Brueckner-Hartree-Fock Calculations of Spherical Nuclei in an Harmonic-Oscillator Basis*

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A method is developed for performing Brueckner-Hartree-Fock (BHF) calculations of spherical nuclei in the harmonic-oscillator representation. Both the Brueckner and the HF self-consistencies are satisfied. The method is applied to the calculation of ^{16}O, ^{40}Ca, ^{48}Ca, and ^{208}Pb with a G matrix derived from the Hamada-Johnston potential. The nuclei are too small and underbound. Various kinds of convergence are studied. It is concluded that the calculations are essentially as easy and as reliable as, though a little longer than, pure Hartree-Fock calculations.

I. INTRODUCTION

THE last few years have seen the emergence of many nuclear-structure calculations performed with two-body potentials obtained from the two-nucleon data. Some of these[1–3] have been shell-model calculations seeking to add many-particle improvements on an assumed single-particle (SP) basis. Others have aimed at laying the foundations of the shell model itself, thereby calculating the SP parameters and exhibiting the nature of the essential (nonperturbative) many-particle correlations. The simplest way to perform a calculation of the second category is the Hartree-Fock (HF) method, in which there are no essential many-particle correlations. It is valid when the two-nucleon interaction is sufficiently smooth. If the interaction is smooth at distances of the order of the average internuclear spacing, but unsmooth at short distances, a modified form of HF is still applicable, in which the essential correlations are those of two nucleons interacting at close range in the average field of the others. This is the Brueckner-Hartree-Fock (BHF) method,[4] which has been much studied in the last decade.[5–7]

The principal change from HF to BHF is the replacement of the two-nucleon interaction V by an effective interaction often known as the G matrix $G(\omega)$ dependent on the available energy ω of the correlated pair. The equation connecting V and $G(\omega)$ is the Bethe-Goldstone equation.[8] The solution of this equation requires the knowledge of the single-particle parameters. The latter cannot be determined, however, without prior knowledge of $G(\omega)$; this is the Brueckner self-consistency. Another self-consistency arises in the HF part of the problem, since the single-particle potential U is calculated in terms of $G(\omega)$ by a sum over occupied single-particle states, but the latter cannot be determined without prior knowledge of U; this is the HF self-consistency.

In light nuclei, harmonic-oscillator wave functions (HOWFs) are well known[9,10] to be good approximations

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to the true single-particle wave functions (SPWFs). Hence the HF self-consistency problem is mostly of academic interest in light spherical nuclei. Calculations using pure HOWFs encounter only the Brueckner self-consistency problem and are called Brueckner calculations. Several such calculations have been performed[11–15] and it has been a pleasant surprise to discover that they are at least as easy as those of nuclear matter.

The situation is altered in heavy nuclei, where HOWFs and self-consistent SPWFs may give noticeably different results. For instance,[16] the radial charge distribution is radically different with the two sets of WFs. One might try to avoid coping with HF self-consistency for heavy nuclei by using a preselected potential well (for instance, Woods-Saxon) to calculate the SPWFs. It turns out, however, that the G matrix is much easier to calculate for HOWFs. Hence the best procedure seems to be to perform BHF calculations in an oscillator basis. Eventually BHF calculations will have to be performed for deformed heavy nuclei, since most heavy nuclei outside of the vicinity of ^{208}Pb seem to be either deformed or highly deformable.

This paper discusses BHF calculations of spherical nuclei in the oscillator representation. Past BHF calculations[17,18] were done in the coordinate representation and they contained many more approximations than the present method. (Very recently BHF calculations have also been done by Irvine[19] using the Reid soft-core potential.[20]) The line of approach of this paper is the same as in the recent series of pure HF calculations.[9,10,16,21] The oscillator basis is chosen because the Talmi-Moshinsky transformation[22] permits easy passage from the c.m.-relative representation to the single-particle representation and because oscillator wave functions are fairly good approximations to the true wave functions, so that the basis needs to have only a fairly small dimensionality. It is found that relatively minor changes are sufficient to transform the HF formalism into a BHF formalism. The HF programs developed by Tarbutton and Davies[16] are readily transformed into BHF programs. The main difference comes in the energy dependence of G. The energy ω occurring in $G(\omega)$ is given in terms of SP energies, which are part of the result of the calculation, and therefore this aspect of Brueckner self-consistency has to be worked into the iteration procedure. The other aspects have to do with the dependence of G on the intermediate-state SP potential and on the Pauli operator. They can be included in a self-consistent manner also, but this complicates the calculations greatly, and since the numerical value of G is rather insensitive to small changes in these variables, they have been kept fixed in the calculations done so far.

