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Note on Interpretation of Fission-Fragment Angular Distributions at Moderate Excitation Energies*

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An approximate theoretical expression is presented for calculation of fission-fragment angular distributions at moderate excitation energies. Theoretical anistropies obtained with this approximate expression are compared with exact theory for different bombarding particles and energies, and the agreement for all cases is very good when the target and projectile spins are zero.

I. THEORY

A LARGE amount of experimental information on fission-fragment angular distributions for particle-induced fission has been published. A large fraction of the experiments, in particular, the charged-particle-induced fission experiments, have been performed at moderately high excitation energies where the excited levels in the transition nucleus are described by statistical theory. The K distribution (where K is the projection of the total angular momentum I on the nuclear symmetry axis) of the levels in the transition nucleus for a given temperature is then predicted to be Gaussian,[1]

$$F(K) \propto \exp(-K^2/2K_0^2), \qquad (1)$$

and the variance of the K distribution (designated as K_0^2) is

$$K_0^2 = \mathcal{J}_{\text{eff}} t/\hbar^2. \qquad (2)$$

The quantity $\mathcal{J}_{\text{eff}}$ is equal to $\mathcal{J}_\perp \mathcal{J}_\parallel/(\mathcal{J}_\perp - \mathcal{J}_\parallel)$, where $\mathcal{J}_\perp$ and $\mathcal{J}_\parallel$ are nuclear moments of inertia about an axis perpendicular and parallel to the symmetry axis, respectively, and t is the temperature of the nucleus at the saddle point (or of the nucleus in the transition state).

If one assumes that the fission fragments separate along the nuclear symmetry axis and that K is a good quantum number in the descent from the nucleus in the transition state to the configuration of separated fragments, then the fission-fragment angular distribution for a transition state with quantum numbers I, K, and M (projection of I on a space-fixed axis which is usually taken as the beam direction) is given by[2]

$$W_{M,K}^I(\theta) = [(2I+1)/4\pi] \, |d_{M,K}^I(\theta)|^2. \qquad (3)$$

The $d_{M,K}^I(\theta)$ functions are defined by[2]

$$d_{M,K}^I(\theta) = [(I+M)!(I-M)!(I+K)!(I-K)!]^{1/2} \sum_X \frac{(-1)^X (\sin\frac{1}{2}\theta)^{K-M+2X} (\cos\frac{1}{2}\theta)^{2I-K+M-2X}}{(I-K-X)!(I+M-X)!(X+K-M)!X!}, \qquad (4)$$

where the sum is over $X=0, 1, 2, \cdots$, and contains all terms in which no negative value appears in the denominator of the sum for any one of the quantities in parentheses.

If the target and projectile spins are included, an exact expression for the fission-fragment angular distribution[3] for a Gaussian K distribution is

$$W(\theta) \propto \sum_{I=0}^{\infty} \sum_{M=-j_{\max}}^{+j_{\max}} \left[\sum_{l=0}^{\infty} \sum_{j=|I_0-s|}^{I_0+s} \sum_{\mu=-I_0}^{+I_0} \left(\frac{(2l+1)T_l \, |C_{M,0,M}^{j,l,I}|^2 \, |C_{\mu,M-\mu,M}^{I_0,s,j}|^2}{\sum_{l=0}^{\infty} (2l+1)T_l} \right) \right]$$

$$\times \sum_{K=-I}^{I} \left[(2I+1) \, |d_{M,K}^I(\theta)|^2 \exp\left(\frac{-K^2}{2K_0^2}\right) \Big/ \sum_{K=-I}^{I} \exp\left(\frac{-K^2}{2K_0^2}\right) \right]. \qquad (5)$$

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[1] I. Halpern and V. M. Strutinski, in *Proceedings of the First United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955* (United Nations, New York, 1958), Vol. 15, p. 408.

[2] J. A. Wheeler, in *Fast Neutron Physics*, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1963), Pt. II. p. 2051.

[3] James J. Griffin, Phys. Rev. **132**, 316 (1963).

