the form factor as a result of assuming Eq. (4.13). Moreover, we make an additional assumption that the parameter $\mu_J{}^2$ is much smaller than the threshold $4m_J{}^2$, i.e.,[39]

$$\kappa_J\equiv4m_J{}^2/\mu_J{}^2\gg1. \qquad (4.14)$$

Then, from Eq. (4.13), we can easily evaluate the in-

[35] Using the data recently given by the Novosibirsk group (Ref. 40), we obtain the ρ^0 resonance contribution as

$$\tfrac{1}{2}(g_\mu-2)_{\pi^+\pi^-(\rho^0\text{ resonance})}=(3.1\pm1.3)\times10^{-8}.$$

[36] S. D. Drell, Ann. Phys. (N. Y.) 4, 75 (1958).

[37] For example, if we make the parametrization for the breakdown of the photon propagator $1/q^2\to1/q^2-1/(q^2+\Lambda^2)$, we obtain the result $\Delta\frac{1}{2}(g_\mu-2)=-(\alpha/3\pi)m_\mu{}^2/\Lambda^2$.

[38] We note the result recently reported by the Novosibirsk group (Ref. 40) for the $e^++e^-\to\pi^++\pi^-$ colliding-beam experiment at the energy region near the ρ^0 resonance. The result is well approximated by

$$|F_\pi(-a)|^2=k\Big/\left|1-\frac{a}{(m_\rho+i\Gamma_\rho/2)^2}\right|^2,$$

where $k=0.59\pm0.15$, $m_\rho=764\pm11$ MeV, and $\Gamma_\rho=93\pm15$ MeV.

[39] For the spacelike form factor of the proton,

$$\kappa_p=4\times0.88\text{ GeV}^2/0.71\text{ GeV}^2\cong4.9\gg1.$$

tegral in Eq. (4.12) as follows:

$$\frac{1}{\Lambda_J^2}=\frac{\alpha}{48\pi}\frac{2J+1}{m_J^2}\frac{1}{\kappa_J^{2l_J}}B(\tfrac{5}{2},\,2l_J+1)\,,\quad \text{for } J=0,1,2,\cdots$$

$$=\frac{\alpha}{48\pi}\frac{2J+1}{m_J^2}\frac{1}{\kappa_J^{2l_J}}B(\tfrac{3}{2},\,2l_J+2)\,,\quad \text{for } J=\tfrac{1}{2},\tfrac{3}{2},\cdots \tag{4.15}$$

where $B(\alpha,\beta)$ is a beta function, which is expressed in terms of Γ functions, viz.,

$$B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha,\beta)\,. \tag{4.16}$$

At the end of this section, it is worth noticing that in the case of K^+-K^- state the assumption Eq. (4.14) will probably be wrong because the ϕ-meson resonance ($m_\phi^2=1.04$ GeV^2) is situated just above the threshold ($4m_K^2=0.98$ GeV^2). Supposing the form factor of the K meson to be

$$|F_K(-a)|^2=k\Bigg/\left|1-\frac{a}{(m_\phi+\frac{1}{2}i\Gamma_\phi)^2}\right|^2, \tag{4.17}$$

we can estimate the contribution to $\frac{1}{2}(g_\mu-2)$ from the $K\bar{K}$ state near the ϕ resonance. The result is

$$\Delta\tfrac{1}{2}(g\mu-2)_{K\bar{K}(\phi\text{ resonance})}\simeq k\left(\frac{\alpha}{\pi}\right)^2\frac{\pi m_\mu^2}{36m_\phi\Gamma_\phi}\left(1-\frac{4m_K^2}{m_\phi^2}\right)^{3/2}, \tag{4.18}$$

which gives a numerical value $2.1k\times10^{-8}$, using the datum $\Gamma_\phi=3.6$ MeV.[1] The parameter k will soon be given by the experiment at Novosibirsk.[40] Supposing $k\simeq1$, the result $\Delta\frac{1}{2}(g_\mu-2)_{K\bar{K}}\simeq2.1\times10^{-8}$ is much larger than that expected in comparison with $\Delta\frac{1}{2}(g_\mu-2)_{\pi^+\pi^-}$, considering only their mass difference.