The G matrix used in this paper is derived[23] from the Hamada-Johnston potential. For the S-wave part, the G matrix is calculated using a combination of the separation method and the reference-spectrum method. The potential V is separated into a short-range and a long-range part, V_S and V_L. Then the G matrix is given by a sum of G_S, V_L, and $V_L(Q/e)V_L$, plus cross terms of G_S and V_L.[23] Here G_S is the reaction matrix deduced from V_S; it is calculated by the reference-spectrum method. Free-particle intermediate states are used in calculating both G_S and $V_L(Q/e)V_L$. The criterion for choosing the separation distance which divides V into V_S and V_L is to minimize the Pauli correction terms due to approximating the physical G_S by the reference spectrum G_S. For the other partial waves, the G matrices are obtained essentially by the reference-spectrum method.

This G matrix differs from that of Kuo and Brown,[1] which has been widely used in shell-model calculations but was never meant for BHF, by the fact that it retains the repulsive contribution of the short-range potential G_S. This contribution was set equal to zero in Ref. 1 because of the following consideration. The Hamada-Johnston potential has a hard repulsive core while one would expect that the short-range repulsion is better represented by a "soft" repulsive core. The repulsion of G_S will be largely reduced upon changing from a hard-core potential to a soft-core potential. Thus for nuclear shell-model calculations where one does not deal with the problem of nuclear saturation, one may as well totally leave out G_S until a better description of the short-range repulsion is available. However, the situation for a BHF calculation is very different, because now we are dealing with the nuclear-saturation problem and G_S has an important saturation effect. G_S is therefore included in the present work, although it should be

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remembered that the G_S derived from the Hamada-Johnston potential probably overestimates the physical short-range repulsion. Calculations with other G matrices are also underway.

A BHF calculation is not a substitute for a shell-model calculation. It is only a beginning, the derivation of SP parameters which are usually chosen empirically. After this, higher-order diagrams may need to be calculated, i.e., a shell-model calculation may have to be performed. For example, the radial charge distribution resulting from BHF calculations needs to be corrected for Brueckner correlations.

II. BRUECKNER SELF-CONSISTENCY

Notations:

q charge quantum number (neutron or proton),
l single-particle orbital angular momentum,
j single-particle total angular momentum,
m z component of j,
$s, t \equiv (q\ l\ j)$ symmetry type.[16]

For harmonic oscillator wave functions:

n radial quantum number (number of radial nodes),
$\alpha, \beta, \gamma, \delta \equiv (n\ q\ l\ j)$ complete specification of HO state,
$a, b, c, d \equiv (n\ q\ l\ j) \equiv (ns)$.

For self-consistent wave functions:

p radial quantum number,
$\xi, \eta, \zeta \equiv (p\ q\ l\ j\ m)$ complete specification of HF state,
$x, y, z \equiv (p\ q\ l\ j) \equiv (ps)$.

This section deals with the method for introducing into the iteration scheme the dependence of $G(\omega)$ upon the available energy ω. The connection between a matrix element $\langle\xi|U|\eta\rangle$ of the SP potential and $G(\omega)$ has been discussed by BBP[24] in the case of nuclear matter. The formula depends on whether ξ and η are holes or particles. The extension to finite nuclei[6] of the BBP theorem yields the following results. If ξ and η are both holes,

$$\langle\xi|U|\eta\rangle = \tfrac{1}{2}\sum_{\zeta}{}' \langle\xi\zeta|G(\epsilon_x+\epsilon_z)+G(\epsilon_y+\epsilon_z)|\eta\zeta\rangle. \quad (1a)$$