The quantities I_0, s, and j are the target spin, projectile spin, and channel spin, respectively. The channel spin j is defined by the relation $\mathbf{j}=\mathbf{I}_0+\mathbf{s}$. The total angular momentum I is given by the sum of the channel spin and orbital angular momentum, $\mathbf{I}=\mathbf{j}+\mathbf{l}$. The projection of I_0 on the space-fixed axis is given by μ, whereas the projection of j (and I) on the space-fixed axis is M. The quantity in the first set of large square brackets gives the weighting factor for a particular (I, M) combination. This value multiplies the angular-dependent term for the allowable K states (K distribution is weighted also) of a particular I. This product is summed first over all M values for a particular I and finally over all I values.

If the target and projectile spins are zero and no particle emission from the initial compound nucleus occurs before fission (i.e., $M=0$), the angular distribution for a particular I with the assumption of a Gaussian K distribution is

$$W_{M=0}{}^I(\theta)\propto\sum_{K=-I}^{I}\left[(2I+1)\mid d_{M=0,K}{}^I(\theta)\mid^2\exp\left(\frac{-K^2}{2K_0{}^2}\right)\Big/\sum_{K=-I}^{I}\exp\left(\frac{-K^2}{2K_0{}^2}\right)\right]. \tag{6}$$

When many I values of the compound nucleus contribute and the transmission coefficients T_l are known, the over-all angular distribution for $M=0$ becomes

$$W(\theta)\propto\sum_{I=0}^{\infty}(2I+1)T_I\sum_{K=-I}^{I}\left[(2I+1)\mid d_{M=0,K}{}^I(\theta)\mid^2\exp\left(\frac{-K^2}{2K_0{}^2}\right)\Big/\sum_{K=-I}^{I}\exp\left(\frac{-K^2}{2K_0{}^2}\right)\right], \tag{7}$$

where the transmission coefficients are written as T_I, since $l=I$ when $M=0$.

Equation (7) is an exact theoretical expression for computation of fission-fragment angular distribution when both the target and projectile spins are zero. Examples of such reactions are α-particle-induced fission of even-even target nuclei (limited to reactions where only first-chance fission occurs, since M is no longer zero if fission follows initial deexcitation by neutron emission). The use of Eqs. (5) or (7), however, requires the evaluation of many $d_{M,K}{}^I(\theta)$ functions; hence neither equation has been used previously at moderate excitation energies.

In the classically allowed region for large I, the $d_{M,K}{}^I(\theta)$ function may be approximated by[2]

$$\mid d_{M,K}{}^I(\theta)\mid^2\cong\pi^{-1}[(I+\tfrac{1}{2})^2\sin^2\theta-M^2-K^2+2MK\cos\theta]^{-1/2}. \tag{8}$$

Substitution of this expression into Eq. (3) for $M=0$ gives

$$W_{M=0,K}{}^I(\theta)=[(2I+1)/4\pi^2][(I+\tfrac{1}{2})^2\sin^2\theta-K^2]^{-1/2}. \tag{9}$$

With a Gaussian K distribution, the angular distribution for a particular I (for $M=0$) is

$$W_{M=0}{}^I(\theta)=(2I+1)\int_0^{(I+1/2)\sin\theta}dK[(I+\tfrac{1}{2})^2\sin^2\theta-K^2]^{-1/2}\exp\left(\frac{-K^2}{2K_0{}^2}\right)\Big/4\pi^2\int_0^{(I+1/2}dK\exp\left(\frac{-K^2}{2K_0{}^2}\right). \tag{10}$$

Both the numerator and denominator can be integrated explicitly to give

$$W_{M=0}{}^I(\theta)=\frac{(\tfrac{1}{2}\pi)(2I+1)\exp[-(I+\tfrac{1}{2})^2(\sin^2\theta)/4K_0{}^2]J_0[i(I+\tfrac{1}{2})^2(\sin^2\theta)/4K_0{}^2]}{4\pi^2(\tfrac{1}{2}\sqrt{\pi})(2K_0{}^2)^{1/2}\operatorname{erf}[(I+\tfrac{1}{2})/(2K_0{}^2)^{1/2}]}, \tag{11}$$

where J_0 is the zero-order Bessel function with an imaginary argument, and $\operatorname{erf}[(I+\tfrac{1}{2})/(2K_0{}^2)^{1/2}]$ is the error function defined by $\operatorname{erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-t^2}dt$.

