Nuclear-Polarization Corrections to the Levels of Muonic Atoms*

R. K. Cole, Jr. †

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

(Received 27 June 1968)

It is shown that information about the nuclear electromagnetic-transition vertex derived from experimental inelastic-scattering cross sections for electrons may be used to evaluate the nuclear-polarization (dispersion) corrections to the levels of muonic atoms. A model-independent result is obtained for the contributions of discrete nuclear states. The most important systematic features of nuclear-excitation spectra, the giant-dipole resonance and the quasielastic peak, are considered in detail. The Goldhaber-Teller model is used for the former, and a simple-harmonic-oscillator shell model for the latter. Numerical estimates are obtained for total level shifts of low-lying muon states in nuclei with closed (harmonic-oscillator) proton shells, using closure approximation for the muon. The muon closure energies are considered in detail. It is felt that the results obtained are probably accurate to a factor of 2. The shifts are estimated to be several keV for the $1s$ state in heavy nuclei, and somewhat less (a few tenths to about 1 keV) for the $2s$ and $2p$ states. These shifts are significant in comparison to the present accuracy of measurement of muonic x-ray spectra, and should be considered in calculations to fit nuclear-charge distributions.

1. INTRODUCTION

In 1949, soon after the properties of the muon began to be understood, it was realized that this particle could be very useful as a probe in measuring the electromagnetic properties of nuclei.[1] The muon has no strong interactions; in fact, except for the weak interactions permitted by its 200 times greater mass, it seems just a heavy electron. In particular, muonic atoms can be formed in exact analogy with electronic atoms. Because of the mass ratio, the muonic Bohr orbit is $\frac{1}{200}$ that for an electron, so that for heavy nuclei such as lead ($Z=82$) a $1s$ muon spends about one-half its time inside the nucleus, which is quite transparent to it. This makes the energy levels, and therefore the transition energies (which are in the x-ray range), very sensitive to the nuclear-charge distribution; for example, the binding energy of a $1s$ muon in lead is reduced by almost half from the point-Coulomb value by the finite size of the nucleus.

A few years later the x-ray spectra of several muonic atoms from titanium ($Z=22$) to lead ($Z=82$) were measured by Fitch and Rainwater.[2] These were analyzed to find the radii of the nuclear-charge distributions, and gave $R\approx 1.2A^{1/3}$ F – markedly smaller than the then accepted $R\approx 1.4 - 1.5A^{1/3}$ F. A companion paper by Cooper and Henley[3] showed that this difference was almost certainly not the result of any neglected effects. There had been earlier suggestions of an electromagnetic radius of about this size,[4] and the electron-scattering experiments of Hofstadter *et al.*[5] gave rapid confirmation.

These two methods, muonic x-ray spectra and elastic electron scattering, have become the standard tools in measurements of nuclear-charge distributions.

With subsequent improvements in experiment and theory (in particular, inclusion of vacuum-polarization effects corresponding to excitation of virtual electron-positron pairs) agreement between the two methods has become extremely good, and detailed enough to fit several parameters of the nuclear-charge distribution. The most familiar form used is the Fermi distribution

$$\rho(r)=\rho_0\{1+\exp[(r-c)/a]\}^{-1}, \tag{1.1}$$

with $c\approx 1.12A^{1/3}$ F, $a\approx 0.5$ F. (1.2)

Occasionally muon and electron experiments give results which appear to differ outside of experimental error. There seem to be two possible explanations:

(1) The two experiments are sensitive to different aspects of the charge distribution, and we may be fitting a distribution which is not really appropriate. As Elton said at Williamsburg:[6] "The Fermi distribution is not God given, it isn't even Fermi given," and quite good fits can also be gotten with shorter-tailed distributions such as that generated using the shell model in a Saxon-Woods potential.[7] Such a distribution is clearly more reasonable physically than the simple Fermi shape.

(2) The two experiments are really measuring *different* distributions, because the electromagnetic fields of the muon or the electron distort the nucleus – presumably differently. These effects are referred to as "nuclear-polarization corrections," and have been called by Ravenhall[8] "the largest contribution to the uncertainty in the theoretical predictions."

Before we can draw any conclusions about (1) we must put some limits on (2).

Nuclear polarization was one of the possible effects considered by Fitch and Rainwater.[2] They estimated that the effects were less than 1.2, 13, and 60 keV for the $1s$ states of aluminum ($Z=13$), copper ($Z=39$), and lead ($Z=82$), respectively. These estimates were deliberately rather high, and more recent calculations have reduced them considerably, to something like 1 to 10 keV for the $1s$ state of lead or bismuth. Contributions

have been estimated for the nuclear core and collective excitations,[9,10] and for valence nucleons such as the odd proton in bismuth 209.[11] Until about 1964 these effects were totally negligible: the NaI scintillation detectors then in use had resolutions no better than a few hundred keV. With the introduction of the Ge(Li) detector, resolutions have been improved by two orders of magnitude, producing a considerable revival of interest in muonic x-ray spectra. Transition energies may now be measured with a precision of 1 keV or better so that nuclear-polarization effects, while small, should still be observable and must be considered if we are to benefit from the full accuracy of the experiments.

One case in which the effect is much larger is the so-called "dynamic quadrupole effect" in nuclei with low-lying rotational bands of energy comparable to muonic fine structure.[12,13] In this case the levels are strongly mixed, and a large quadrupole splitting (100 keV or more in favorable cases) can be observed, even in nuclei with *no* ground state electric-quadrupole moment.

Our aim will be to estimate the remaining nuclear-polarization shifts – particularly core effects – for the ground state and for excited states as generally as possible. We attempt to evaluate the most important contributions in what is in principle a model-independent way, by observing that the only unknown quantity in the problem is the nuclear electromagnetic vertex, about which a great deal is already known from inelastic electron scattering. The actual calculations will be somewhat schematic, using typical average properties of nuclei rather than those appropriate to some particular nuclei. The results are still approximate enough (probably at best a factor of 2) so that more detail seems unjustified, and the use of average parameters is intended to aid interpolation to intermediate nuclei. A preliminary version of part of this work has been published previously.[14]

As an aid to visualizing the situation, we give in Table I some approximate values of sizes and energies for muonic atoms, and their relations to electronic atoms.

2. THEORY

An exact solution of the muonic-atom problem would require diagonalization of the total Hamiltonian of the coupled system of muon and nucleus

$$\hat{H} = \hat{H}_{0N} + \hat{t}_\mu + \hat{H}_{em} , \tag{2.1}$$

where $\hat{H}_{0N}$ is the total nuclear Hamiltonian, $\hat{t}_\mu$ is the muon kinetic energy, and $\hat{H}_{em}$ is the electromagnetic interaction of muon and nucleus. This interaction is given by[15]

$$\hat{H}_{em} = -e\int d^3x \hat{j}_\alpha(\vec{x})\hat{A}^\alpha(x) \equiv -e\int d^3x(\hat{\rho}_\mu\hat{\phi} - \hat{\vec{j}}\cdot\hat{\vec{A}}) . \tag{2.2}$$

$\hat{A}_\alpha = (\hat{\vec{A}}, \hat{\phi})$ is the electromagnetic potential of the nucleus. It is convenient to work in the Coulomb gauge, $\vec{\nabla}\cdot\hat{\vec{A}} = 0$, separating the three vector $\vec{J}$ into longitudinal and transverse parts, defined in general by

$$\vec{v} = \vec{v}^T + \vec{v}^L, \quad \vec{\nabla}\times\vec{v}^L = 0, \quad \vec{\nabla}\cdot\vec{v}^T = 0 . \tag{2.3}$$

Then the potential is related to the nuclear current by

$$\nabla^2\hat{\phi} = -e\hat{\rho}_N, \quad \Box\hat{\vec{A}} = -e\hat{\vec{J}}^T . \tag{2.4}$$

TABLE I. Sizes and energies of muonic atoms.

Quantity	Ratio to electronic atom[a]	$Z=1$	$Z\sim80$ Point nucleus	$Z\sim80$ Extended charge
$\langle r^2\rangle_{1s}^{1/2} \propto (mZ)^{-1}$	$(200)^{-1}$	260 F	~3 F	~7 F
Nuclear radius	...	~2 F	~7 F	
Binding energy $\epsilon_{1s}, \propto mZ^2$	200	3 keV	~20 MeV	~10 MeV
Fine structure, $2p, \propto mZ^4$	200	0.01 eV		150–200 keV
Hyperfine $M1$ splitting $\propto m^2Z^3$	$(200)^2$			~2 keV
Hyperfine $E2$ splitting $\propto m^3Z^3$	$(200)^3$			~10–100 keV (deformed)
Vacuum polarization, $(1s)$, $\propto Z^4m$	200			~80 keV

[a] $m_\mu/m_e \approx 200$.

This assumes a local nuclear current, but does not further restrict its form. What has been done is to separate the electromagnetic interaction into an instantaneous Coulomb part and a retarded transverse part.

In lowest order, we may replace the nuclear current by its ground-state expectation to get

$$\hat{H}_{\mathrm{em}0} = \langle 0 | \hat{H}_{\mathrm{em}} | 0 \rangle \ , \tag{2.5}$$

where $|0\rangle$ is the nuclear ground state. The most important part of $\hat{H}_{\mathrm{em}0}$ is a spherically symmetric Coulomb well

$$\hat{V}_C = -\alpha \int d^3x \ \hat{\rho}_\mu(\vec{x})(4\pi)^{-1} \int d\Omega_x \int d^3y \ [\langle 0|\hat{\rho}_N(\vec{y})|0\rangle / |\vec{x}-\vec{y}|] \ . \tag{2.6}$$

There may also be nonzero (even) electric- and (odd) magnetic-nuclear-moment terms, but we will neglect these for simplicity, imagining them to be treated if necessary by a separate perturbation calculation.

We can now write, in anticipation of a perturbation expansion,

$$\hat{H} = \hat{H}_0 + \hat{H}', \quad \hat{H}_0 = \hat{H}_{0\mathrm{N}} + \hat{t}_\mu + \hat{V}_C \ , \quad \hat{H}' = \hat{H}_{\mathrm{em}} - \langle 0 | \hat{H}_{\mathrm{em}} | 0 \rangle \ . \tag{2.7}$$

$\hat{H}'$ is simply the coupling to the nuclear transition current, and is responsible for the nuclear polarization. It has diagonal matrix elements equal to zero,[16] and off-diagonal elements equal to those of $\hat{H}_{\mathrm{em}}$, since $\langle 0|\hat{H}_{\mathrm{em}}|0\rangle$ is a nuclear c number.

$$\langle N' | \hat{H}' | N \rangle = \langle N' | \hat{H}_{\mathrm{em}} - \langle 0 | \hat{H}_{\mathrm{em}} | 0 \rangle | N \rangle = \langle N' | \hat{H}_{\mathrm{em}} | N \rangle \ , \quad N' \neq N \ . \tag{2.8}$$

$\hat{H}_0$ does not couple nuclear and muon coordinates ($\hat{V}_C$ is a nuclear c number), and the unperturbed solutions are therefore direct-product states $|\gamma IM\rangle|nljm\rangle$, where

$$\hat{H}_{0\mathrm{N}}|\gamma I\rangle = E_{\gamma I}|\gamma I\rangle \ , \quad (\hat{t}_\mu + \hat{V}_C)|nljm\rangle = \epsilon_{nlj}|nljm\rangle \ . \tag{2.9}$$

We should really consider states of total angular momentum $\vec{F} = \vec{I} + \vec{j}$ [this is the lowest-order effect of the nuclear-moment terms omitted in (2.5)],

$$|\gamma I, nlj; FM_F\rangle = \sum_{Mm} (IMjm | IjFM_F) | \gamma IM\rangle | nljm\rangle \ .$$

For simplicity we will average matrix elements of any operator in the coupled system over hyperfine structure by taking $(2I+1)^{-1}(2j+1)^{-1}\sum_F(2F+1)$. The orthogonality of the Clebsch-Gordan coefficients shows this to be equivalent to individually averaging over nuclear and muon orientations by taking $(2I+1)^{-1}\sum_M(2j+1)^{-1}\sum_m$ in the uncoupled basis.

Since the diagonal matrix elements of $\hat{H}'$ are zero, the first contribution to the energy comes in second order in $\hat{H}'$. The retarded nature of the interaction makes it natural to use Feynman perturbation theory; the two graphs which contribute are shown in Fig. 1, where the nuclear vertex is understood to have no diagonal matrix elements. (The analysis may be performed in time-ordered perturbation – with the same results –, but it is necessary to treat Fig. 1 as fourth order in the interaction of the currents with the electromagnetic field, and allow intermediate states to contain real photons. There are clearly many time-ordered processes.)

The energy shift is just the real part of the diagonal matrix element of the transition matrix T, which is related to the scattering matrix S by

$$S_{FI} = \delta_{FI} - 2\pi i \delta(E_F - E_I) T_{FI} \ . \tag{2.10}$$

S_{FI} may be written down directly from Fig. 1 using the Feynman rules for an external field representation[17] as

$$S_{FI}^{(2)} = -(4\pi\alpha)^2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 [D_{\mu\rho}(x_4 - x_2) D_{\nu\sigma}(x_3 - x_1) + D_{\mu\sigma}(x_3 - x_2) D_{\nu\rho}(x_4 - x_1)]$$

$$\times \bar{\Psi}_0(x_2)\Gamma_\mu i S_F^{\ e}(x_2, x_1)\Gamma_\nu \Psi_0(x_1)\bar{\phi}_i(x_4)\gamma_\rho i s_F^{\ e}(x_4, x_3)\gamma_\sigma \phi_i(x_3) \ . \tag{2.11}$$

Here Ψ is a nuclear wave function, and ϕ is a muon wave function. $S_F^{\ e}$ is the external-field Feynman propagator and $D_{\alpha\beta}$ the electromagnetic propagator. Γ_μ is the nuclear-transition vertex (equal to the total vertex but with diagonal matrix elements equal to zero).