If ξ is a hole and η a particle,

$$\langle\xi|U|\eta\rangle = \sum_{\zeta}{}' \langle\xi\zeta|G(\epsilon_x+\epsilon_z)|\eta\zeta\rangle. \quad (1b)$$

If ξ and η are both particles,

$$\langle\xi|U|\eta\rangle = \tfrac{1}{2}\sum_{\zeta}{}' \langle\xi\zeta|G(\bar{\epsilon}_x+\epsilon_z)+G(\bar{\epsilon}_y+\epsilon_z)|\eta\zeta\rangle. \quad (1c)$$

The primed sums run over hole states only. The single-particle energies, eigenvalues of $T+U$, have been denoted by ϵ. The quantities $\bar{\epsilon}$ differ from ϵ and their definition is somewhat uncertain. It is discussed below. In fact, the particle-particle matrix elements have been the subject of much study in the literature.[25] One extreme proposal has been simply to set them equal to zero; another one has been to use Eq. (1c) with $\bar{\epsilon}$ replaced by ϵ, the "on-shell" prescription. Somewhere in between lies the "best" prescription, and the arguments of the last few years seem to have produced a rough consensus as to its nature: The potential should vanish for the high particle states, but for the low ones it should be attractive and almost continuous with the hole potential. The distinction between "high" and "low" in nuclear matter comes for momenta of the order of $2k_F$. For the low particle states, $\bar{\epsilon}$ should be somewhat off the energy shell, i.e., lower than ϵ. This off-shell prescription makes U slightly less attractive than the on-shell prescription does, producing a small gap at the Fermi surface for the SP energies. However, for "low" states, the off-shell effect is far smaller than the effect of setting $U=0$. This is readily checked for nuclear matter, and in finite nuclei it has been demonstrated in some recent calculations of Becker *et al.*[11] A reasonable rough formula for $\bar{\epsilon}$ might be

$$\bar{\epsilon}_x = 2\epsilon_0 - \epsilon_x, \quad (2)$$

where ϵ_0 is the average energy of occupied SP states.

Since BHF calculations are concerned only with hole states and low-lying particle states, the particle-particle matrix elements of U should definitely be retained in a form such as (1c), and not be set equal to zero. This is fortunate because the similarity of treatment between particles and holes makes the formalism simpler. Note that the SP potential defined by Eqs. (1) is Hermitian. It has sometimes been said, because of the dependence of the right-hand side of (1) on SP energies, that U is "state-dependent" and not a true potential, and that it must be averaged before it is used. This is false. U is an ordinary potential which is completely specified by a Hermitian matrix, with no additional parameter; but the relationship of U to the G matrix involves a delicate self-consistency problem.

In view of the uncertainty associated with the treatment of the particle energies, we have taken the liberty of modifying Eqs. (1) slightly. This is not necessary; the equations could be used as they are. We argue that one is mostly interested in the SP energies and wave functions for the occupied states; hence the formula for hole-hole elements, Eq. (1a), should not be changed. The hole-particle elements, Eq. (1b), are less important because a small change in them affects the occupied states in second order only; this formula can be altered

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a little. The particle-particle elements, Eq. (1c), come in third order; they can be modified even more. In our calculations so far, we have used the following formula, which replaces all three Eqs. (1) and is exact as far as hole-hole elements are concerned:

$$\langle\xi|U|\eta\rangle=\tfrac{1}{2}\sum_{\zeta}{}'\langle\xi\zeta|G(\epsilon_x'+\epsilon_z)+G(\epsilon_y'+\epsilon_z)|\eta\zeta\rangle, \quad (3)$$

with

$$\begin{aligned}\epsilon_x'&=\epsilon_x \text{ for levels below the Fermi surface}\\&=e_s \text{ for levels above the Fermi surface.}\end{aligned} \quad (4)$$

The quantity e_s is taken to depend only on the symmetry type. The choice which makes Eq. (3) as accurate as possible for hole-particle elements is

$$e_s{}^{(0)}=\text{average value of } \epsilon \text{ for the occupied levels of symmetry type } s. \quad (5)$$

However we have used other choices of e_s and their effects will be discussed in Sec. 4. Choice (5) also produces the desired off-shell effect in the particle-particle elements, although it is not exactly that given by Eq. (2).