V. MASS SPECTRUM OF PARTICLES

In the previous sections, we have made all the preparations for examining various mass spectra of particles in this section. Our criterion is that we must forbid any mass spectra of particles that would make the sum of all the contributions to $\frac{1}{2}(g_\mu-2)$ larger than ϵ even if every contribution is smaller than ϵ. We can apply this criterion to mass spectra of resonances as well as to those of particles because every state of a resonance and its antiresonance is independent of other states; then we never fall into double counting of the contributions.

Let us consider all the particles and resonances already known in the mass regions $m_B<1.8$ GeV (boson) and $m_F<3.3$ GeV (fermion).[1] Supposing $l_J=0$ for the sake of simplicity, we calculate from Eqs. (4.8) and (4.15) the sum of all the contributions to $\frac{1}{2}(g_\mu-2)$ from all the particles and resonances already known. The numerical result is

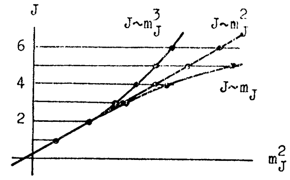


FIG. 2. Various spin-mass spectra of particles.

$$\tilde{\Delta}\equiv\sum\Delta\tfrac{1}{2}(g_\mu-2)=0.51\times10^{-8},\quad\text{from all bosons other than the } \pi \text{ meson}$$
$$=1.2\times10^{-8},\quad\text{from all fermions other than leptons.} \tag{5.1}$$

This shows that the mass spectrum of particles known at present certainly satisfy the criterion mentioned above. Here we note that in the mass spectrum there are some series in which the masses of members become heavier according as their spins become higher. Furthermore, these series can be fitted to a good approximation on the Regge trajectories or Chew-Frautschi plots[41] which are rising linearly (Fig. 2), i.e., $J\sim m_J^2$. Since in the case of $l_J=0$ every contribution to $\frac{1}{2}(g_\mu-2)$ has the form const$\times(2J+1)/m_J^2$, it is clearly seen that the sum of all the contributions grows linearly and would break the criterion at a mass region of about several hundred BeV if the trajectories should rise linearly to that region.

Next, we shall examine possible types of infinitely rising spin-mass spectra of charged particles more generally and formally.[42,43] Let us consider a series expressed by the spin-mass spectra $m_J=f(J)$ infinitely rising, i.e., $f(J)\to\infty$ as $J\to\infty$. Supposing the same simple forms of the timelike form factors $\mathfrak{F}_J(a)$ as in Eq. (4.13), we can find from Eqs. (4.8) and (4.15) that a contribution of every member in the series to $\frac{1}{2}(g_\mu-2)$ has the following form:

$$\Delta_J=\text{const}\times\frac{2J+1}{m_J^2}\frac{1}{\kappa_J^{2l_J}}B(\tfrac{5}{2},\,2l_J+1)\,,\quad\text{for } J=0,1,2,\cdots$$

$$=\text{const}\times\frac{2J+1}{m_J^2}\frac{1}{\kappa_J^{2l_J}}B(\tfrac{3}{2},\,2l_J+2)\,,\quad\text{for } J=\tfrac{1}{2},\tfrac{3}{2},\cdots. \tag{5.2}$$

For the sake of simplicity, we make the further

[40] V. L. Auslander, G. I. Budker, Ju. N. Pestov, V. A. Sidorov A. N. Skrinsky, and A. G. Khabakhpashev, Phys. Letters **25B** 433 (1967).

[41] T. Regge, Nuovo Cimento **18**, 947 (1960); G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[42] R. C. Brower and J. Harte, Phys. Rev. **164**, 1841 (1967).