Since the muonic atom is not very relativistic, we will use the nonrelativistic limit of the Feynman propagators

$$iS_F^{\ e}(x, x') = \sum_n \psi_n(x)\psi_n^*(x')\theta(t - t'), \quad \psi_n(x) = \psi_n(\vec{x}) e^{-iE_n t} \ . \tag{2.12}$$

for both the muon and the nucleus. This is "nonrelativistic" in the sense of omitting particle-antiparticle pairs in time-ordered perturbation theory, where they are high-energy excitations (twice a particle mass)

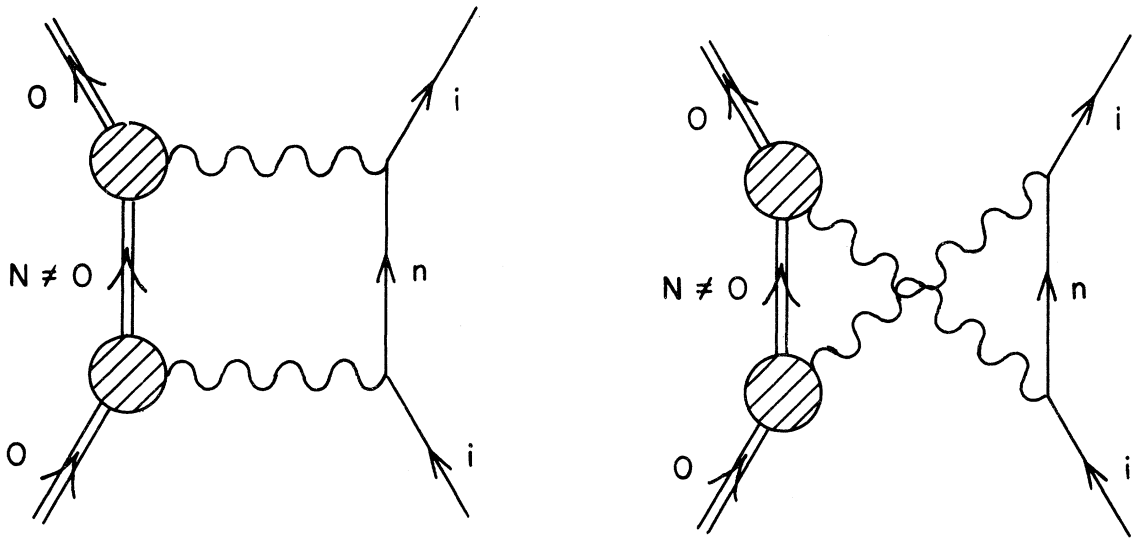

FIG. 1. Diagrams contributing to nuclear polarization in lowest order.

and correspondingly unimportant.

We will also separate the electromagnetic propagator into Coulomb and transverse parts

$$D(x) = D^C(x) + D^T(x) , \tag{2.13}$$

where the nonzero components are

$$D_{00}{}^C(x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{iqx}}{q^2 - i\eta}, \quad D_{ij}{}^T(x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{iqx}}{q^2 - i\eta} \left(\delta_{ij} - \frac{q_i q_j}{\vec{q}^2}\right); \quad i,j = 1, 2, 3 . \tag{2.14}$$

Use of the Fourier transform of the θ function

$$\theta(t) = (2\pi i)^{-1} \int_{-\infty}^{\infty} ds\, e^{ist}/(s - i\alpha) \tag{2.15}$$

allows us to do the time integrals in S_{FI}, getting a δ function for each. The spatial integrals all have the form

$$\int d^3x\, \bar{\psi}_2(\vec{x}) \gamma_\mu \psi_1(\vec{x}) e^{-i\vec{q}\cdot\vec{x}} = J_\mu(q)_{21} \tag{2.16}$$

and $\delta ij - q_i q_j / |\vec{q}|^2$ in D^T just projects out the transverse parts of both currents. We then identify

$$\begin{aligned} T_{II} = &- (4\pi\alpha)^2 \sum_{N\neq 0} \sum_n \int \frac{d^3q'}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi i} \frac{1}{\Delta E + \omega - i\alpha} \\ &\times \{(\Delta\epsilon - \omega - ia)^{-1} [\rho_N(\vec{q}')_{0N} \rho_\mu(-\vec{q}')_{in}/q'^2 - \vec{J}^T(\vec{q}')_{0N} \cdot \vec{j}^T(-\vec{q}')_{in}/(q'^2 - \omega^2 - i\eta)] \\ &\times [\rho_N(\vec{q})_{N0} \rho_\mu(-\vec{q})_{ni}/q^2 - \vec{J}^T(\vec{q})_{N0} \cdot \vec{j}^T(-\vec{q}')_{ni}/(q^2 - \omega^2 - i\eta)] \\ &+ (\Delta\epsilon + \omega - i\alpha)^{-1} [\rho_N(\vec{q}')_{0N} \rho_\mu(-\vec{q}')_{ni}/q'^2 - \vec{J}^T(\vec{q}')_{0N} \cdot \vec{j}^T(-\vec{q}')_{ni}/(q'^2 - \omega^2 - i\eta)] \\ &\times [\rho_N(\vec{q})_{N0} \rho_\mu(-\vec{q})_{in}/q^2 - \vec{J}^T(\vec{q})_{N0} \cdot \vec{j}^T(-\vec{q})_{in}/(q^2 - \omega^2 - i\eta)]\} , \end{aligned} \tag{2.17}$$

where $\Delta E = E_N - E_0$ and $\Delta\epsilon = \epsilon_n - \epsilon_i$.

It is convenient to make a multipole expansion of the Fourier transform of the currents into angular-momentum components, using the Coulomb and transverse-multipole operators defined by[18]

$$\begin{aligned} \hat{M}_{KQ}(q) &= i^K \int d^3x\, j_K(qx) Y_{KQ}(\Omega_x) \hat{\rho}(\vec{x}) , \\ \hat{T}_{KQ}^{\mathrm{el}}(q) &= i^K q^{-1} \int d^3x\, \vec{\nabla} \times [j_K(qx) \vec{Y}_{KK1}{}^Q(\Omega_x)] \cdot \hat{\vec{J}}(\vec{x}) , \end{aligned} \tag{2.18}$$

$$\hat{T}_{KQ}{}^{\mathrm{mag}}(q) = i^K \int d^3x j_K(qx) \vec{Y}_{KK1}{}^{Q}(\Omega_x) \cdot \hat{\vec{J}}(\vec{x}) ,$$

and $\hat{m}_{KQ}$, $\hat{t}_{KQ}{}^{\mathrm{el}}$, and $\hat{t}_{KQ}{}^{\mathrm{mag}}$ similarly defined for the muon. Here

$$\vec{Y}_{KK1}{}^{Q} \equiv \sum_{Mm} (KM1m \mid K1KQ) Y_{KM}(\Omega_x) \vec{e}_m , \tag{2.19}$$

and the $\vec{e}_m$ are related to an orthogonal right-hand triple of unit vectors $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z = \vec{q}/|\vec{q}|$ by

$$\vec{e}_0 = \vec{e}_z , \quad \vec{e}_{\pm 1} = \mp 2^{-1/2} (\vec{e}_x \pm i\vec{e}_y) . \tag{2.20}$$

The multipole operators in (2.18) are irreducible-tensor operators of rank K. In terms of them, we have for the currents

$$\rho(\vec{q})_{fi} = \sum_{KQ} (-)^{K+J_f-M_f} \begin{pmatrix} J_f & K & J_i \\ -M_f & Q & M_i \end{pmatrix} 4\pi Y_{KQ}{}^{*}(\Omega_q) \langle f \| \hat{M}_K(q) \| i \rangle ,$$

$$\vec{J}^T(\vec{q})_{fi} = \sum_{KQ} (-)^{K+J_f-M_f} \begin{pmatrix} J_f & K & J_i \\ -M_f & Q & M_i \end{pmatrix} \sum_{\lambda=\pm 1} e_\lambda{}^{\dagger} \mathfrak{D}_{Q\lambda}{}^{K}(-\phi_q, -\theta_q, \phi_q) \tag{2.21}$$

$$\times [2\pi(2K+1)]^{1/2} \langle f \| \hat{T}_K{}^{\mathrm{el}}(q) - \lambda \hat{T}_K{}^{\mathrm{mag}}(q) \| i \rangle .$$

We can now substitute these expressions into T_{II} and average over hyperfine structure by taking $(2I_0+1)^{-1} \times \sum M (2j+1)^{-1} \sum m$, since T is evaluated in an uncoupled basis. The sums on magnetic quantum numbers m, M and m', M' (of n and N) eliminate the three-j symbols, and the integrals over the angles of $\vec{q}$ and $\vec{q}'$ can then be done using the orthogonalities of the Y's and of the $\mathfrak{D}$'s. Use is made of the following relations

$$\rho(-\vec{q})_{fi} = \rho(\vec{q})_{if}{}^{*} , \quad \langle i \| \hat{M}_K(q) \| f \rangle^{*} = (-)^{j_f - j_i + K} \langle f \| \hat{M}_K(q) \| i \rangle ,$$

$$\vec{j}^T(-\vec{q})_{fi} = \vec{j}^T(\vec{q})_{if}{}^{*} , \quad \langle i \| \hat{T}_K(q) \| f \rangle^{*} = (-)^{j_f - j_i + K + 1} \langle f \| \hat{T}_K(q) \| i \rangle . \tag{2.22}$$

The ω integral can also be done and its real part taken to give the energy shift.[19]

$$\overline{\Delta E}_{nlj} = -(4\pi\alpha)^2 \frac{1}{2I_0+1} \sum_{N \neq 0} \frac{1}{2j+1} \sum_{\nu} \sum_{K} \frac{1}{2K+1} \mathcal{P} \left[\frac{1}{\Delta E + \Delta\epsilon} \left| \frac{2}{\pi} \int_0^\infty dq \langle N \| \hat{M}_K(q) \| 0 \rangle \langle \nu \| \hat{m}_K(q) \| nlj \rangle \right|^2 \right.$$

$$+ \frac{1}{\Delta E + \Delta\epsilon} \left(\frac{2}{\pi} \int_0^\infty dq' \frac{q'^2}{(q' + \Delta E)(q' + \Delta\epsilon)} \langle N \| \hat{T}_K(q') \| 0 \rangle^{*} \langle \nu \| \hat{t}_K(q') \| nlj \rangle^{*} \frac{2}{\pi} \int_0^\infty dq \langle N \| \hat{M}_K(q) \| 0 \rangle \langle \nu \| \hat{m}_K(q) \| nlj \rangle + \text{c. c.} \right)$$

$$+ \frac{2}{\pi} \int_0^\infty dq' \frac{2}{\pi} \int_0^\infty dq \frac{q'q}{(q' + \Delta E)(q' + \Delta\epsilon)(q + \Delta E)(q + \Delta\epsilon)} \left(\Delta E + \Delta\epsilon + q + q' + \frac{q'q}{\Delta E + \Delta\epsilon} + \frac{\Delta E \Delta\epsilon}{q + q'} \right)$$

$$\left. \times \langle N \| \hat{T}_K(q') \| 0 \rangle^{*} \langle \nu \| \hat{t}_K(q') \| nlj \rangle^{*} \langle N \| \hat{T}_K(q) \| 0 \rangle \langle \nu \| \hat{t}_K(q) \| nlj \rangle \right] . \tag{2.23}$$

Here $\mathcal{P}$ denotes the Cauchy principal value of any improper integral (the q and q' integrals may be improper if $\Delta\epsilon < 0$). We have also employed a shorthand

$$\hat{T}_K \hat{t}_K \equiv \hat{T}_K{}^{\mathrm{el}} \hat{t}_K{}^{\mathrm{el}} + \hat{T}_K{}^{\mathrm{mag}} \hat{t}_K{}^{\mathrm{mag}} . \tag{2.24}$$

No confusion can arise as $\hat{T}_K{}^{\mathrm{el}}$ and $\hat{T}_K{}^{\mathrm{mag}}$ have opposite parities.