Coming now to the way in which this self-consistency is achieved, one must transform Eq. (3) from the unknown ξ, η, ζ representation to the known α, β, γ, δ harmonic-oscillator representation. This is done by the following steps

$$\begin{aligned}\langle\alpha|U|\beta\rangle&=\tfrac{1}{2}\sum_{\xi\eta}\sum_{\zeta}{}'\langle\alpha|\xi\rangle\langle\xi\zeta|G(\epsilon_x'+\epsilon_z)\\&\qquad+G(\epsilon_y'+\epsilon_z)|\eta\zeta\rangle\langle\eta|\beta\rangle\\&=\tfrac{1}{2}\sum_{\alpha'\beta'\gamma\delta}\sum_{\xi\eta}\sum_{\zeta}{}'\langle\alpha|\xi\rangle\langle\xi|\alpha'\rangle\langle\zeta|\gamma\rangle\langle\delta|\zeta\rangle\langle\beta'|\eta\rangle\\&\qquad\times\langle\eta|\beta\rangle\langle\alpha'\gamma|G(\epsilon_x'+\epsilon_z)+G(\epsilon_y'+\epsilon_z)|\beta'\delta\rangle\\&=\tfrac{1}{2}\sum_{\alpha'\beta'\gamma\delta}\sum_{\xi\eta}\sum_{\zeta}{}'\langle\alpha|\rho_\xi|\alpha'\rangle\langle\delta|\rho_\zeta|\gamma\rangle\langle\beta'|\rho_\eta|\beta\rangle\\&\qquad\times\langle\alpha'\gamma|G(\epsilon_x'+\epsilon_z)+G(\epsilon_y'+\epsilon_z)|\beta'\delta\rangle,\end{aligned}$$

with the definition

$$\langle\alpha|\rho_\xi|\alpha'\rangle=\langle\alpha|\xi\rangle\langle\xi|\alpha'\rangle. \quad (6)$$

If the two terms of the two-body matrix element are separated, the sum over either ξ or η becomes trivial, and one obtains

$$\begin{aligned}\langle\alpha|U|\beta\rangle&=\tfrac{1}{2}\sum_{\alpha'\gamma\delta}\sum_{\xi}\sum_{\zeta}{}'\langle\alpha|\rho_\xi|\alpha'\rangle\langle\delta|\rho_\zeta|\gamma\rangle\\&\qquad\times\langle\alpha'\gamma|G(\epsilon_x'+\epsilon_z)|\beta\delta\rangle+\tfrac{1}{2}\sum_{\beta'\gamma\delta}\sum_{\eta}\sum_{\zeta}{}'\langle\delta|\rho_\zeta|\gamma\rangle\\&\qquad\times\langle\beta'|\rho_\eta|\beta\rangle\langle\alpha\gamma|G(\epsilon_y'+\epsilon_z)|\beta'\delta\rangle.\end{aligned} \quad (7)$$

The next step is to do the geometry. This has been described in Ref. 16. A matrix F_J is defined by

$$\begin{aligned}\langle\alpha\gamma|G(\omega)|\beta\delta\rangle&=-\sum_{JM}F_J(abdc;\omega)(-)^{j_b-m_\beta}(-)^{j_c-m_\gamma}\\&\qquad\times C(j_aj_bJ;m_\alpha,-m_\beta,M)C(j_dj_cJ;m_\delta,-m_\gamma,M).\end{aligned} \quad (8)$$

The matrices U and ρ have the form

$$\langle\alpha|U|\beta\rangle=\delta_{s_\alpha s_\beta}\delta_{m_\alpha m_\beta}(n_\alpha|U_s|n_\beta), \quad (9a)$$

$$\langle\alpha|\rho_\xi|\beta\rangle=\delta_{s_\alpha s_\xi}\delta_{s_\beta s_\xi}\delta_{m_\alpha m_\xi}\delta_{m_\beta m_\xi}(n_\alpha|\rho_{sp}|n_\beta). \quad (9b)$$