[43] R. W. Childers, Phys. Rev. Letters **21**, 868 (1968).

assumption:

$$l_J = l = \text{const.}^{44} \quad (5.3)$$

Then,

$$\Delta_J = \text{const} \times [(2J+1)/m_J^2] \kappa_J^{-2l}. \quad (5.4)$$

In order to obey the criterion, i.e., $\sum_J \Delta_J < \epsilon$, the mass spectrum $m_J = f(J)$ must satisfy at least the condition that the sum of the infinite series $\sum_J \Delta_J$ should be convergent. From Eq. (5.4), this weak restriction leads to the following condition by which the spin-mass spectra of particles should be bounded asymptotically. We illustrate three typical cases for the timelike form factors:

Case I. $l=0$; then $m_J > O(J)$. (5.5)

Case II. $l>0$, $\kappa_J = \kappa = \text{const}$; then $m_J > O(J)$. (5.6)

Case III. $l>0$, $\mu_J = \mu = \text{const}$; then $m_J > O(J^{1/(2l+1)})$. (5.7)

Some of the models[4–7] predict *space*like form factors like the *time*like ones in case II, while the vector-meson-dominance model[16,45,46] suggests form factors like those in case III. We, however, consider these cases only to see a few typical results from the criterion.

For a charged Regge trajectory rising linearly and infinitely, i.e., $m_J \sim J^{1/2}$, the conditions (5.5)–(5.7) lead us to the result:

Case I and case II are forbidden and, in case III, $l > \frac{1}{2}$. (5.8)

Similarly, for a model of the spin-mass spectrum in which there are parallel infinite series rising linearly, infinitely, and at equal distances as illustrated in Fig. 3, the conditions lead to the result:

Of course, case I and case II are forbidden and, in case III, $l > 1$. (5.9)

VI. RESULTS AND DISCUSSION

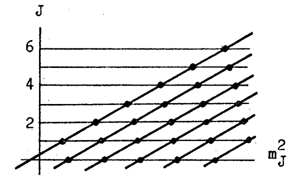
We have obtained the following simple results: the asymptotic condition of the timelike form factors, Eq. (3.14), and the conditions, Eqs. (5.5)–(5.9), which the infinitely rising spin-mass spectra and their timelike form factors must follow. It should be noted that the latter conditions have been derived from the weakest condition that the contributions to $\frac{1}{2}(g_\mu - 2)$ must be finite. We may, therefore, give a stronger restriction on the observable quantities if we apply faithfully the criterion stated in Sec. V. Moreover, a much stronger restriction will be obtained when a more precise experimental value for $\frac{1}{2}(g_\mu - 2)$ is available in the near future.

[44] As far as the asymptotic behavior of the form factors is concerned, the result given by Amati *et al.* shows that this is the case. D. Amati, R. Jengo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Letters **27B**, 38 (1968); D. Amati, L. Caneschi, and R. Jengo, Report (unpublished).

[45] J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966).

[46] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

FIG. 3. A model of the spin-mass spectrum of particles.



From the purely theoretical point of view, we have poor information about the timelike form factors of particles. We can only find a prediction[47] by using the hypothetical crossing symmetries of the form factors, which have not been able to give definite predictions. More studies about the timelike form factors are required.

On the other hand, a few $e^+ - e^-$ colliding beam machines are beginning to work in the energy region of several hundred MeV.[40] It seems, however, to be somewhat difficult at present to observe the timelike form factors of hadrons (e.g., π, K, p, n, Λ, etc.) at energy regions far from some vector-meson resonances. We wish to obtain experimental information about the timelike form factors in the broad regions which contain energies near the thresholds and the vector-meson resonances, far from them, and, furthermore, the highest energies available.

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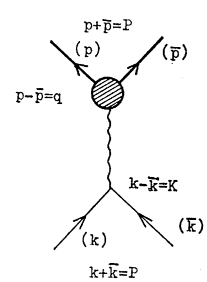
APPENDIX

For the purpose of experimental observation of the timelike form factors and the spectral function of the photon discussed in the earlier sections, we shall consider the colliding-beam experiments $e^+ + e^-$ (or $\mu^+ + \mu^-$) $\rightarrow \lambda + \bar{\lambda}$ [where λ is a particle, with mass m_J and an arbitrary spin J, other than leptons (l)]. These reactions proceed through a one-photon intermediate state in the lowest order of α^2 (Fig. 4). It has been

FIG. 4. Diagram for the reaction $l^+ + l^- \rightarrow \lambda + \bar{\lambda}$ through a one-photon intermediate state.