The unpolarized double-differential cross section for inelastic scattering of an electron from $(\vec{k}_1, \epsilon_1)$ to $(\vec{k}_2, \epsilon_2)$ through angle θ may be expressed in terms of the multipole operators as[18]

$$\frac{d^2\sigma}{d\Omega_2 d\epsilon_2} = \frac{8\pi\alpha^2}{q_\mu{}^4} \frac{k_2}{k_1} \frac{1}{2I_0+1} \sum_K \sum_N \delta(E_N - E_0 + \epsilon_2 - \epsilon_1) [V_L(\theta) |\langle N \| \hat{M}_K(q) \| 0 \rangle|^2 + V_T(\theta) |\langle N \| \hat{T}_K(q) \| 0 \rangle|^2] , \tag{2.25}$$

where $q_\mu = (\vec{k}_2 - \vec{k}_1, \epsilon_2 - \epsilon_1)$, $V_L(\theta) = \frac{1}{2}(q_\mu{}^4/\vec{q}^4)[(\epsilon_1 - \epsilon_2)^2 - \vec{q}^2]$, $V_T(\theta) = \frac{1}{4}|\vec{k}_1 - \vec{k}_2|^2 - \frac{1}{2}(k_1{}^2 - k_2{}^2)^2/\vec{q}^2 + \frac{1}{2}q_\mu{}^2$. (2.26)

The two components of the cross section may be separated by the different angular dependences of V_L and V_T, allowing experimental measurement of two response functions

$$R^C(q,\omega) = [4\pi/(2I_0+1)] \sum_N \sum_K |\langle N \| \hat{M}_K(q) \| 0 \rangle|^2 \delta(E_N - E_0 - \omega),$$

$$R^T(q,\omega) = [4\pi/(2I_0+1)] \sum_N \sum_K |\langle N \| \hat{T}_K(q) \| 0 \rangle|^2 \delta(E_N - E_0 - \omega). \tag{2.27}$$

We see that since the reduced matrix elements are large where the cross section is large, the same nuclear excitations which are important in inelastic electron scattering will be important in nuclear polarization.

For the case of a discrete excitation with angular momentum K of a spin-zero nucleus, only one multipole contributes to the cross section and to the nuclear polarization. We can then make the replacements

$$\langle N\|\hat{M}_K(q)\|0\rangle \to \left(\frac{2I_0+1}{4\pi}\right)^{1/2}[R^C(q,\omega)]^{1/2},\quad \langle N\|\hat{T}_K^{\,i}(q)\|0\rangle \to \left(\frac{2I_0+1}{4\pi}\right)^{1/2}[R^T(q,\omega)]^{1/2}, \tag{2.28}$$

in (2.23) and integrate ω over the line width. Here i is el or mag as the parity of the excitation is $\pm(-)^K$, and we are to take $\hat{t}K^i$ for the muon. In principle we can evaluate the energy shift due to this excitation without resorting to any nuclear model. The muon matrix elements are known (the muon wavefunctions are just solutions of the Dirac equation), and the nuclear matrix elements may be determined from experimentally determined scattering cross sections (the response functions will have to be extrapolated into the region $0 \leq q < \omega$ using the known threshold behavior since $q^2 \leq \omega^2$ for physical electron scattering). The square root is unambiguous: the phases of the reduced matrix elements in R^C and R^T are known by time reversal.[18]

Even for low-lying states, the cross sections are not really known with sufficient accuracy to use Eq. (2.28), and for higher states or for nuclei with nonzero spin, the contributions of the various multipoles cannot be resolved at all, so we must use other methods to actually estimate the energy shifts. We may still make use of the relation to electron scattering by computing the nuclear matrix elements in (2.23) using models which have proved good for electron scattering and including in the sum the states which are important in scattering.

A typical double-differential cross section is shown in Fig. 2, reproduced from deForest and Walecka.[18] We see an elastic peak (at $\omega = 0$) which is not relevant to the polarization problem, various low-lying states and resonances among which the giant dipole (GD) is outstanding, and a quasielastic (QE) peak at $\omega \approx q^2/2M$. The rising cross section above $\omega = m_\pi$ represents pion production which is of little interest to us since the excitation energy is so large. The GD and QE peaks are systematic features while the other low-lying peaks vary from nucleus to nucleus.

We will attempt to estimate the contributions to nuclear polarization of these systematic features of the excitation spectrum, the GD resonance and the high-lying excitations responsible for the QE peak, keeping only the Coulomb part of the interaction. The transverse contribution will be shown to be much smaller. A calculation of this type, separating out the GD contribution, has been quite successful in muon capture.[20]

A great simplification may be effected in our estimate of the energy shift by employing closure approximation for the muon: taking an average muon excitation energy outside the sum on intermediate muon states and observing that $\sum_\nu |\nu\rangle\langle\nu| = 1$. This also aids in demonstrating the relative importance of Coulomb and transverse terms.

It is convenient to work with the muon operators explicitly in coordinate representation. Since the muon is a spin-$\frac{1}{2}$ point particle

$$\hat{\rho}(\vec{x}) = \delta(\vec{x} - \vec{z})\ . \tag{2.29}$$

The current operator is, in general

$$\hat{\vec{j}}(\vec{x}) = \hat{\vec{j}}^{\,C}(\vec{x}) + \vec{\nabla}\times\hat{\vec{\mu}}(\vec{x})\ . \tag{2.30}$$

Here $\hat{\vec{j}}^{\,C}$ is the convection part, $\hat{\vec{j}}^{\,C} \approx m^{-1}\hat{\vec{p}}$ with $\hat{\vec{p}}$ the momentum density, and $\vec{\nabla}\times\hat{\vec{\mu}}$ is the (divergenceless) magnetization part with $\hat{\vec{\mu}}$ the magnetization density (divided by e), and in coordinate representation,

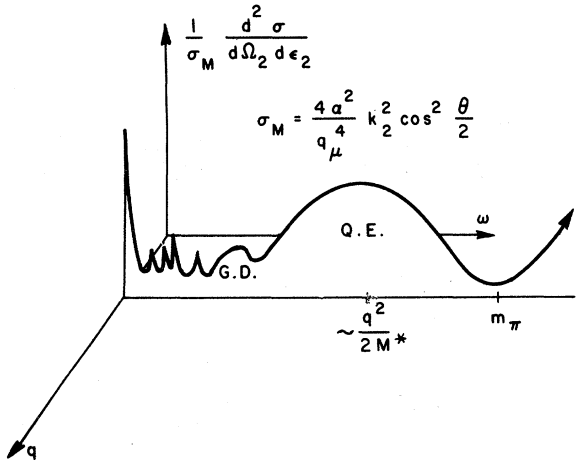

FIG. 2. Typical double-differential cross section for inelastic electron scattering, from deForest and Walecka, Ref. 18.

$$\hat{\vec{j}}^{c}(\vec{x}) = (im)^{-1}[\delta(\vec{x}-\vec{z})\vec{\nabla}_{z}]_{\text{sym}}, \tag{2.31}$$

$$\hat{\vec{\mu}}(\vec{x}) = \delta(\vec{x}-\vec{z})(2m)^{-1}\vec{\sigma}, \tag{2.32}$$

where $\vec{\sigma}$ is a Pauli matrix.

Then

$$\langle \nu|\hat{\rho}(\vec{q})|i\rangle = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\varphi_{\nu}^{*}(\vec{x})\varphi_{i}(\vec{x}), \tag{2.33}$$

and if we denote by $\vec{\vec{T}}$ the tensor which projects out transverse parts in a Fourier transform,

$$T_{ij} = \delta_{ij} - q_i q_j/|\vec{q}|^2, \tag{2.34}$$

we have

$$\langle \nu|\hat{\vec{j}}^{T}(\vec{q})|i\rangle = \int d^3x\, e^{i\vec{q}\cdot\vec{x}}[\varphi_{\nu}^{*}(2im)^{-1}\vec{\nabla}\varphi_{i} - \varphi_{i}(2im)^{-1}\vec{\nabla}\varphi_{\nu}^{*} + \vec{\nabla}\times\varphi_{\nu}^{*}(2m)^{-1}\vec{\sigma}\varphi_{i}]\cdot\vec{\vec{T}} \tag{2.35}$$

or, after integrating by parts,

$$\langle \nu|\hat{\vec{j}}^{T}(\vec{q})|i\rangle = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\varphi_{\nu}^{*}[(im)^{-1}\vec{\nabla} + i\vec{q}\times(2m)^{-1}\vec{\sigma}]\varphi_{i}\cdot\vec{\vec{T}}. \tag{2.36}$$

Then, after using closure in the form $\sum_{\nu}\varphi_{\nu}^{*}(\vec{x}')\varphi_{\nu}(\vec{x}) = \delta(\vec{x}-\vec{x}')$, we have

$$\sum_{\nu}\langle \nu|\rho(\vec{q}')|i\rangle^{*}\langle \nu|\rho(\vec{q})|i\rangle = \int d^3x\, e^{i(\vec{q}'-\vec{q})\cdot\vec{x}}\varphi_{i}^{*}\varphi_{i}. \tag{2.37}$$

The cross term is

$$\sum_{\nu}\langle \nu|\rho(\vec{q}')|i\rangle^{*}\langle \nu|\hat{\vec{j}}^{T}(\vec{q})|i\rangle = \int d^3x\, e^{i(\vec{q}'-\vec{q})\cdot\vec{x}}\varphi_{i}^{*}[(im)^{-1}\vec{\nabla} + i\vec{q}\times(2m)^{-1}\vec{\sigma}]\varphi_{i}\cdot\vec{\vec{T}} \tag{2.38}$$

and the transverse-transverse term

$$\sum_{\nu}\langle \nu|\hat{\vec{j}}^{T}(\vec{q}')|i\rangle^{*}\langle \nu|\hat{\vec{j}}^{T}(\vec{q})|i\rangle = \int d^3x\, e^{i(\vec{q}'-\vec{q})\cdot\vec{x}}$$

$$\vec{\vec{T}}'\cdot\{-(im)^{-1}\vec{\nabla}\varphi_{i}^{*} - i\vec{q}'\times\varphi_{i}^{*}(2m)^{-1}\vec{\sigma}\}\{(im)^{-1}\vec{\nabla}\varphi_{i} + i\vec{q}\times(2m)^{-1}\vec{\sigma}\varphi_{i}\}\cdot\vec{\vec{T}}. \tag{2.39}$$

We are to average over fine and hyperfine structure by taking

$2^{-1}(2l+1)^{-1}\sum_{jm}$ which is the same as $(2l+1)^{-1}\sum_{m_l}2^{-1}\sum_{m_s}$ nonrelativistically.

Any term linear in $\vec{\sigma}$ will go to zero because the trace of $\vec{\sigma}$ is zero.

Then, dropping the $\vec{\vec{T}}$'s and estimating the magnitude of the tensor $\vec{j}\,\vec{j}$ by taking its trace, the three terms (2.37)–(2.39) become

$$\langle i|\hat{\rho}\hat{\rho}|i\rangle = \int d^3x\, e^{i(\vec{q}'-\vec{q})\cdot\vec{x}}\varphi_{i}^{*}\varphi_{i}, \quad \langle i|\hat{\rho}\hat{j}^{T}|i\rangle = (-i/m)\int d^3x\, e^{i(\vec{q}'-\vec{q})\cdot\vec{x}}\partial_{x}^{-1}\varphi_{i}^{*}(\partial/\partial x)\varphi_{i},$$

$$\mathrm{Tr}\langle i|\hat{\vec{j}}\hat{\vec{j}}|i\rangle = m^{-2}\int d^3x\, e^{i(\vec{q}'-\vec{q})\cdot\vec{x}}(\vec{\nabla}\varphi_{i}^{*}\cdot\vec{\nabla}\varphi_{i} + \tfrac{1}{2}\vec{q}'\cdot\vec{q}\,\varphi_{i}^{*}\varphi_{i}). \tag{2.40}$$

Nuclear-current conservation $\vec{q}\cdot\vec{J}(\vec{q}) = \omega\rho(\vec{q})$ implies that

$$|\vec{J}^{c}(\vec{q})|_{fi} \sim (\Delta E/q)\rho(\vec{q})_{fi} \tag{2.41}$$

and a partial integration of the magnetization term gives

$$|\vec{\nabla}\times\vec{\mu}(\vec{q})|_{fi} \sim q\mu\rho(\vec{q})_{fi} \sim (q^2\lambda/2M\Delta E)|\vec{J}^{c}(\vec{q})|_{fi}. \tag{2.42}$$

Here we have used

$$\mu = \lambda/2M, \tag{2.43}$$

where $\lambda \approx 5$ is the isovector moment of the nucleon and M its mass.

If R is the nuclear radius and $\langle r^2\rangle$ the mean square radius of the muon orbit, we would expect the important range of momentum transfers to be between about $\min(R^{-1}, \langle r^2\rangle^{-1/2})$ and at most a few times R^{-1}. Outside these limits the matrix elements are small.

Rough values for large and small nuclei are

	a_B^{-1} (MeV)	$\langle r^2\rangle_{1s}^{-1/2}$ (MeV)	R^{-1} (MeV)	$q_{\min}$ (MeV)
$Z=8$	6	6	50	6
$Z\sim 80$	60	30	30	30

$m \sim 100$ MeV, $M \sim 1000$ MeV, and a typical ΔE is ~10–20 MeV.