If each I value is weighted by $(2I+1)T_I$, the over-all fission-fragment angular distribution in this approximation $(M=0)$ is

$$W(\theta)\propto\sum_{I=0}^{\infty}\frac{(2I+1)^2T_I\exp[-(I+\tfrac{1}{2})^2(\sin^2\theta)/4K_0{}^2]J_0[i(I+\tfrac{1}{2})^2(\sin^2\theta)/4K_0{}^2]}{\operatorname{erf}[(I+\tfrac{1}{2})/(2K_0{}^2)^{1/2}]}. \tag{12}$$

Equation (12) is an approximate form of Eq. (7). In the derivation of Eq. (12), the $d_{M,K}{}^I(\theta)$ functions are approximated by the relation given by Eq. (8). The advantage of Eq. (12), as compared to Eq. (7), is its much simpler mathematical form. Both Eqs. (7) and (12) assume that $M=0$, i.e., that both the target and projectile spins are zero.

If, in Eq. (10), one replaces $(I+\frac{1}{2})$ with I and again weights each I value by $(2I+1)T_I$, the following modified form of Eq. (12) results:

$$W(\theta) \propto \sum_{I=0}^{\infty} \frac{(2I+1)T_I I \exp[-I^2(\sin^2\theta)/4K_0^2]J_0[iI^2(\sin^2\theta)/4K_0^2]}{\operatorname{erf}[I/(2K_0^2)^{1/2}]}. \quad (13)$$

Halpern and Strutinsky[1] have introduced into the literature still a different approximate equation. In Eq. (10), they replace $(I+\frac{1}{2})$ with I and integrate this equation, with the result

$$W_{M=0}{}^I(\theta) = \left(\frac{2}{\pi}\right)^{1/2} \frac{N}{2\pi} \frac{I}{2K_0} \exp\left(\frac{-I^2 \sin^2\theta}{4K_0^2}\right) J_0\left(\frac{iI^2 \sin^2\theta}{4K_0^2}\right). \quad (14)$$

Halpern and Strutinsky define N as a "normalization constant which is very close to 1 as long as I is somewhat larger than K_0." For the angular distribution, where many I values are involved, they imply that one integrates Eq. (14) over I with the appropriate weighting factor for each I. If a weighting factor $(2I+1)T_I$ is used, the expression for the angular distribution becomes

$$W(\theta) \propto \sum_{I=0}^{\infty} (2I+1)T_I I \exp\left(\frac{-I^2 \sin^2\theta}{4K_0^2}\right) J_0\left(\frac{iI^2 \sin^2\theta}{4K_0^2}\right). \quad (15)$$

Implicit in Eq. (15) is the assumption that the error function of Eq. (12) is a normalization constant independent of I. While this is a reasonably good approximation for the terms with I much larger than K_0, it is incorrect for the other terms with smaller I values. This approximation then leads to a sizeable error in many cases.

Most of the analyses of experimental angular distributions of fission fragments at moderate excitation energies which have been reported in the literature have been performed with Eq. (15). We have recently observed quite different values of K_0^2 calculated with expressions which do and do not include the error function. Therefore it is necessary to compare the various formulas with exact calculations in order to determine their accuracies and establish which, if any, of the approximate expressions can be used in place of the more cumbersome exact expressions.

II. COMPARISON OF VARIOUS THEORETICAL EXPRESSIONS

Several fission reactions are chosen to illustrate the dependence of the derived value of K_0^2 on the form of the mathematical expression used in the theoretical analysis. Optical-model transmission coefficients are used in all calculations, and the deuteron[4] and α-particle[5] transmission coefficients are kept fixed for the calculations with the different equations.