Then Eq. (7) becomes

$$\begin{aligned}(n_1|U_s|n_2)&=\tfrac{1}{2}\sum_{n_1'n_3n_4}\sum_{p_1}\sum_{tp_3}{}'(n_1|\rho_{sp_1}|n_1')(n_4|\rho_{tp_3}|n_3)\\&\qquad\times[-(2j_t+1)^{1/2}(2j_s+1)^{-1/2}\\&\qquad\times F_0(sn_1',sn_2,tn_4,tn_3;\epsilon_{sp_1}'+\epsilon_{tp_3})]\\&\qquad+\tfrac{1}{2}\sum_{n_2'n_3n_4}\sum_{p_2}\sum_{tp_3}{}'(n_4|\rho_{tp_3}|n_3)(n_2'|\rho_{sp_2}|n_2)\\&\qquad\times[-(2j_t+1)^{1/2}(2j_s+1)^{-1/2}\\&\qquad\times F_0(sn_1,sn_2',tn_4,tn_3;\epsilon_{sp_2}'+\epsilon_{tp_3})] \quad (10)\\&=\tfrac{1}{2}\sum_{n_1'p_1}(n_1|\rho_{sp_1}|n_1')(n_1'|\Gamma_s(\epsilon_{sp_1}')|n_2)\\&\qquad+\tfrac{1}{2}\sum_{n_2'p_2}(n_1|\Gamma_s(\epsilon_{sp_2}')|n_2')(n_2'|\rho_{sp_2}|n_2), \quad (11)\end{aligned}$$

with the definition

$$\begin{aligned}(n_1|\Gamma_s(\epsilon')|n_2)&=\sum_{n_3n_4}\sum_{tp_3}{}'(n_4|\rho_{tp_3}|n_3)[-(2j_t+1)^{1/2}\\&\qquad\times(2j_s+1)^{-1/2}F_0(sn_1,sn_2,tn_4,tn_3;\epsilon'+\epsilon_{tp_3})].\end{aligned} \quad (12)$$

We are now in a position to explain how an iteration is actually performed. The previous iteration has yielded a set of single-particle levels sp whose energies are ϵ_{sp} and whose wave functions can be specified by

$$(n|\rho_{sp}|n')=(sn|sp)(sp|sn'). \quad (13)$$

The complete iteration consists of three steps: (1) for each s and for each ϵ' in s, compute a matrix $\Gamma_s(\epsilon')$ by Eq. (12); (2) for each s, compute a matrix U_s by Eq. (11), which may be simplified as explained in the next paragraph; (3) for each s, find the eigenvalues and eigenvectors of the matrix $T+U_s$, which are the new SP energies ϵ_{sp} and wave functions $(sn|sp)$. This procedure is actually very close to that followed in a pure HF calculation, the difference being that, in HF, Γ_s is independent of ϵ' and is therefore identical to U_s. Furthermore, in BHF, one must have easy access to the matrix elements of $F_0(\omega)$ or $G(\omega)$ for any value of the available energy, since this value changes with each iteration. Since it would be prohibitive to solve the Bethe-Goldstone equation all over for each iteration, the method that is followed is to solve it ahead of time

for a few values of ω and to interpolate. The convergence of the iteration procedure toward self-consistency has been found to be almost as good as in the HF case.

From a programming point of view, Eq. (11) can be considerably simplified, especially for large dimensionalities, if it is rewritten as

$$\langle n_1|U_s|n_2\rangle=\langle n_1|\Gamma_s(e_s)|n_2\rangle+\tfrac{1}{2}\sum_{p_1}{}'\sum_{n_1'}\langle n_1|\rho_{sp_1}|n_1'\rangle \times\langle n_1'|\Gamma_s(\epsilon_{sp_1})-\Gamma_s(e_s)|n_2\rangle+\tfrac{1}{2}\sum_{p_2}{}'\sum_{n_2'}\langle n_1|\Gamma_s(\epsilon_{sp_2}) -\Gamma_s(e_s)|n_2'\rangle\langle n_2'|\rho_{sp_2}|n_2\rangle. \quad (14)$$

III. CALCULATIONAL DETAILS

The method of calculation of the two-body matrix elements has been described previously.[23] The energy denominators occurring in $V_{TL}(Q/e)V_{TL}$ and $V_{CL}(Q/e)V_{CL}$ were approximated by the constant values 210 and 420 MeV, respectively. The separation distance d was chosen as 1.05 fm and the Fermi momentum $k_F=1.3$ fm^{-1} was used in the second-order terms.