[47] M. Conversi, T. Massam, Th. Muller, and A. Zichichi, in *Fifth International Conference on High-Energy Accelerators, 1965* (Centre Nationale d'Etudes Nucléaire, Saclay, 1966).

shown by Gatto[48] that interferences between the one-photon and the two-photon channels vanish on account of charge-conjugation invariance for experiments that treat the charges symmetrically. In order to calculate the differential cross sections for the reactions, we need the following two quantities[8]:

$$\langle j_\mu{}^* j_\nu\rangle \equiv \sum_{\text{spin}} \langle 0|j_\mu|k;\bar{k}\rangle\langle k;\bar{k}|j_\nu|0\rangle$$

and

$$\langle J_\mu{}^* J_\nu\rangle \equiv \sum_{\text{spin}} \langle 0|J_\mu|p;\bar{p}\rangle\langle p;\bar{p}|J_\nu|0\rangle, \qquad \text{(A1)}$$

where j_μ and J_μ are the electromagnetic currents of the lepton and the particle λ, respectively. k ($\bar{k}$) and p ($\bar{p}$) are the momenta of the lepton (antilepton) and the particle λ ($\bar{\lambda}$), respectively ($k+\bar{k}=p+\bar{p}=P$ and $-P^2=a>0$). From Eqs. (2.7) and (2.8), these quantities are written in the special frame in terms of the form factors, $\mathcal{A}(m;a)$ and $\mathcal{B}_\pm(m;a)$ for the lepton, and $A^{(J)}(m;a)$ and $B_\pm{}^{(J)}(m;a)$ for the particle λ, as follows:

$$\begin{aligned}\langle j_\mu{}^* j_\nu\rangle_{\text{special frame}} &= \bar{\mathfrak{A}}(a), && \text{for } \mu=\nu=3\\ &= \bar{\mathfrak{B}}(a), && \text{for } \mu=\nu=1 \text{ or } \mu=\nu=2\\ &=0, && \text{otherwise}\end{aligned} \qquad \text{(A2)}$$

where

$$\bar{\mathfrak{A}}(a)=\sum_m |\mathcal{A}(m;a)|^2$$

and

$$\bar{\mathfrak{B}}(a)=\tfrac{1}{2}\sum_m |\mathcal{B}_+(m;a)|^2. \qquad \text{(A3)}$$

Similar relations hold for $\langle J_\mu{}^* J_\nu\rangle$. For lower spins, from Eqs. (2.16)–(2.18), $\bar{A}^{(J)}(a)$ and $\bar{B}^{(J)}(a)$ are given by

$$\bar{A}^{(0)}(a)=e^2(a/4m_0{}^2-1)|F(-a)|^2, \quad \text{for } J=0 \qquad \text{(A4)}$$

$$\bar{A}^{(1/2)}(a)=2e^2|F_1(-a)+\kappa(a/4m_{1/2}{}^2)F_2(-a)|^2, \qquad \text{(A5)}$$

and

$$\bar{B}^{(1/2)}(a)=2e^2(a/4m_{1/2}{}^2)|F_1(-a)+\kappa F_2(-a)|^2, \quad \text{for } J=\tfrac{1}{2}. \qquad \text{(A6)}$$

We give the generalization of $\langle j_\mu{}^* j_\nu\rangle$ and $\langle J_\mu{}^* J_\nu\rangle$ to an arbitrary Lorentz frame[8,9] as follows:

$$\langle j_\mu{}^* j_\nu\rangle=\bar{\mathfrak{A}}(a)\frac{K_\mu K_\nu}{K^2}+\bar{\mathfrak{B}}(a)\left(\delta_{\mu\nu}-\frac{P_\mu P_\nu}{P^2}-\frac{K_\mu K_\nu}{K^2}\right)$$

and

$$\langle J_\mu{}^* J_\nu\rangle=\bar{A}^{(J)}(a)\frac{q_\mu q_\nu}{q^2}+\bar{B}^{(J)}(a)\left(\delta_{\mu\nu}-\frac{P_\mu P_\nu}{P^2}-\frac{q_\mu q_\nu}{q^2}\right), \qquad \text{(A7)}$$