The experimental anisotropy $W(174°)/W(90°)$ for the Bi^{209} (d,f) reaction with 21-MeV deuterons is 1.446.[6] The different lines of Fig. 1 illustrate the theoretical dependence of the anisotropy on K_0^2 for the various theoretical expressions. The results of exact theoretical calculations with Eq. (5), which includes the spins of both the target and projectile, are shown by curve (d). The results obtained with Eq. (7), which assumes that both the target and projectile spins are zero, are plotted as curve (c). The value of K_0^2 derived with Eq. (7) is 13% too large and demonstrates the error introduced by neglecting the target and projectile spins. For the par-

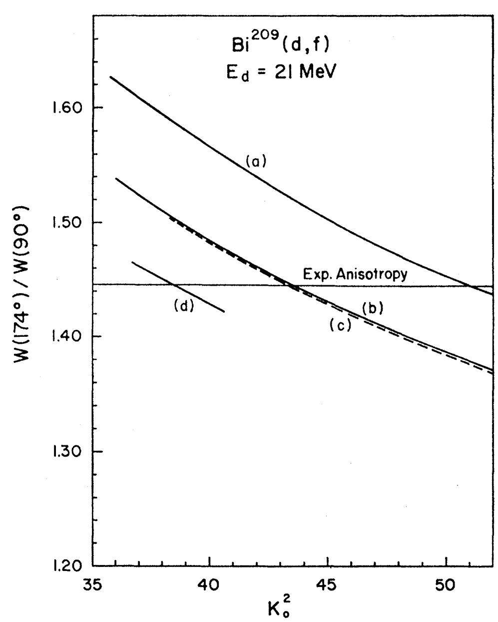

FIG. 1. Anistropy, $W(174°)/W(90°)$, of fission fragments for the Bi^{209} (d,f) reaction with 21-MeV deuterons. The theoretical lines are calculated with the following equations: (a) with Eq. (12) without error function in denominator, (b) with Eq. (12), (c) with Eq. (7), and (d) with Eq. (5).

[4] M. A. Melkanoff, T. Sawada, and N. Cindro, Phys. Letters **2**, 98 (1962).
[5] J. R. Huizenga and G. Igo, Nucl. Phys. **29**, 462 (1962).
[6] G. L. Bate, R. Chaudhry, and J. R. Huizenga, Phys. Rev. **131**, 722 (1963).

ticular reaction under consideration, the target spin (Bi^{209}) is $\frac{9}{2}$ and the projectile spin (deuteron) is 1. The value of K_0^2 derived with Eq. (12) is also 13% too large. In addition to neglecting the target and projectile spins, Eq. (12) also contains an approximate form of the $d_{M=0,K}^I(\theta)$ function given by Eq. (8). The close agreement of curves (b) and (c) shows that, for this reaction, Eq. (8) is a very good approximation. The results plotted in curve (a) were calculated with an equation identical to Eq. (12) except without the error function in the denominator. This leads to a value of K_0^2 which is 33% too large.

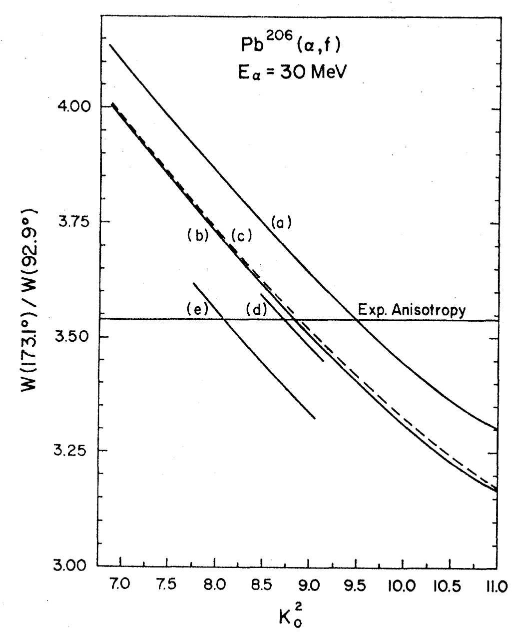

FIG. 2. Anistropy, $W(173.1°)/W(92.9°)$, of fission fragments for the Pb^{206} (α, f) reaction with 30-MeV α particles. The theoretical lines are calculated with the following equations: (a) with Eq. (12) without error function in denominator, (b) with Eq. (12), (c) with Eq. (7), (d) with Eq. (15), and (e) with Eq. (13).