The relative matrix elements can be approximated[23] by linear functions of ω and $2N+L$, i.e.,

$$g\simeq g_0+\omega g_1+(2N+L)g_2, \quad (15)$$

where g_0, g_1, g_2, depend on S, T, n, l, n', l', j, and $b=(\hbar/m\Omega)^{1/2}$. Within this linear approximation, there is a simple relation between g_1 and g_2

$$g_2=-\tfrac{1}{2}\hbar\Omega g_1, \quad (16)$$

for the odd and singlet-even states since ω and $2N+L$ enter only through the combination

$$\gamma^2=(2m/\hbar^2)[\tfrac{1}{2}(2N+L+\tfrac{3}{2})\hbar\Omega-\omega]. \quad (17)$$

There is no such relation for the triplet-even states, since the second-order tensor term introduces an additional (linear) ω dependence.

Substituting Eq. (15) into the general expression for two-body matrix elements,[16] we obtain functions which are linear in ω; each matrix element is then stored in the computer memory in the form of two numbers, from which the value of this element at any ω can be obtained when needed during the BHF iteration process.

The calculation of two-body matrix elements takes only slightly longer than with smooth forces, *once the relative matrix elements have been computed.* The biggest increase in time comes in the iteration part of the calculation. For example, in ^{208}Pb, when the dimensionality of each symmetry type equals the number of occupied levels plus one, the time needed on the IBM-360/75 for 17 iterations is 1½ min for HF, 5 min for BHF.

TABLE I. Binding energy per nucleon, neutron radius r_n, and charge radius r_c of the four nuclei studied. The experimental values, in parentheses, are for the binding energies[a] and for the charge radii.[b] The parameters for ^{16}O are $b=1.7$ fm, D (dimensionality of a given symmetry type) $=5$; for ^{40}Ca and ^{48}Ca, $b=1.9$ fm, $D=P$ (number of occupied levels of a given symmetry type)$+3$; for ^{208}Pb, $b=2.2$ fm, $D=P+2$. In all cases the maximum relative l is 3 and the prescription $e_s^{(0)}$, Eq. (5), is used.

Nucleus	$-H_0/A$ (MeV)	r_n (fm)	r_c (fm)
^{16}O	3.84 (7.98)	2.46	2.61 (2.75)
^{40}Ca	4.20 (8.55)	2.97	3.12 (3.50)
^{48}Ca	3.73 (8.67)	3.24	3.14 (3.49)
^{208}Pb	2.82 (7.87)	4.74	4.60 (5.49)

[a] Reference 26.
[b] Reference 27.

IV. RESULTS AND DISCUSSION

The main results on binding energies and radii are given in Table I.[26,27] All nuclei come out too small (most likely because the saturating effect of the tensor force has not been treated properly in Ref. 23) and underbound. Presumably this might have been predicted if nuclear-matter calculations had been done with the same type of G matrix, but they have not been. As for nuclear-matter calculations using other types of G matrices also based on potentials derived from the two-body data, none of those published at the time of this writing has succeeded in yielding the desired binding energy and density; hence it is not surprising that the finite nuclei calculations fail also. It simply means that the search for the "true" two-nucleon potential and for the best Brueckner-type theory must go on. Very encouraging results have been reported recently[28] with the Reid soft-core potential.[20] In the future, when a new two-nucleon potential is being checked for agreement with experimental data on many-nucleon systems, it would be desirable to calculate both nuclear matter (in the usual way) and finite nuclei (by the present method).