Using these values, and estimating $|\vec{\nabla}\phi_i|$ by $\langle r^2\rangle^{-1/2}\phi_i$, we see that for the important ranges of q and q', the convection part of the nuclear current dominates and that $q'q/m^2$ is a fair bound on the muon $\vec{j}\vec{j}$ term. Then the three contributions to (2.23) are in the ratio

$$\text{Coulomb: cross: transverse} \approx 1: \quad \frac{2}{m}\langle r^2\rangle^{-1/2}\left\langle \frac{q\Delta E}{(q+\Delta E)(q+\langle\Delta\epsilon\rangle)}\right\rangle \left(\frac{\Delta E}{m}\right)^2 \left\langle \frac{qq'}{(q'+\Delta E)(q'+\langle\Delta\epsilon\rangle)(q+\Delta E)(q+\langle\Delta\epsilon\rangle)}\right.$$

$$\left.\times\,[(\Delta E+\langle\Delta\epsilon\rangle+q')(\Delta E+\langle\Delta\epsilon\rangle+q)+\Delta E\langle\Delta\epsilon\rangle\,(\Delta E+\langle\Delta\epsilon\rangle)/(q+q')]\right\rangle, \tag{2.44}$$

where $\langle\ldots\rangle$ denotes some average value for q and q' both in the range from $\langle r^2\rangle^{-1/2}$ to a few times R^{-1}. Both $\langle\ldots\rangle$ terms are less than or equal to 1 for any q and q' in this range and any reasonable ΔE and (ΔE). The quantity $2\langle r^2\rangle^{-1/2}/m \lesssim 0.6$, and is smaller for small nuclei, and for $\Delta E \lesssim 20$ MeV, $(\Delta E/m)^2 \lesssim 0.05$. $\langle\Delta\epsilon\rangle$ may be negative, but certainly $|\langle\Delta\epsilon\rangle| < Z\alpha\langle r^2\rangle^{-1/2}/2$, so that the singularities in q and q' are well outside the important ranges. The importance of the singularity is usually further reduced by cancellation in taking the principal values of the integrals, but there may be exceptional cases where the effect is large. Note that the existence of nearly degenerate nuclear and muon excitations, making $(\Delta E+\Delta\epsilon)^{-1}$ very large for the upper muon state, does not affect the ratio of Coulomb to transverse contributions much. It does make the Coulomb and cross terms large, and such excitations should really be included in the initial diagonalization of the Hamiltonian as has been done with the dynamic quadrupole effect, using the methods of nearly degenerate perturbation theory.

Our estimates have been crude, but we have tried to overestimate the transverse current at every point. Thus, despite the uncertainties of making approximations in oscillating integrals, and the fact that (2.44) is not really a rigorous upper bound, we feel that our estimates are realistic, and that the transverse current probably contributes at most half as much to the energy shift as the Coulomb interaction (although there may be exceptional cases, particularly if $\langle\epsilon\rangle$ is negative). The transverse interaction can probably be neglected in a calculation of this accuracy, and we will do so. This result is not surprising since in the interaction with the ground-state nuclear current the transverse interaction (magnetic hyperfine structure) is typically much smaller than the Coulomb (electric hyperfine structure).

Expressing muon closure for the Coulomb term in (2.23) in terms of reduced matrix elements

$$(2j+1)^{-1}\sum_\nu \langle\nu\|\hat{A}_K\|nlj\rangle^*\langle\nu\|\hat{B}_K\|nlj\rangle = \sum_\nu\sum_Q \langle\nu|\hat{A}_{KQ}|nlj\rangle^*\langle\nu|\hat{B}_{KQ}|nlj\rangle = \sum_Q\langle nlj|\hat{A}_{KQ}{}^\dagger\hat{B}_{KQ}|nlj\rangle, \tag{2.45}$$

and then using

$$\sum_Q Y_{KQ}{}^* Y_{KQ} = (2K+1)/4\pi$$

we have

$$(2j+1)^{-1}\sum_\nu \langle\nu\|\hat{m}_K(q')\|nlj\rangle^*\langle\nu\|\hat{m}_K(q)\|nlj\rangle = [(2K+1)/4\pi]\int d^3x\, j_K(q'x)j_K(qx)\langle nlj|\hat{\rho}(\vec{x})|nlj\rangle. \tag{2.46}$$

We can then write the energy shift in a particularly simple form by separating the muon closure energy out in a "closure factor" $\mathcal{K}$ as

$$\overline{\Delta E}_{nlj} = \sum_K \mathcal{K}_{Knlj}\langle nlj|\hat{V}_{\text{pol},K}|nlj\rangle, \tag{2.47}$$

where in coordinate representation

$$V_{\text{pol},K}(z) = -\alpha^2\frac{1}{2I_0+1}\sum_{N\neq 0}\frac{1}{E_N-E_0}4\pi\left|\frac{2}{\pi}\int_0^\infty dq\, j_K(qz)\langle N\|\hat{M}_K(q)\|0\rangle\right|^2$$

$$= -\alpha^2\frac{1}{2I_0+1}\sum_{N\neq 0}\frac{1}{E_N-E_0}\frac{4\pi}{(2K+1)^2}\left|\langle N\|\int d^3x\,\hat{\rho}_N(\vec{x})(r_<^K/r_>^{K+1})\vec{Y}_K(\Omega_x)\|0\rangle\right|^2. \tag{2.48}$$

Here $(r_<, r_>) = (x, z)$. $\mathcal{K}$ is chosen to give (2.47) the correct value so that it is just

$$\mathcal{K}_{K,nlj} = \frac{\sum_{N\neq 0}\sum_\nu (E_N-E_0+\epsilon_\nu-\epsilon_{nlj})^{-1}|\langle\nu|\langle N|\hat{H}_K'|0\rangle|nlj\rangle|^2}{\sum_{N\neq 0}\sum_\nu (E_N-E_0)^{-1}|\langle\nu|\langle N|\hat{H}_K'|0\rangle|nlj\rangle|^2} \equiv \frac{E}{E+\langle\epsilon\rangle_{K,nlj}}, \tag{2.49}$$

where $\hat{H}_K'$ is the Kth multipole component of the interaction. E is a typical nuclear excitation energy. $\langle\epsilon\rangle_{K,nlj}$ is the "average" muon excitation, and (2.49) serves to define the appropriate average. In the sum

$$\sum_\nu (E_N-E_0+\epsilon_\nu-\epsilon_{nlj})^{-1}|\langle\nu|\langle N|\hat{H}_K'|0\rangle|nlj\rangle|^2$$

the nuclear matrix element is just $v_K(z)\vec{Y}_K(\Omega_z)$ in coordinate representation for the moun. If we make a nonrelativistic reduction for the muon, separating spin from orbital angular momentum, the Coulomb interaction does not affect the muon spin. We may simply drop the spin and label states by nl rather than nlj, or alternatively use the reduction for a tensor operator operating only on one part $\vec{l}$ of a coupled scheme $\vec{l}+\vec{s}=\vec{j}$. The radial wave functions are complete (in z) for each value of l' permitting us to take angular matrix elements first and then use closure approximation sepatately for each l'. This allows us to define a different closure energy for each. The angular matrix elements are just

$$\langle l'\|\vec{Y}_K\|l\rangle = (-)^{l'}\left(\frac{(2l'+1)(2K+1)(2l+1)}{4\pi}\right)^{1/2}\begin{pmatrix} l' & K & l \\ 0 & 0 & 0\end{pmatrix} \tag{2.50}$$

so that each permitted value of l' contributes a fraction

$$(2l'+1)\begin{pmatrix} l' & K & l \\ 0 & 0 & 0 \end{pmatrix}^2$$

of $V_{\text{pol},K}$ and (2.49) is replaced by

$$\mathcal{K}_{K,nlj} \to \sum_{l'} (2l'+1)\begin{pmatrix} l' & K & l \\ 0 & 0 & 0 \end{pmatrix}^2 \mathcal{K}_{K,nlj\to l'}, \tag{2.51}$$

where

$$\mathcal{K}_{K,nlj\to l'} = \frac{\sum_{N\neq 0}\sum_{\alpha}(E_N - E_0 + \epsilon_{\alpha l'} - \epsilon_{nlj})^{-1}|\langle \alpha l'|\langle N|\hat{H}_K|0\rangle|nlj\rangle|^2}{\sum_{N\neq 0}\sum_{\nu}(E_N - E_0)^{-1}|\langle \alpha l'|\langle N|\hat{H}_K{}'|0\rangle|nlj\rangle|^2} \equiv \frac{E}{E + \langle\epsilon\rangle_{K,nlj\to l'}}. \tag{2.52}$$

$\langle\epsilon\rangle_{K,nlj\to l'}$ is now the "average" muon excitation energy for multipole K to excited states with angular momentum l'. Averaging (2.51), (2.52) over l' restores (2.49), but we will find later that the variation of $\mathcal{K}_{K,nlj\to l'}(\langle\epsilon\rangle_{K,nlj\to l'})$ with l' is often quite significant, particularly when there is an available intermediate muon state of lower energy than nlj (in these cases $\langle\epsilon\rangle_{nlj\to l'}$ may be negative).

$\sum_K \mathcal{K}_{K,nlj} V_{\text{pol},K}$ can be interpreted as the change in the spherically symmetric part of the nuclear Coulomb potential caused by the distortion (polarization) of the nucleus by the Coulomb field of a muon in the state nlj.

The second form of (2.48) is exactly what one would derive more directly by treating the instantaneous Coulomb interaction

$$\hat{H}_c{}' = -\alpha\int d^3x\int d^3y[\hat{\rho}_N(\vec{x}) - \langle 0|\hat{\rho}_N(\vec{x})|0\rangle]\hat{\rho}_\mu(\vec{y})/|\vec{x}-\vec{y}| \tag{2.53}$$

in simple second-order perturbation theory, and ignoring the possibility of hyperfine structure (the reduced matrix elements for the nucleus would come from a simple average over its possible orientations).

3. DIPOLE EXCITATIONS IN GOLDHABER-TELLER MODEL

The first contribution to be considered is that of the GD resonance. Rather than attempt to use detailed experimental cross sections, we will use the familiar Goldhaber-Teller (GT) model. This model has the virtue of being extremely simple while still providing a reasonable fit to experiment.[21]

We assume for this model that the protons move together, and oscillate against the neutrons (also moving together) with a harmonic restoring force. The Hamiltonian for the relative motion is

$$\hat{H} = (2\mu)^{-1}\hat{\vec{P}}^2 + \tfrac{1}{2}\mu\omega^2\hat{\vec{R}}^2, \tag{3.1}$$

where $\hat{\vec{R}}$ and $\hat{\vec{P}}$ are the relative displacement and momentum operators, $\mu = (NZ/A)M$ the reduced mass, and ω the energy.

We quantize the system in the usual way, using the creation and destruction operators $\hat{a}_m{}^\dagger$ and $\hat{a}_m (m = 0, \pm 1)$, where

$$[\hat{a}_m, \hat{a}_{m'}{}^\dagger] = \delta_{mm'}, \quad \hat{a}_m|0\rangle = 0 \tag{3.2}$$

and $$\hat{R}_m = (2\mu\omega)^{-1/2}[\hat{a}_m{}^\dagger + (-)^m\hat{a}_{-m}], \quad \hat{P}_m = i^{-1}(\mu\omega/2)^{1/2}[\hat{a}_m - (-)^m\hat{a}_{-m}{}^\dagger]. \tag{3.3}$$

The protons are displaced by $(N/A)\vec{R}$ relative to the nuclear center of mass so that

$$\hat{\rho}_N(\vec{x}) = \rho_0(|\vec{x} - (N/A)\hat{\vec{R}}|). \tag{3.4}$$

The GT state corresponds to the lowest dipole excitation ($1P$) of the system $|\text{GT}, m\rangle = \hat{a}_m{}^\dagger|0\rangle$, and is coupled to the ground state only by the dipole operator. Then

$$\hat{M}_{1M}(q) = \int d^3x j_1(qx)Y_{1M}(\Omega_x)\rho_0(|\vec{x} - \frac{N}{A}\hat{\vec{R}}|) = -\frac{N}{A}\hat{\vec{R}}\cdot\int d^3x j_1(qx)Y_{1M}(\Omega_x)x^{-1}\vec{x}\frac{\partial}{\partial x}\rho_0(x) \tag{3.5}$$

keeping only the lowest order in $\hat{\vec{R}}$, which is assumed to be small. We can do the angular integral at once, and then integrate by parts to find

$$\hat{M}_{1M}(q) = (N/A)q(\tfrac{1}{3}\pi)^{1/2}\hat{R}_M\int_0^\infty dx\, x^2\rho_0(x)j_0(qx). \tag{3.6}$$

Then the reduced matrix element is

$$\langle \text{GT}\|\hat{M}_1(q)\|0\rangle = 3^{1/2}\langle \text{GT}\, M=0|\hat{M}_{10}(q)|0\rangle = (N/A)q(4\pi/2\mu\omega)^{1/2}\int_0^\infty dx\, x^2\rho_0(x)j_0(qx). \tag{3.7}$$

We substitute this into the first part of (2.48) and do the integral on q, observing that $j_1(qz) = -q^{-1}(d/dz)j_0(qz)$, and

$$(2/\pi)\int_0^\infty dq\, j_0(qz)j_0(qz) = r_>^{-1}, \quad (r_<, r_>) = (x, z). \tag{3.8}$$

The x integral is then simply

$$-\int_0^\infty dx\, x^2\rho_0(x)r_>^{-1} = (4\pi\alpha)^{-1}V_c(z) \tag{3.9}$$

and $-V_{\text{pol, GT}}(z) = (2AME_{\text{GD}}{}^2)^{-1}\frac{N}{Z}\left|\frac{\partial}{\partial z}V_C(z)\right|^2$. (3.10)

This has been derived as if $I_0 = 0$, but is true in general if we average over the orientation of the ground state. The crucial observation is that $\hat{\mathbf{R}}$, and therefore $\hat{\rho}(\vec{\mathbf{x}})$, operates only on the collective part of the motion, and does not change the relative motion responsible for I_0.