Comparison of curve (a) with curves (b) and (c) gives the magnitude of the error in K_0^2 introduced by neglecting the error function in Eq. (12).

The experimental anisotropies for the Pb^{206} (α, f) reaction[7] with 30 and 38 MeV α particles are $W(173°)/W(92.9°) = 3.54$ and $W(171.7°)/W(93.8°) = 2.56$, respectively. For this reaction both the target and projectile spins are zero. Hence the exact theoretical calculations are performed with Eq. (7). Again, the agreement between the results of the exact theory and the approximate theory of Eq. (12) is good, as shown in Figs. 2 and 3. The values of K_0^2 deduced with the approximate theory of Eq. (12), except

[7] L. G. Moretto, R. C. Gatti, S. G. Thompson, J. R. Huizenga, and J. O. Rasmussen, Phys. Rev. (to be published).

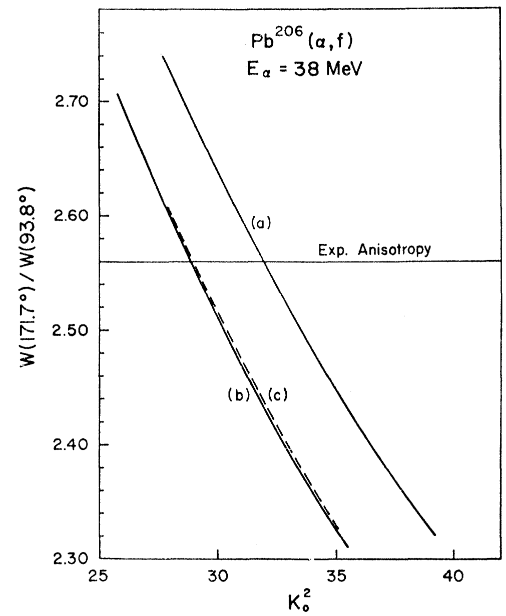

FIG. 3. Anistropy, $W(171.7°)/W(93.8°)$, of fission fragments for the Pb^{206} (α, f) reaction with 38-MeV α particles. The theoretical lines are calculated with the following equations: (a) with Eq. (12) without error function in denominator, (b) with Eq. (12), and (c) with Eq. (7).

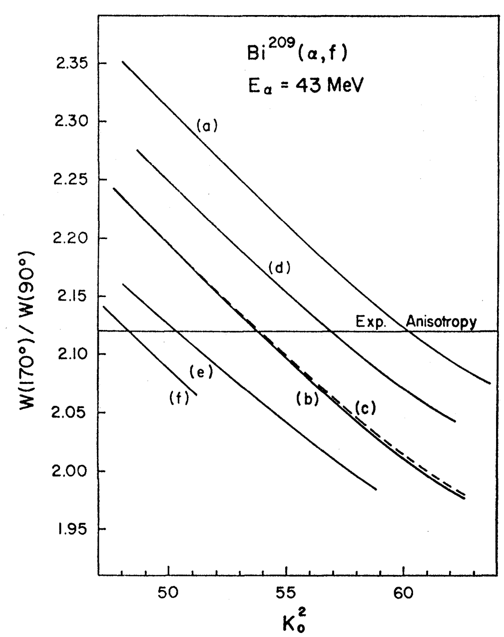

FIG. 4. Anistropy, $W(170°)/W(90°)$, of fission fragments for the Bi^{209} (α, f) reaction with 43-MeV α particles. The theoretical lines are calculated with the following equations: (a) with Eq. (12) without error function in denominator, (b) with Eq. (12), (c) with Eq. (7), (d) with Eq. (15), (e) with Eq. (13), and (f) with Eq. (5).