where $K=k-\bar{k}$ and $q=p-\bar{p}$. Then we can easily obtain the general form of the differential cross section ($l^+ + l^- \to \lambda+\bar{\lambda}$) in the c.m. system:

$$\frac{d\sigma(l^+ + l^- \to \lambda+\bar{\lambda})}{d(\cos\theta)}=\frac{\pi\alpha^2}{8E^2}\left(1-\frac{m_\lambda{}^2}{E^2}\right)^{1/2}\frac{m_\lambda{}^2}{2e^2E^2}\frac{m_l{}^2}{2e^2E^2}\times[\bar{\mathfrak{A}}\bar{A}^{(J)}\cos^2\theta+\bar{\mathfrak{A}}\bar{B}^{(J)}\sin^2\theta+\bar{\mathfrak{B}}\bar{A}^{(J)}\sin^2\theta+\bar{\mathfrak{B}}\bar{B}^{(J)}(1+\cos^2\theta)], \qquad \text{(A8)}$$

where E and θ are the energy of the incident lepton and the scattering angle of the final particle λ in the c.m. system, respectively ($a=4E^2$). It is worth noticing that the angular distributions are explicitly shown in Eq. (A8) for any high-spin final pairs. Assuming the lepton to have a point structure, i.e., $F_1(-a)=1$ and $F_2(-a)=0$, we get, from Eqs. (A5) and (A6), the following more convenient formula:

$$\frac{d\sigma(l^+ + l^- \to \lambda+\bar{\lambda})}{d(\cos\theta)}=\frac{\pi\alpha^2}{8E^2}\left(1-\frac{m_\lambda{}^2}{E^2}\right)^{1/2}\frac{m_\lambda{}^2}{2e^2E^2}\times\left[\bar{A}^{(J)}(4E^2)\left(\sin^2\theta+\frac{m_l{}^2}{E^2}\cos^2\theta\right)+\bar{B}^{(J)}(4E^2)\left(1+\cos^2\theta+\frac{m_l{}^2}{E^2}\sin^2\theta\right)\right]. \qquad \text{(A9)}$$

In the case of the $e^+ - e^-$ colliding beams, a further approximation $m_e \ll E$ gives much simpler formula:

$$\frac{d\sigma(e^+ + e^- \to \lambda+\bar{\lambda})}{d(\cos\theta)}=\frac{\pi\alpha^2}{8E^2}\left(1-\frac{m_\lambda{}^2}{E^2}\right)^{1/2}\frac{m_\lambda{}^2}{2e^2E^2}\times[\bar{A}^{(J)}(4E^2)\sin^2\theta+\bar{B}^{(J)}(4E^2)(1+\cos^2\theta)]. \qquad \text{(A10)}$$

Using these formulas (A9) and (A10), we can determine the two independent form factors $\bar{A}^{(J)}(a)$ and $\bar{B}^{(J)}(a)$ if we observe the angular distributions of the final particles. We should note that more than two independent form factors can never be determined by the experiment although various formulas can be obtained by introducing many form factors of multiple-pole moments for higher-spin particles. If we wish to determine them, we must observe the spin correlation of the scattered particles.

The total cross section of the reaction $l^+ + l^- \to \lambda+\bar{\lambda}$, $\sigma_\lambda(a)$, is related to the contribution to the spectral function of the photon from the $\lambda-\bar{\lambda}$ state, $\Pi_\lambda(a)$, as follows:

$$\sigma_\lambda(a)=(\pi e^2/a)(1+2m_l{}^2/a)\Pi_\lambda(a), \qquad \text{(A11)}$$

which can be derived from the combination of Eqs. (3.5), (3.6), (A3), and (A9) or from Eq. (3.3) directly. Using this relation, we can observe the spectral function directly from the total cross section. Finally we note that the asymptotic behavior of the total cross section is bounded by $O(a^{-1})$ from the unitarity limit of the spectral function[13,14] and by $O(a^{-2})$ from the hypothesis proposed by Kroll, Lee, and Zumino.[16]

[48] R. Gatto, Springer Tracts in Mod. Phys. 39, 106 (1965).