The intuitive interpretation of (3.10) is simple. The transition charge is the derivative of the ground-state charge (just a factor of q in the Fourier transform), so that the transition potential is the derivative of the ground-state potential V_C. The nuclear-polarization potential can be easily evaluated in this model for any ground-state charge distribution.

4. EXCITATIONS IN SHELL MODEL

The second systematic feature of inelastic-scattering cross sections is the quasielastic peak. Here the interaction approximates scattering from individual quasifree nucleons,[22,23] and leads to intermediate nuclear states where one nucleon has been raised to a higher energy level. For the polarization corrections we expect the same states to be important. At the higher excitation energies, one nucleon is struck hard enough that the second interaction must be with the same nucleon to get a reasonable overlap with the ground-state.

We would therefore expect a simple independent-particle shell model to give a reasonable description of the relevant states. Such a model has been found to work reasonably well for electron scattering, with even the simple harmonic-oscillator model giving a fairly good over-all fit[22] although it fails in details. This is also the model used by Foldy and Walecka[20] for nondipole excitations in muon capture.

We will restrict our attention to nuclei with closed proton shells for which the single-particle levels are filled up to some Fermi energy. Excited states for the polarization problem correspond to excitation of a single particle out of the Fermi sphere ($\hat{J}$ is taken as a sum of single-particle operators, and cannot excite more than one particle at a time), and will be denoted by $|N_2L_2M_2(N_1L_1M_1)\rangle$ for a particle excited from $N_1L_1M_1$ to $N_2L_2M_2$.

The Coulomb transition potential of the nucleus is

$$V_T(\vec{\mathbf{z}}) = \langle N_2L_2M_2(N_1L_1M_1)|-\alpha\int d^3x[\hat{\rho}_N(\vec{\mathbf{x}})/|\vec{\mathbf{x}}-\vec{\mathbf{z}}|]|0\rangle = \sum_{KQ} v_{KQ}(z)Y_{KQ}(\Omega_z), \tag{4.1}$$

where use of $\rho(\vec{\mathbf{x}}) = \sum_{i=1}^{Z}\delta(\vec{\mathbf{x}}-\vec{\mathbf{x}}_i)$ gives

$$v_{KQ}(z) = -\frac{4\pi\alpha}{2K+1}(-)^{M_2}\left(\frac{(2L_1+1)(2L_2+1)(2K+1)}{4\pi}\right)^{1/2}\begin{pmatrix}L_1 & K & L_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L_1 & K & L_2\\ M_1 & Q & -M_2\end{pmatrix}$$
$$\times\int_0^\infty dx\,x^2 R_{N_2L_2}{}^*(x)(r_<{}^K/r_>{}^{K+1})R_{N_1L_1}(x). \tag{4.2}$$

Here R_{NL} is the radial part of the single-particle wave function and $(r_>, r_<) = (x, z)$. The contributions to V_{pol} of the nuclear excitations degenerate in M_1 and M_2 must be summed with a factor of 2 for spins. A final summation over possible N_1, L_1, N_2, and L_2 gives

$$V_{\text{pol}}(z) = \sum_K\sum_{N_2L_2}\sum_{N_1L_1}\frac{-2\alpha^2}{E_2-E_1}\frac{CG(L_1, L_2, K)}{(2K+1)^2}\left|I_{N_2L_2, N_1L_1; K}(z)\right|^2, \tag{4.3}$$

where $CG(L_1, L_2, L_3) \equiv (2L_1+1)(2L_2+1)(2L_3+1)\begin{pmatrix}L_1 & L_2 & L_3\\ 0 & 0 & 0\end{pmatrix}^2$ (4.4)

and $I_{N_2L_2, N_1L_1; K}(z) \equiv \int_0^\infty dx\,x^2 R^*_{N_2L_2}(x)(r_<{}^K/r_>{}^{K+1})R_{N_1L_1}(x)$. (4.5)

For simplicity we will use a simple-harmonic-oscillator model with oscillator parameter b defined by

$$H = (\hbar^2/2m)(-\vec{\nabla}^2 + b^{-4}\vec{\mathbf{r}}^2) \tag{4.6}$$

for which the level spacing is

$$\omega_0 = (Mb^2)^{-1}. \tag{4.7}$$

The eigenfunctions are well known, and are

$$\psi_{NLM}(\vec{\mathbf{r}}) = R_{NL}(r)Y_{LM}(\Omega_r),\quad R_{NL}(r) = C_{NL}(r/b)^L e^{-r^2/2b^2}L_{N-1}{}^{L+\frac{1}{2}}(r^2/b^2),$$
$$N = 1, 2, 3, \ldots,\quad L = 0, 1, 2, \ldots, \tag{4.8}$$

with energies $E_{NL} = (2N+L-1)\omega_0$ (4.9)

referred to the lowest (1s) level. The $L_n{}^\alpha$ are generalized Laguerre polynomials, and are explicitly

$$L_n{}^\alpha(x) = \sum_{m=0}^{n=1}(-)^m\binom{n+\alpha}{n-m}\frac{x^m}{m!}. \tag{4.10}$$

C_{NL} is a normalizing constant,

$$C_{NL} = [2^{N+L+1} N!/(2N+2L-1)!!]^{1/2} (\pi^{1/2} b^3)^{-1/2} \equiv C_{NL}' (\pi^{1/2} b^3)^{-1/2} . \tag{4.11}$$

The product of two Laguerre polynomials is a polynomial whose coefficients we will denote by a,

$$L_{N'-1}^{L'+\frac{1}{2}}(y) L_{N-1}^{L+\frac{1}{2}}(y) = \sum_{S=0}^{N'+N-2} a_{N'L', NL; S} y^S . \tag{4.12}$$

If we let $z = \xi b$, we have that

$$I_{N'L', NL; K}(z) = \frac{C_{N'L'}' C_{NL}'}{2\pi^{1/2} b} \sum_{S=0}^{N'+N-2} a_{N'L', NL; S} \left[\frac{1}{\xi^{K+1}} \gamma\left(\frac{L'+L+K+2S+3}{2}, \xi^2 \right) + \xi^K \Gamma\left(\frac{L'+L-K+2S+3}{2}, \xi^2 \right) \right] . \tag{4.13}$$

In this equation, γ and Γ are the incomplete gamma function and its conjugate defined by

$$\gamma(\nu, x) = \Gamma(\nu) - \Gamma(\nu, x) = \int_0^x dy\, e^{-y} y^{\nu-1} . \tag{4.14}$$

We note that $L'+L+K$ must be even [otherwise $CG(L', L, K) = 0$], so that the first argument of the Γ is always integer and that of the γ always half-integer. These incomplete gamma functions may be computed from the recursion relation

$$\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x} \tag{4.15}$$

starting from

$$\gamma(\tfrac{1}{2}, \xi^2) = \pi^{1/2} \operatorname{erf}(\xi), \quad \Gamma(1, \xi^2) = e^{-\xi^2}, \tag{4.16}$$

where $\operatorname{erf}(x)$ is the error function. For small x the recursion involves repeated subtraction of nearly equal numbers with attendant loss of accuracy. This may be avoided by starting with an approximation for large a, such as a few terms in the expansion

$$\gamma(a, x) = x^a e^{-x} \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+n+1)} x^n \tag{4.17}$$

and recursing downward.

This model errs in that the nucleon coordinates are referred to a fixed center of potential rather than to the center of mass. This gives rise to spurious states in which the center of mass is itself excited. These spurious states have been included in Eq. (4.3) and should be subtracted out. Their contributions may be evaluated as shown in Appendix A, and we find that only one spurious state is important; the rest in aggregate produce a correction of only a percent or two.

The significant state has the center of mass in a $1P$ state with internal coordinates of the ground state. The charge distribution for this state is like the Goldhaber-Teller state, but the reduced mass is AM rather than $(NZ/A)M$ and the charge is displaced by the full oscillator coordinate rather than N/A of it. Thus we must subtract the spurious dipole contribution

$$V_{\text{pol, sp}}(z) = \frac{Z}{N} V_{\text{pol, GT}}(z) = -(2AM\omega_0^2)^{-1} \left| \frac{d}{dz} V_C(z) \right|^2 \tag{4.18}$$

which will be almost half of the uncorrected dipole distortion. (This is rigorously derived in Appendix A.)

The contributions to the various multipole terms in $V_{\text{pol}}(z)$ from excitations of different energies are shown in Fig. 3 with the closure factor [defined by (2.49), (2.51), and (2.52)]

$$\mathcal{K} = E/(E + \langle \epsilon \rangle) \tag{4.19}$$

containing the muon closure energy separated out. They have been evaluated using a computer program based on the method outlined above, for a nucleus with $Z = 20$ (calcium), $A = 43$, and $b = 2.055$ F corresponding to a uniform sphere radius of 4.60 F. We see that convergence as a function of excitation energy $n\omega_0$ is quite rapid, particularly outside the nucleus.

5. MUON WAVE FUNCTIONS

In the nonrelativistic approximation, the muon wave functions are solutions of the Schrödinger equation

$$[-(2m)^{-1} \vec{\nabla}^2 + V_C(z) - \epsilon] \psi(\vec{z}) = 0 . \tag{5.1}$$

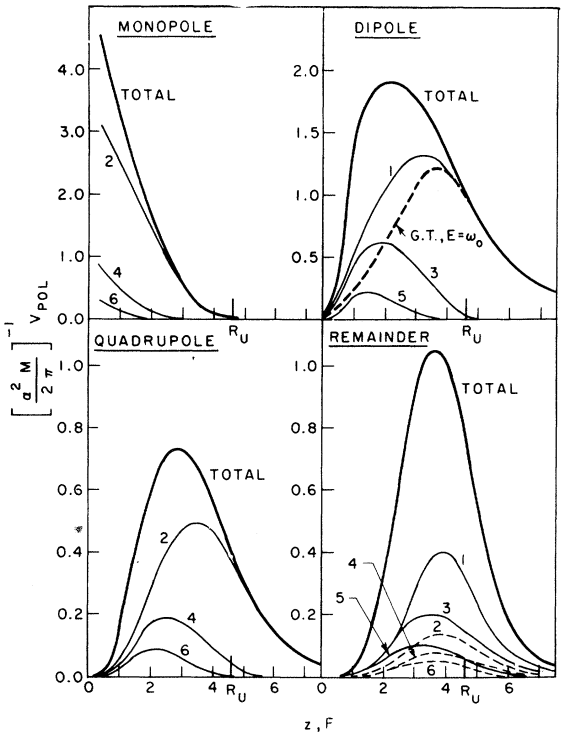

FIG. 3. Contributions to the nuclear-polarization potential in shell model labeled by n, and $n\omega_0$ is the excitation energy. "Total" is the sum through $n=10$.

If we approximate $V_C(z)$ by the Coulomb potential of a uniform spherical charge distribution of radius R (R will be chosen to give the right value for $\langle r^2\rangle$), we have

$$\begin{aligned} V_C(z) &= -(Z\alpha/2R)[3-(z/R)^2], \quad z \le R, \\ &= -(Z\alpha/z), \qquad\qquad z \ge R. \end{aligned} \tag{5.2}$$

This potential is just a harmonic oscillator inside the nucleus and a point Coulomb field outside. The solutions may be given in closed form for each region in terms of confluent hypergeometric functions as

$$\begin{aligned} &\psi_{nlm}(\vec{z}) = R_{nl}(z)Y_{lm}(\Omega_z), \\ &R_{nl}(z) = z^l e^{-t^{1/2}x^2/2} M(\tfrac{1}{4}[2l+3+t^{1/2}(t\gamma^{-2}-3)],\ l+\tfrac{3}{2},\ t^{1/2}x^2), \quad z \le R \\ &\qquad = z^l e^{-tx/\gamma} U(l+1-\gamma, 2l+2, 2tx/\gamma), \qquad\qquad z \ge R, \end{aligned} \tag{5.3}$$

where $t = R/a_B \equiv m\alpha ZR$ (a_B is the Bohr radius), $x = z/R$, and γ is the "effective quantum number" defined by

$$\epsilon = (Z\alpha)^2 m/2\gamma^2 . \tag{5.4}$$

Note that γ reduces to n in the hydrogenic (point Coulomb, $t \to 0$) limit, and that we can use this n to label the solutions.

Inside the nucleus, R_{nl} can be easily computed from a few terms of the series

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{5.5}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. However, outside the nucleus the converging series for U are extremely inconvenient to work with. (U has, in general, a singularity at the origin.) We can use the asymptotic (semiconvergent) expansion

$$U(a,b,z) \sim z^{-a}\left(\sum_{n=0}^{S-1} \frac{(a)_n (1+a-b)_n}{n!}(-z)^{-n} + O(|z|^{-S})\right), \quad \text{as } z \to \infty, \tag{5.6}$$

for large z, where the error term is approximately equal to the first term neglected. We then simply numerically integrate the radial equation from a point where this series is sufficiently accurate back in to

the nuclear surface. The eigenvalues γ are found by requiring that

$$\frac{d}{dz}[\ln R_{nl}(R_+)] = \frac{d}{dz}[\ln R_{nl}(R_-)]. \tag{5.7}$$

This is essentially the method used by Pustovalov[24] with the exception of the representation inside the nucleus. We have used a computer to find solutions to (5.7) for the $n = 1$, 2, 3 states of the closed-proton-shell nuclei listed in Table III. The values found for γ agree with Pustovalov's to four or five places.