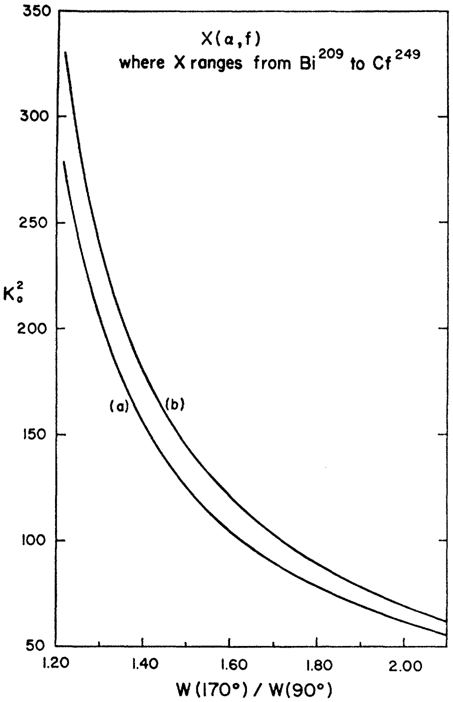

FIG. 5. Values of K_0^2 deduced from (α,f) reactions with 43-MeV α particles on targets ranging from Bi^{209} to Cf^{249}; value of anistropies, $W(170°)/W(90°)$, used in the calculations were published recently (Ref. 8). (a) Values of K_0^2 calculated with Eq. (12); (b) with Eq. (12) without error function in denominator.

neglecting the error function, are too large by 8 and 12% for the 30- and 38-MeV bombardments, respectively. Curve (e) in Fig. 2 is calculated with the classical form of the approximate equation with the error function as given in Eq. (13). Curve (d) is computed with Eq. (15), which neglects the error function. Again, if one neglects the error function, the value of K_0^2 which is extracted from theory is too large.

The measured anisotropy $W(170°)/W(90°)$ for the Bi^{209} (α,f) reaction with 43-MeV α-particle projectiles is 2.12.[8] The values of K_0^2 derived from Eqs. (7) and (12) are nearly the same as shown by curves (c) and (b) of Fig. 4. It should be emphasized that the spin of the target (Bi^{209}) is $\frac{9}{2}$ for this reaction, and the value of K_0^2 derived from Eq. (7) [curve (c)] is 11% larger than the value derived from exact theory including spin with Eq. (5) [curve (f)]. Neglect of the error function in Eq. (12) leads to curve (a) and a value of K_0^2, which is 25% larger than exact theory [curve (f)] and 12% larger than the values deduced from Eq. (7) [curve (c)] and Eq. (12) [curve (b)]. Curve (d) is calculated with Eq. (15) and curve (e) with Ep. (13). The classical form of this approximate equation (i.e., substituting I for $I+\frac{1}{2}$) gives values of K_0^2 which are some 6–10% smaller for the reactions of Figs. 2 and 4.

In Fig. 5, values of K_0^2 deduced from the (α,f) reaction with 43-MeV α particles are shown for targets ranging from Bi^{209} to Cf^{249}. The targets and their experimental anisotropies $W(170°)/W(90°)$ used in the present calculations are those reported earlier.[8] One curve in Fig. 5 shows the values of K_0^2 determined with Eq. (12), which included the error function in the denominator. The other curve in Fig. 5 makes use of Eq. (12) without the error function. As can be seen from a comparison of these two curves, the neglect of the error function leads to values of K_0^2 which are too large by several percent for all cases.

III. CONCLUSIONS

If $M=0$, the results displayed in Figs. 1–4 show that the approximate theory of fission-fragment angular distribution as given by Eq. (12) agrees very nearly with the exact theory of Eq. (7). Neglect of the error function in Eq. (12) is not warranted and results in a sizeable error as shown by the calculations displayed in Figs. 1–5. When the target and/or projectile spins are larger than zero and the assumption $M=0$ is no longer valid, values of K_0^2 calculated with Eqs. (7) and (12) are too large by several percent. In this case, the exact theory including target and projectile spins as given in Eq. (5) is required.

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