We are excluding from the above considerations the "smooth potentials," one of which[29] has given rather good agreement[16] with the experimental nuclear radii. This agreement was built into the potential from the start by fitting the properties of nuclear matter. Although such potentials have been found satisfactory in almost all respects in nuclear calculations so far,[30] they do not have the strong short-range repulsion which theory seems to suggest, and therefore it is expected that, some time in the future, they are bound to fail.

Since this paper does not claim to obtain completely satisfactory shell-model parameters, its main interest lies in the method used and in the fact that the calculations can be repeated with relative ease whenever a

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new two-body force looks hopeful. With a view to illuminating the method, we shall discuss briefly some questions of convergence.

The convergence with dimensionality is critical for large nuclei such as ^{208}Pb, for which calculations of high-dimensionality are prohibitively long. The rapidity of this convergence is, in turn, very dependent on the choice of the harmonic-oscillator length b. We demonstrate this behavior for the light nucleus ^{16}O. Figures 1 and 2 show the convergence with dimensionality of the total energy and the proton radius in ^{16}O for various b. A similar graph for the kinetic energy would look very much like that for r_p. The best convergence is obtained with $b=1.7$ fm. This is close to the b value that would give the best results with pure HOWFs. As a rule, BHF calculations should be done with the b that yields the right size with pure HOWFs, because this also simplifies the treatment of the Pauli operator and the particle spectrum in the calculation of G. This is different from the rule followed in HF calculations, where the best b was that which minimized H_0. If the HF rule were followed in the present case, one would find with dimensionality 3, for instance, that the b at minimum is 2.1 fm, for which Figs. 1 and 2 show the convergence to be much slower. Such a mixtake could be fatal in ^{208}Pb. The HF rule should not be used, since the minimum principle does not hold in Brueckner theory.

Once b has been chosen in the way just said, the convergence with dimensionality is about as good as with smooth potentials.[16]

Note also in Fig. 1 that, although H_0 converges satisfactorily with dimensionality, the limiting value depends on b. This is due to the intrinsic b dependence of the G matrix and would be eliminated if Brueckner self-consistency were properly taken into account in the intermediate state propagator. It is one of the reasons for using the b which is best with pure HOWFs. It can be corrected for by perturbation theory after the self-consistent part of the calculation has been done, but this is not worth doing at the moment.

The convergence with relative l is quite comparable to that found with the smooth potentials.[16] All calculations reported here stop at relative f waves, and no perceptible change of results is obtained by going further.

Table II gives the single-particle energies of the four nuclei studied. The most unsatisfactory ones are the $i_{13/2}(n)$ and $h_{11/2}(p)$ of ^{208}Pb, which should lie among the levels of opposite parity belonging to the harmonic-oscillator shell below them. Instead, these two levels are about 8 MeV higher. The same phenomenon occurred with the smooth potential of Ref. 16, and it was predicted there that the discrepancy would persist in BHF calculations. Its resolution must involve higher-order processes or a better treatment of the Pauli operator in the G matrix.

The calculations are fairly insensitive to the exact choice of e_s in Eq. (4). Prescriptions for e_s other than

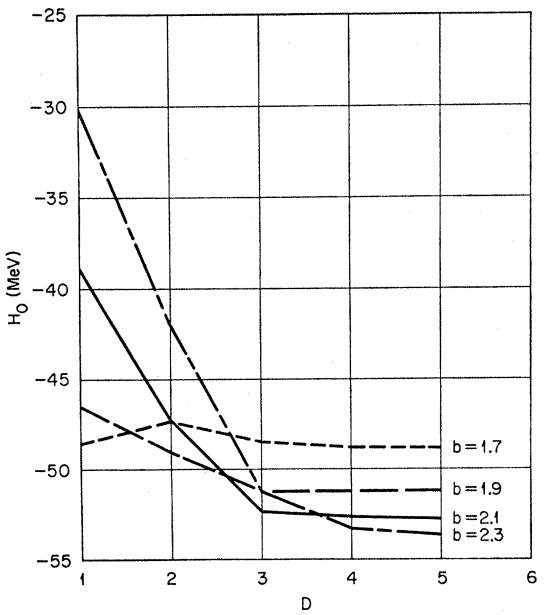

FIG. 1. ^{16}O total energy H_0 as a function of dimensionality D, for various b values. The maximum relative l is 2.