As a check on relativistic effects and the accuracy of our other approximations, the energies for $Z = 70$ were compared with those computed by Barrett[25] using the Dirac equation for a Fermi shape nucleus and including vacuum-polarization effects. In no case did the difference exceed 3% (the center of gravity was used for the fine structure multiplets of p and d states). For a second check, the overlaps of the $1s$ wave functions with the nucleus were computed and compared with the relativistic results of Sens.[26] Here agreement was to 5%. These nonrelativistic wave functions should therefore be sufficiently accurate for this calculation, and have been used in the results which follow.

For later reference, the energies of the $n = 1$, 2, and 3 bound states are plotted in Fig. 4 as a function of Z.

6. MATRIX ELEMENTS OF THE POLARIZATION POTENTIALS

The polarization potentials and energy shifts have been evaluated for the $1s$, $2s$, $2p$, and $3d$ muon states for nuclei with closed (harmonic-oscillator) proton shells; $Z = 8$, 20, 40, and 70. As the results are rather approximate, and we may wish to interpolate to estimate the core contributions for nonclosed-shell nuclei, typical parameters (rather than those for any particular nuclei) have been used. In particular, the charge distribution is taken to match (in $\langle r^2 \rangle$) a Fermi distribution

$$\rho(r) = \rho_0\{1 + \exp[(x - c)/a]\}^{-1}, \tag{6.1}$$

where the parameters

$$c = 1.12A^{1/3}\,\mathrm{F}, \quad a = 0.50\ \mathrm{F} \tag{6.2}$$

have been chosen to give a good over-all fit to experimental values. We ignore the fact that $Z = 70$ lies in a range of strongly deformed nuclei. Typical n-p asymmetries and GD energies[27] have been used, and the nucleon effective mass taken equal to its free mass. The various parameters are given in Table II.

It is well known that the harmonic-oscillator shell model gives a poor description of dipole excitations – the dipole strength is at too low an energy, mostly at ω_0 rather than $E_{GD} \approx 2\omega_0$ (the GT state exhausts the dipole sum rule). This gives roughly a factor of 4 difference in ΔE. If we use the oscillator-model form factor but put the dipole strength at E_{GD}, Fig. 3 shows that the polarization potential matches the GT form outside the nucleus, but is significantly larger inside. This is not surprising – the GT model is essentially a long wavelength model and is expected to err at short distances. The difference in computed dipole-energy shifts (with $\omega_0 = E_{GD}$) is less than about 20% with the exception of 35% for the $2s$ state of the largest nucleus considered, Yb; it is much less for small Z or large l (where the part of V_{pol} inside the nucleus is relatively less important).

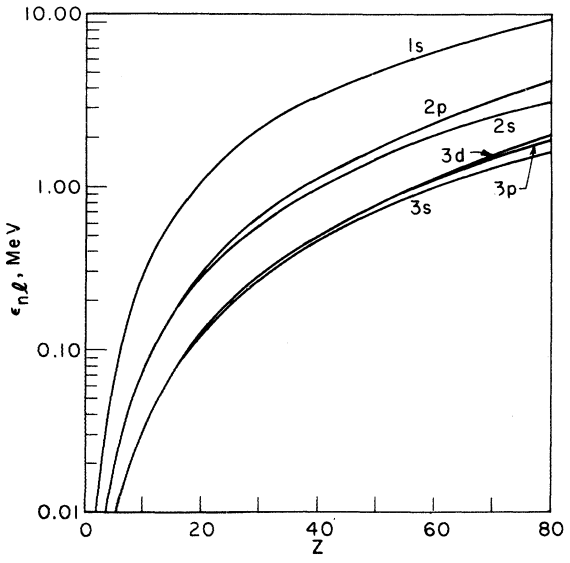

FIG. 4. Energy levels of muonic atoms.

TABLE II. Properties of typical nuclei. The entries are a_B, the Bohr radius, $\langle r^2\rangle_N$, the mean-square radius of the nuclear charge distribution, and R_u, the equivalent uniform radius. N_F is the number of filled harmonic-oscillator proton shells, b the oscillator parameter, and ω_0 the oscillator energy ($M^*=M$). E_{GD} is the giant dipole energy. These are all typical average values fitted to the general trend of stable nuclei, not the experimental values for the specific nuclei listed.

Z	A	Element	a_B (F)	$\langle r^2\rangle_N$ (F^2)	R_u (F^2)	N_F	b (F)	ω_0, shell (MeV)	E_{GD} (MeV)
8	16	O	32.0	8.225	3.70	2	1.912	11.4	21
20	43	Ca	12.7	12.674	4.60	3	2.055	9.9	19.7
40	91	Zr	6.39	18.695	5.58	4	2.233	8.4	16.6
70	173	Yb	3.65	26.742	6.68	5	2.438	7.0	14.7

We have used the GT model for dipole transitions and the harmonic-oscillator shell model for all others, summing through $\Delta E = 10\omega_0$, and computed separately the energy shifts due to monopole (M), dipole (D), quadrupole (Q), and remainder, $K \geq 3$, (R) transitions. This exactly parallels the approach of Foldy and Walecka[20] in analysis of muon capture. The muon closure energy has again been separated out as a closure factor as in Fig. 3 [see (2.49), (2.51), (2.52)]

$$\mathcal{K} = E/(E + \langle\epsilon\rangle) \,. \tag{6.3}$$

Here E is to be taken as E_{GD} for D, as about ω_0 for other odd multipoles (part of R), and as about $2\omega_0$ for even multipoles (M, Q, and the rest of R) since most of the shell-model contribution comes at these energies. The results $\mathcal{K} - 1\overline{\Delta E}_{K,nl}{}^C$, are given in Table III. They are not yet directly comparable; the muon closure factors K vary considerably.

7. MUON CLOSURE FACTOR

To estimate the muon closure energies and therefore $\mathcal{K}$'s, we recall that $\mathcal{K}$ is defined by (2.49)

$$\mathcal{K}_{K,nl} \equiv \sum_N \sum_\nu (E_N - E_0 - \epsilon_\nu - \epsilon_{nl})^{-1} |\langle\nu|\langle N|\hat{H}_K{}'|0\rangle|nl\rangle|^2 \Big/ \sum_N \sum_\nu (E_N - E_0)^{-1} |\langle\nu|\langle N|\hat{H}_K{}'|0\rangle|nl\rangle|^2 \,, \tag{7.1}$$

where $\hat{H}'_K$ is the Kth multipole component of the interaction. We may approximate this by

$$\mathcal{K}_{K,nl} \approx \sum_\nu E(E + \epsilon_\nu - \epsilon_{nl})^{-1} |\langle\nu|[\hat{V}_{\text{pol},K}]^{1/2}|nl\rangle_{\text{rad}}|^2 \Big/ \sum_\nu |\langle\nu|(\hat{V}_{\text{pol},K})^{1/2}|nl\rangle_{\text{rad}}|^2 \,, \tag{7.2}$$

TABLE III. Matrix elements of the polarization potentials in eV. M, D, Q, and R refer to monopole, dipole, and quadrupole transitions, and to the remainder (octupole and higher).

		M	D	Q	R
$Z=8$	$1s$	4.65	9.01	9.59	9.41
	$2s$	0.581	1.11	1.20	1.18
	$2p$	0.000 2	0.014 1	0.002 8	0.001 8
	$3d$	$\sim 10^{-9}$	0.000 206	0.000 001	$\sim 10^{-7}$
$Z=20$	$1s$	120	145	215	294
	$2s$	15.7	17.7	25.6	37.4
	$2p$	0.08	1.36	0.72	0.75
	$3d$	$\sim 10^{-5}$	0.024 6	0.001 2	0.000 4
$Z=40$	$1s$	906	749	1330	2140
	$2s$	133	90.6	173	270
	$2p$	5.6	40.3	36.1	51.6
	$3d$	0.004 91	1.13	0.24	.17
$Z=70$	$1s$	3150	1680	3640	6380
	$2s$	565	204	489	809
	$2p$	119	387	505	906
	$3d$	0.6	24.2	13.4	16.5

where E is the energy of the most important nuclear excitations and "rad" means the radial part of the matrix element. This is exact for the D contribution (only one state $N=\mathrm{GD}$ contributes) and not unreasonable for the others. If we keep track of the angular momentum of the states ν as in (2.51) and (2.52) this becomes

$$\mathcal{K}_{K,nl} \equiv \sum_{l'} (2l'+1)\begin{pmatrix} l' & K & l \\ 0 & 0 & 0 \end{pmatrix}^2 \mathcal{K}_{K,nl\to l'}$$

$$\mathcal{K}_{K,nl\to l'} \approx \sum_{\alpha} E(E+\epsilon_{\alpha l'}-\epsilon_{nl})^{-1} \left|\langle \alpha l' | (\hat{V}_{\mathrm{pol},K})^{1/2} | nl\rangle_{\mathrm{rad}}\right|^2 / \sum_{\alpha} \left|\langle \alpha l' | (\hat{V}_{\mathrm{pol},K})^{1/2} | nl\rangle_{\mathrm{rad}}\right|^2 . \tag{7.3}$$

Here α must run over both bound and continuum states. High bound states are not important since they are small near the origin. We have roughly evaluated the sums in (7.3) numerically using our numerical wave functions for low bound states and spherical components of plane waves (spherical Bessel functions) for continuum states. (This is really too many states, but it turns out that usually either most of the sum comes from low bound states, or most of it comes from relatively high-energy plane waves so that the duplication is unimportant.)

For plane waves, the radial wave functions are $(2/\pi)^{1/2} k j_l(kx)$ with the continuum normalization

$$\langle k'l' | kl\rangle = \delta_{ll'}\delta(k-k') . \tag{7.4}$$

$$\text{Then } \langle kl' | \hat{V}_K | nl\rangle_{\mathrm{rad}} = (2/\pi)^{1/2} \int_0^\infty dr\, r^2 k j_{l'}(kr) V_K(r) R_{nl}(r) , \tag{7.5}$$

where R_{nl} is the radial part of the bound-state wave function, and

$$\mathcal{K}_{K,nl\to l'} = \int_0^\infty dk\, E[E+(k^2/2m)-\epsilon_{nl}]^{-1} \left|\langle kl' | \hat{V}_K | nl\rangle\right|^2 / \int_0^\infty dk \left|\langle kl' | \hat{V}_K | nl\rangle\right|^2 . \tag{7.6}$$

For bound intermediate states $|n'l'\rangle$, the matrix elements are explicitly evaluated and the sum on α in (7.3) taken over low bound states.

As an example, we will treat muon closure for the dipole contribution in detail. For the $1s$ and $2s$ states the only intermediate states are p states. Less than 10% of the sum comes from bound states, with the exception of about 30% for the $1s$ state of Yb, the highest Z nucleus discussed, so that we should be able to do the sum with plane waves using (7.6). $\mathcal{K}$ is evaluated twice, first assuming that (7.6) applies to all intermediate states, and then assuming that it applies only to continuum states while bound-state contributions are treated separately. $E=E_{\mathrm{GD}}$ is taken from Table II and ϵ_{nl} from Fig. 4. In no case do the results differ by more than about 25%, in most cases by less than 5%. A simple average is taken, and the results entered in Table IV.

Evaluating the factors

$$(2l'+1)\begin{pmatrix} l' & K & l \\ 0 & 0 & 0 \end{pmatrix}^2$$

in (7.3), we see that the $2p$ state is coupled $\frac{1}{3}$ to s states and $\frac{2}{3}$ to d states. We find that 80 to 90% of the coupling to s states is to the bound $1s$ state, while almost all of the rest and of the coupling to d states is to the continuum. Then $\mathcal{K}_{D,\,2p} = \frac{1}{3}\mathcal{K}_{D,\,2p\to s} + (\frac{2}{3})\mathcal{K}_{D,\,2p\to d}$, and we compute $\mathcal{K}_{D,\,2p\to s}$ using bound states (with a plane wave estimate of the 10 to 20% of the sum which comes from the continuum), and $\mathcal{K}_{D,\,2p\to d}$ using plane waves. The values obtained are given in Table IV. We note that $\mathcal{K}_{D,\,2p\to s}$ is greater than one for $Z \geq 20$. This is because the sum on intermediate states is dominated by the $1s$ state, a downward transition from the $2p$ state.

Closure factors for the $3d$ state are computed exactly as for the $2p$. The coupling is $\frac{2}{5}$ to p states (95 to 99% to the bound $2p$ state) and $\frac{3}{5}$ to d states (almost all continuum). Numerical values for these partial- and total-closure factors are also given in Table IV.

The remaining total-closure factors are computed in the same manner as $\mathcal{K}_D$ and entered in Table V. Most of the contributions come from the continuum except for $\mathcal{K}_{Q,\,3d\to s}$ (85 to 95% to bound states), $\mathcal{K}_{R,\,3d\to p}$ (20% to bound states for $Z=40$, 50% for $Z=70$), and $\mathcal{K}_{M,\,1s\to s}$, $\mathcal{K}_{M,\,2s\to s}$ (both 10% to bound states for $Z=40$ and 35% for $Z=70$). For $\mathcal{K}_R$ we note that the largest contributions to $V_{\mathrm{pol},R}$ are from $E_N=\omega_0$ and $E_N=2\omega_0$, so we would expect E in (7.2), (7.3), and (7.6) to satisfy $\omega_0 \lesssim E \lesssim 2\omega_0$. This gives a range in computed $\mathcal{K}$'s as great as a factor of two. However, $\mathcal{K}_R V_{\mathrm{pol},R}$ contributes at most 10 to 20% to the total energy shift so that our final results are not very sensitive to $\mathcal{K}_R$. The values in Table V correspond to $E \approx 1.5\omega_0$. A number of blanks have been left for factors which will not be needed because the corresponding matrix elements in Table III are small.

Our computed $\mathcal{K}$'s are certainly not accurate to the full two or three places given. The initial approximation (7.2) of replacing individual nuclear transition potentials by $(V_{\mathrm{pol},K})^{1/2}$ is probably reasonably good: closure energies are not very sensitive to the detailed shape of the potentials, and $(V_{\mathrm{pol},K})^{1/2}$ is very like the individual potentials (see Fig. 3). The muon closure energy is affected to some extent by the value of E, the nuclear excitation energy, used in (7.2), (7.3), and (7.6), and we might go so far as to use different values for each different nuclear excitation energy. This would seem, however, to be placing too much faith in the details of the nuclear model, in violation of the original philosophy of our calculation. The values of $\mathcal{K}_K$ obtained are almost certainly good to a factor of two, and are probably somewhat better, perhaps

TABLE IV. Partial closure factors for dipole excitations, $\mathcal{K} = E/(E + \langle \epsilon \rangle)$, to intermediate states of various angular momenta.

	$\mathcal{K}_{D,nl\to s}$	$\mathcal{K}_{D,nl\to p}$	$\mathcal{K}_{D,nl\to d}$	$\mathcal{K}_{D,nl\to f}$	$\mathcal{K}_{D,nl}$
$Z=8$ $1s$		0.49			0.49
$2s$		0.48			0.48
$2p$	0.99		0.85		0.90
$3d$		1.00		0.98	0.99
$Z=20$ $1s$		0.65			0.65
$2s$		0.61			0.61
$2p$	1.08		0.69		0.82
$3d$		1.01		0.92	0.95
$Z=40$ $1s$		0.42			0.42
$2s$		0.39			0.39
$2p$	1.15		0.51		0.72
$3d$		1.03		0.74	0.86
$Z=70$ $1s$		0.40			0.40
$2s$		0.36			0.36
$2p$	1.43		0.40		0.75
$3d$		1.13		0.54	0.78

within 20 to 50% for the most important ones. We believe, therefore, that they are sufficiently accurate for the calculation at hand.

8. RESULTS AND DISCUSSION

The changes in the muonic energy levels may now be found by multiplying the matrix elements tabulated in Table V by the closure factors in Table V. The various contributions, M, D, Q, and R, and their sum are plotted against Z in Fig. 5 for each of the muonic levels considered. The total shifts for each state are plotted together in Fig. 6.

We see from Fig. 6 that the $1s$ state is shifted by 1 to 3 keV for medium to heavy nuclei, while the $2s$ and $2p$ levels are shifted somewhat less – a few tenths to 1 keV or so. These results are very similar to those we obtained previously[14] retaining only dipole terms and setting all muon closure factors equal to unity, and in reasonable agreement with other existing estimates[9–11] discussed in Sec. 1.

With regard to the accuracy of these results, we feel that the greatest uncertainty is probably in the muon closure factors, which may be in error by as much as a factor of two. The nuclear physics is probably accurate to 30% or so. Our calculation has assumed that the giant-dipole resonance exhausts the T. R. K. dipole sum rule, which is not strictly true, particularly in light nuclei. For example, estimates obtained by integration of the photodisintegration cross section[28] indicate that the GD resonance accounts for about 68% of the sum rule for oxygen and for about 82%

TABLE V. Total closure factors, $\mathcal{K} = E/(E+\langle \epsilon \rangle)$. M, D, Q, and R refer to monopole, dipole, and quadrupole transitions, and the remainder (octupole and higher).

	M	D	Q	R
$Z=8$ $1s$	0.26	0.49	0.27	0.12
$2s$	0.26	0.48	0.27	0.12
$2p$	• • •	0.90	0.48	0.18
$3d$	• • •	0.99	• • •	• • •
$Z=20$ $1s$	0.32	0.65	0.17	0.07
$2s$	0.32	0.61	0.17	0.07
$2p$	0.19	0.82	0.36	0.12
$3d$	• • •	0.95	0.71	• • •
$Z=40$ $1s$	0.42	0.42	0.22	0.10
$2s$	0.40	0.39	0.21	0.10
$2p$	0.22	0.72	0.37	0.16
$3d$	0.15	0.86	0.63	0.26
$Z=70$ $1s$	0.54	0.40	0.19	0.10
$2s$	0.57	0.36	0.17	0.09
$2p$	0.28	0.75	0.34	0.15
$3d$	0.17	0.78	0.66	0.30

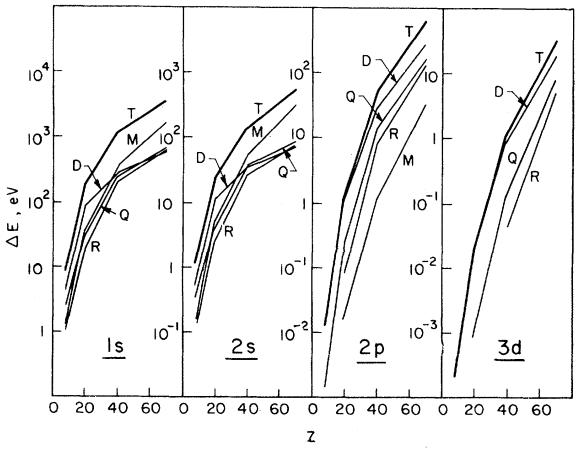

FIG. 5. Various multipole contributions to the muon level shifts: Monopole (M), Dipole (D), Quadrupole (Q), and Remainder (R). T is the total shift.

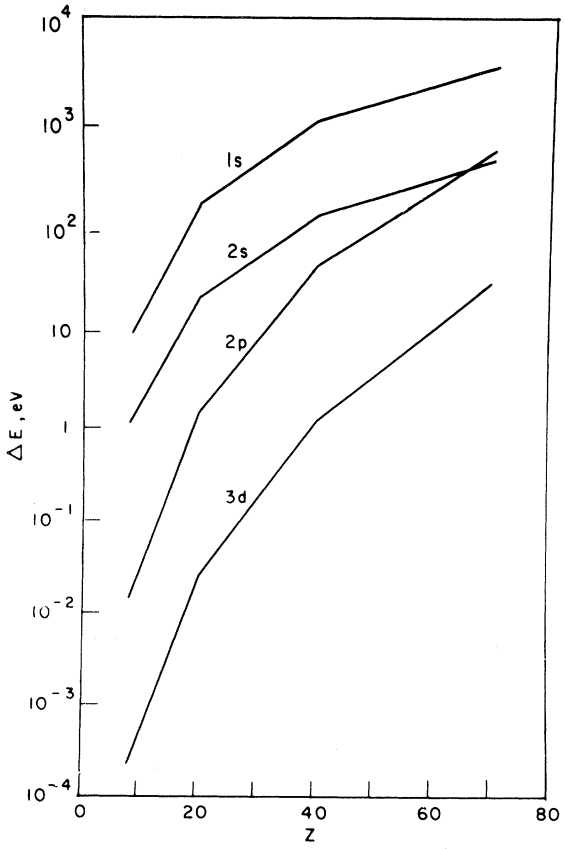

FIG. 6. Total level shifts caused by nuclear polarization.

for calcium. We note that Foldy and Walecka's[20] muon-capture calculation, which uses the same nuclear models but with the dipole strength normalized to the experimental photodisintegration value, produces capture rates within 10% of the experimental values for oxygen and calcium.

The neglect of the transverse interaction is probably no more important than the uncertainties in the nuclear physics. Our aim from the beginning has been to obtain estimates of the level shifts accurate to a factor of two or better, and we feel that each part of the calculation is at least this good. Therefore, while it is not impossible that the errors are larger, we believe that Fig. 6 is accurate within a factor of two.

The nuclear-polarization corrections are therefore clearly of sufficient size (one to several keV) for heavy nuclei to be significant in the analysis of muonic x-ray spectra. A calculation by Barrett *et al.*[25] which corrects the spectrum of ^{209}Bi for nuclear-polarization effects of this order of magnitude seems to show a slight worsening of agreement with electron-scattering results. We can draw no conclusions from this: no account has been taken of nuclear polarization in interpreting the electron-scattering data. A calculation very similar to the one just performed could clearly be made for electron scattering, although transverse terms would almost certainly be relatively more important. Until some such calculation is performed, we will still be unsure of how to interpret any apparent discrepancies between nuclear charge distributions fitted to muonic x-ray spectra and ones fitted to elastic electron scattering.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor J. D. Walecka who suggested this problem, for his continued help and encouragement during this work. The author is also grateful to Dr. R. C. Barrett for calculations pertaining to the relativistic muon wave functions.

APPENDIX A. SPURIOUS STATES IN HARMONIC-OSCILLATOR SHELL MODEL

The nuclear Hamiltonian for a harmonic-oscillator shell model of the nucleus is, with a fixed center of potential,

$$H=\frac{1}{M}\sum_{i=1}^{A}(-\vec{\nabla}_i^{\,2}+b^{-4}\vec{x}_i^{\,2}),\quad \omega_0=(Mb^2)^{-1}. \tag{A1}$$

The states of this system include spurious center-of-mass excitations, which we wish to exclude. This problem has been considered previously, for example by Elliott and Skyrme,[29] but we will reformulate it in a manner particularly convenient to the situation at hand.

If we change coordinates to use center-of-mass and relative coordinates

$$\vec{R}=A^{-1}\sum_{i=1}\vec{x}_i,\quad \vec{\xi}_i=\vec{x}_i-\vec{R} \tag{A2}$$

(the $\vec{\xi}_i$ are not all independent) we can separate the Hamiltonian as

$$H=(2MA)^{-1}(-\vec{\nabla}_R^{\,2}+A^2b^{-4}\vec{R}^2)+H_{\rm rel}(\{\vec{\xi}_i,\vec{\nabla}_{\xi_i}\}). \tag{A3}$$

The first term is just a harmonic-oscillator Hamiltonian for the center of mass with mass MA, oscillator parameter $bA^{-1/2}$ and energy ω_0. The second term describes the motion in the center of mass frame. The solutions are

$${}_N\Psi_m(\{\vec{x}_i\})=\phi_m(\{\vec{\xi}_i\})\Phi_N^{(n)}(A^{\frac{1}{2}}\vec{R}/b), \tag{A4}$$

where $\Phi_N^{(n)}$ is a harmonic-oscillator function with energy $n\omega_0$. The "good" states are those for which $N=0$ $(1s)$, with the center-of-mass wave function

$$\Phi_0^{(0)} = (A/\pi b^2)^{\frac{3}{4}} e^{-AR^2/2b^2}. \tag{A5}$$

We should really use $\phi_m(\{\xi_i\}) = (\pi b^2/A)^{3/4}\, e^{AR^2/2b^2}\, {}_0\Psi_m(\{x_i\})$ for the allowed shell-model wave function referred to a fixed center of mass. However, Tassie and Barker[30] have shown that matrix elements of charges and currents computed with ϕ_m's are related to those computed with ${}_0\Psi_m$'s by a multiplicative factor of $\exp[(qb/2)^2 A^{-1}]$. Because of the A^{-1} in the exponential the effect is small except for very large q. Further, the increase at large momentum transfer tends to be offset by the decrease in the form factor of the individual nucleons which must also be included. For nuclei of reasonable size we may simply neglect both effects without serious error.

Any excited state of the center of mass with energy $n\omega_0$ may be represented in the form

$$\Phi_{NLM}^{(n)} = (A/\pi b^2)^{\frac{3}{4}} \mathcal{O}_{NLM}^{(n)} [e^{-AR^2/b^2}] e^{AR^2/2b^2}, \tag{A6}$$

where $\mathcal{O}_{NLM}^{(n)}$ is a differential operator proportional to n gradients $\vec{\nabla}_R$ coupled up to a tensor of rank L component M. This $\Phi_{NLM}^{(n)}$ clearly has the desired angular momentum, and the simple observation that $[\vec{R}\cdot\vec{\nabla}_R, (\partial/\partial R_i)] = -(\partial/\partial R_i)$ implies that

$$[\vec{R}\cdot\vec{\nabla}_R, \mathcal{O}_{NLM}^{(n)}] = -n\mathcal{O}_{NLM}^{(n)} \tag{A7}$$

which makes it easy to verify directly that $\Phi_{NLM}^{(n)}$ is an eigenfunction of $\hat{H}$ with eigenvalue $(n+\frac{3}{2})\omega_0$.

In particular, we have

$$\mathcal{O}_{1PM}^{(1)} = -\frac{b}{\sqrt{2A}}\nabla_M, \quad \mathcal{O}_{2S}^{(2)} = -\frac{1}{\sqrt{6}}\frac{b^2}{2A}\vec{\nabla}^2, \quad \mathcal{O}_{1DM}^{(2)} = \frac{1}{\sqrt{2}}\frac{b^2}{2A}\sum_{m_1 m_2}(1m_1 1m_2 | 112M)\nabla_{m_1}\nabla_{m_2}. \tag{A8}$$

We may write any of these in the form

$$\mathcal{O}_N^{(n)}(\vec{R}) = \sum_{k=1}^{3} a_{N,k}\frac{\partial^n}{\partial R_k^n} + \sum_{k\neq l=1}^{3}\sum_{\mu+\nu=n} b_{N,kl,\mu\nu}\frac{\partial^\mu}{\partial R_k^\mu}\frac{\partial^\nu}{\partial R_l^\nu} + \sum_{\mu+\nu+\sigma=n} c_{N,\mu\nu\sigma}\frac{\partial^\mu}{\partial R_1^\mu}\frac{\partial^\nu}{\partial R_2^\nu}\frac{\partial^\sigma}{\partial R_3^\sigma}. \tag{A9}$$

The transition potential between a spurious state and the ground state is

$$\begin{aligned} V_{Nm,00} &= \langle {}_N\Psi_m | \int d^3x\, \hat{\rho}(\vec{x})/|\vec{x}-\vec{z}|\, {}_0\Psi_0\rangle \\ &= \int d^3r_1\cdots d^3r_A\, d^3x\, |\vec{x}-\vec{z}|^{-1}\phi_m(\{\vec{\xi}_i\})\phi_0(\{\vec{\xi}_i\})\left(\frac{A}{\pi b^2}\right)^{3/2}\mathcal{O}_{NLM}(\vec{R})\, e^{-AR^2/b^2}\sum_{j=1}^{Z}\delta(\vec{x}-\vec{r}_j) \end{aligned} \tag{A10}$$

using $\rho(x) = \sum_{j=1}^{Z}\delta(\vec{x}-\vec{r}_j)$. But the derivatives in $\mathcal{O}(\vec{R})$ commute with any function of the $\vec{\xi}_i$, and $\partial/\partial R_k = \sum_{i=1}^{A}\partial/\partial(r_i)_k$ giving

$$V_{Nm,00} = \int d^3x d^3r_1\cdots d^3r_A \sum_{j=1}^{Z}[\delta(\vec{x}-\vec{r}_j)\,\mathcal{O}_{NLM}^{(n)}(\vec{r}_j)\{{}_0\Psi_m\, {}_0\Psi_0\}]\frac{1}{|\vec{x}-\vec{z}|} \tag{A11}$$

plus terms which are perfect derivatives and integrate to zero at once.

Integrating by parts, we get the derivatives $\vec{\nabla}_{r_j}$ in $\mathcal{O}_{NLM}^{(n)}(\vec{r}_j)$ onto the δ functions, where they may be replaced by $\vec{\nabla}_x$ (there are no surface terms)

$$V_{Nm,00} = -\int\frac{d^3x}{|\vec{x}-\vec{z}|}(-)^n\mathcal{O}_{NLM}^{(n)}(\vec{x})\int d^3r_1\cdots d^3r_A\, {}_0\Psi_m[\sum_{j=1}^{Z}\delta(\vec{x}-\vec{r}_j)]{}_0\Psi_0. \tag{A12}$$

The sum on j in (A11) is again $\rho(\vec{x})$. Another integration by parts puts the derivatives $\vec{\nabla}_x$ onto $(\vec{x}-\vec{z})^{-1}$ where they may be replaced by $-\vec{\nabla}_z$ to give

$$V_{Nm,00}(\vec{z}) = (-)^{n+1}\mathcal{O}_{NLM}^{(n)}(\vec{z})V_{0m,00}(\vec{z}). \tag{A13}$$

This simply relates the transition potential for transition to a spurious state to that for the related "good" state. An overestimate of the contribution of spurious states may be made by removing the restriction of good states on the right of (A13) as

$$V_{\text{pol, sp}}(z) \leq \sum_{NLM} \sum_{S} [(n+s)\omega_0]^{-1} |\mathcal{O}_{NLM}^{(n)}(\vec{z}) V_{S,0}(\vec{z})|^2, \tag{A14}$$

where NLM has energy $n\omega_0$ and the sum on shell-model states S (energy $s\omega_0$) runs over all states (including spurious ones and the ground state).

Dipole excitations of the center of mass (by $\mathcal{O}_{1p}^{(1)}$) will be the most important: they have the smallest energy denominators and a factor of A in V_{pol} relative $n=2$ excitations. The most important such state is the one derived from the ground state since the total transition charge is Ze rather than just e. For this state

$$\sum_M |\mathcal{O}_{1PM}^{(1)}(\vec{z}) V_{0,0}(\vec{z})|^2 = (b^2/2A) |(\partial V_C(z)/\partial z)|^2 \tag{A15}$$

and $$V_{\text{pol, sp}\,1P,0}(z) = (2AM\omega_0^2)^{-1} |(\partial V_C(z)/\partial z)|^2 \tag{A16}$$

very much like the Goldhaber-Teller state.

For the other spurious $1P$ center-of-mass states, the gradient formula gives

$$-\frac{b}{(2A)^{1/2}} \nabla_\mu \Big[\sum_{kQ} v_{KQ}(z) Y_{KQ}(\Omega_z) \Big] = -\frac{b}{(2A)^{1/2}} \sum_{KQ} \Bigg[\left(\frac{K+1}{2K+3} \right)^{1/2} (KQ\,1\mu \,|\, K\,1\,K+1\,Q+\mu) \left(\frac{\partial}{\partial z} - \frac{K}{z} \right) v_{KQ}$$

$$\times Y_{K+1\,Q+\mu}(\Omega_z) - \left(\frac{K}{2K-1} \right)^{1/2} (KQ\,1\mu \,|\, K\,1\,K-1\,Q+\mu) \left(\frac{\partial}{\partial z} + \frac{K+1}{z} \right) v_{KQ} Y_{K-1\,Q+\mu}(\Omega_z) \Bigg]. \tag{A17}$$

After a little algebra, the contribution to V_{pol} is found to be, in the notation of Sec. 4,

$$V_{\text{pol, sp, other}\,1P}(z) = \sum_K \sum_{N_2 L_2} \sum_{N_1 L_1} \frac{2\alpha^2}{E_2 - E_1 + \omega_0} \frac{CG(L_1, L_2, K)}{(2K+1)^2} \frac{b^2}{2A}$$

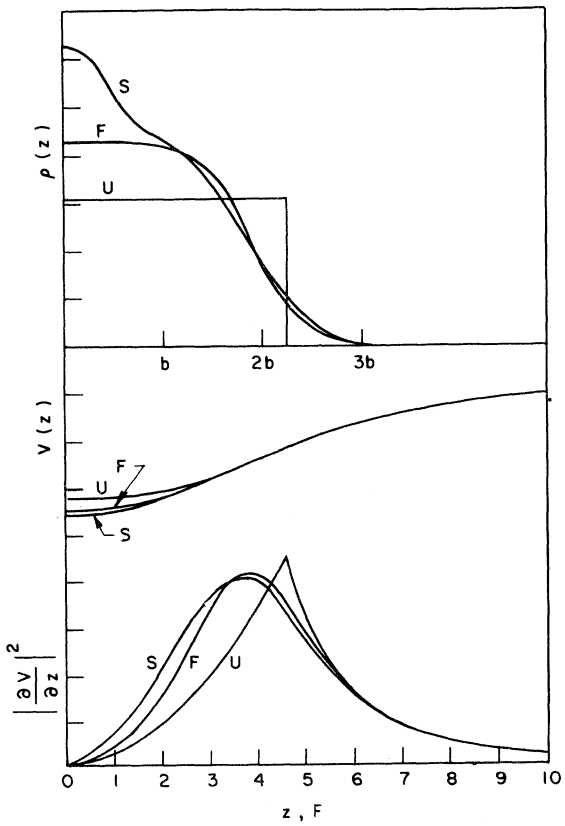

FIG. 7. Charge density, potential, and $|dV/dz|^2$ for three model nuclear charge distributions for calcium: uniform (U); Fermi shape (F); and harmonic-oscillator shell model (S).

$$\times\left[\frac{K+1}{2K+1}\left|\left(\frac{\partial}{\partial z}-\frac{K}{z}\right)I_{N_2L_2,N_1L_1;K}(z)\right|^2+\frac{K}{2K+1}\left|\left(\frac{\partial}{\partial z}+\frac{K+1}{z}\right)I_{N_2L_2,N_1L_1;K}(z)\right|^2\right]\ . \tag{A18}$$

Here the first term in the brackets comes through multipole $K+1$ and the second through $K-1$. These corrections have been evaluated on a computer in exactly the same manner used for the naive shell model of Sec. 4, and found to be negligible (contributing in aggregate no more than a percent or so) with the exception of the pure center-of-mass $1P$ state in (A14).

Of the remaining spurious states the most important are the ${}_N\Psi_0$ with N above the $1P$ state since the total charge Ze contributes to matrix elements. Their contributions to $V_{\rm pol}$ may be evaluated as the $1P$ state was, and are found to be negligible.

APPENDIX B. NUCLEAR-CHARGE MODELS

To illustrate the similarities and differences of the three charge models used in this paper, Fig. 7 shows $Z^{-1}\rho(z)$, $(Z\alpha)^{-1}V_C(z)$, and $(Z\alpha)^{-2}|\partial V_C/\partial z|^2$ for each of the three models used: (1) uniform charge density to a radius $R(U)$; (2) Fermi shape (F); and (3) independent-particle harmonic-oscillator shell model (S). The parameters used are those for $Z=20$ in Table V. We note that except for the charge density near the origin, the S and F models are *very* similar and U not too different. The U model was used only in computing muon wave functions, and should be adequate there since the potential is not much different from U or F.

---

*Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Contract No. F44620-68-C-0075.

†Present address: Sandia Laboratory, Albuquerque, New Mexico 87115.

[1]J. A. Wheeler, Rev. Mod. Phys. 21, 133 (1949).

[2]V. Fitch and J. Rainwater, Phys. Rev. 92, 789 (1953).

[3]L. Cooper and E. Henley, Phys. Rev. 92, 801 (1953).

[4]For example, E. M. Lyman, A. O. Hanson, and M. B. Scott, Phys. Rev. 84, 626 (1951).

[5]R. Hofstadter, B. Hahn, A. W. Knudsen, and J. A. McIntyre, Phys. Rev. 84, 626 (1954).

[6]L. R. B. Elton, in Proceedings of the Williamsburg Conference on Intermediate Energy Physics (College of William and Mary, Williamsburg, Va., 1966), p. 48.

[7]F. G. Perey and J. P. Schiffer, Phys. Rev. Letters 17, 328 (1966).

[8]D. G. Ravenhall, in Proceedings of the Williamsburg Conference on Intermediate Energy Physics (College of William and Mary, Williamsburg, Va., 1966), p. 38.

[9]W. Lakin, Carnegie Institute of Technology Technical Report No. 2, OOR No. 116–53, 1954 (unpublished).

[10]W. Pieper and W. Greiner, Phys. Letters 24B, 377 (1967).

[11]F. Scheck, Z. Physik 172, 239 (1963).

[12]B. Jacobsohn, Phys. Rev. 96, 1637 (1954).

[13]L. Wilets, Kgl. Danske Videnskab. Selskab Mat. Fys. Medd. 29, (1954).

[14]R. K. Cole, Jr., Phys. Letters 25B, 178 (1967).

[15]We use units with $\hbar=c=1$, a caret to denote an operator in Hilbert space, capital and lower-case letters when in Latin alphabet for nuclear and muonic quantities, respectively, and a metric where $qx=\vec{q}\cdot\vec{x}-\omega t$. Angular momentum notation follows A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1957).

[16]This is strictly true only for states containing the nucleus in its ground state, but these are the states of immediate interest. It may be considered a reasonable approximation for other states if the nuclear charge distribution is not changed much from the ground state.

[17]J. M. Jauch and F. Rohrlich, The Theory of Photons and Electrons (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955).

[18]T. deForest and J. D. Walecka, Advan. Phys. 15, 1 (1966).

[19]In the preliminary version of this work, Ref. 14, the transverse interaction was treated incorrectly. Since its effect was then neglected in comparison to the Coulomb term, the conclusions of that paper are not affected.

[20]L. L. Foldy and J. D. Walecka, Nuovo Cimento 34, 1026 (1964).

[21]D. B. Isabelle and G. R. Bishop, Nucl. Phys. 45, 209 (1963).

[22]P. Bounin and G. R. Bishop, J. Phys. Radium 22, 555 (1961).

[23]W. Czyz, Phys. Rev. 131, 2141 (1963).

[24]G. E. Pustovalov, Zh. Eksperim. i Teor. Fiz. 36, 1806 (1959) [English Transl.: Soviet Phys. – JETP 36, 1288 (1959)].

[25]R. C. Barrett, private communication, and R. C. Barrett, S. J. Brodsky, G. W. Erickson, and M. H. Goldhaber, Phys. Rev. 166, 1589 (1968).

[26]J. C. Sens, Phys. Rev. 111, 940 (1959).

[27]E. G. Guller and E. Hayward, in Nuclear Reactions edited by P. M. Endt and P. B. Smith (North-Holland Publishing Company, Amsterdam, 1962), Vol. 2.

[28]E. Hayward, Rev. Mod. Phys. 35, 324 (1963).

[29]J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London) A232, 561 (1955).

[30]L. J. Tassie and F. C. Barker, Phys. Rev. 111, 940 (1958).