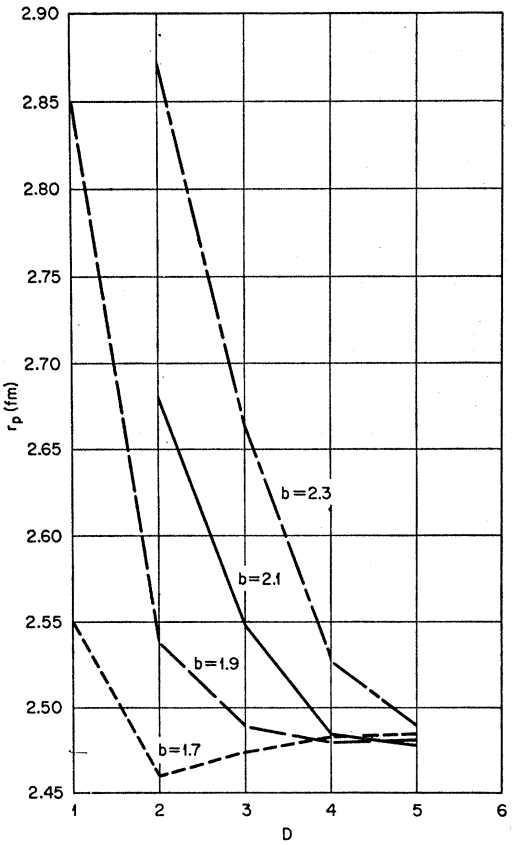

FIG. 2. ^{16}O proton radius r_p as a function of dimensionality D, for various b values. The maximum relative l is 2.

TABLE II. Single-particle energies, in MeV. For the unoccupied symmetry types in ^{16}O, ^{40}Ca, and ^{48}Ca, the parameters $D=4$ and $\epsilon_x'=-20$ MeV are chosen; all other parameters are the same as for Table I.

Nucleus	^{16}O		^{40}Ca		^{48}Ca		^{208}Pb	
States	Neutrons	Protons	Neutrons	Protons	Neutrons	Protons	Neutrons	Protons
$0s_{1/2}$	−43.8	−40.4	−66.1	−58.2	−67.9	−62.9	−91.6	−75.4
$0p_{3/2}$	−20.3	−17.0	−44.7	−37.0	−46.8	−43.1	−78.6	−63.3
$0p_{1/2}$	−15.9	−12.6	−41.1	−33.5	−45.3	−41.3	−78.0	−62.8
$0d_{5/2}$	0.43	3.8	−23.7	−16.2	−25.6	−22.6	−64.7	−50.0
$0d_{3/2}$	4.6	7.6	−17.7	−10.5	−22.8	−19.8	−64.0	−49.4
$1s_{1/2}$	0.95	3.8	−20.5	−13.1	−22.8	−19.1	−62.1	−47.1
$0f_{7/2}$			− 3.2	4.5	− 6.4	− 1.3	−50.0	−35.8
$0f_{5/2}$			3.4	10.2	− 0.37	3.0	−49.1	−35.0
$1p_{3/2}$			− 0.22	6.2	− 1.5	3.0	−46.1	−31.6
$1p_{1/2}$			1.1	7.3	− 0.43	4.1	−45.3	−31.0
$0g_{9/2}$							−34.9	−21.2
$0g_{7/2}$							−33.4	−19.8
$1d_{5/2}$							−30.5	−16.3
$1d_{3/2}$							−28.9	−15.0
$2s_{1/2}$							−28.5	−14.1
$0h_{11/2}$							−20.0	− 6.5
$0h_{9/2}$							−17.4	
$1f_{7/2}$							−15.6	
$1f_{5/2}$							−13.2	
$2p_{3/2}$							−13.1	
$2p_{1/2}$							−12.0	
$0i_{13/2}$							− 5.7	

(5) can easily be used, e.g.,

$e_s^{(+)}$ = the highest occupied level of symmetry type s, (18)

$e_s^{(-)}$ = the lowest occupied level of symmetry type s, (19)

$e_s^{(w)}$ = a constant w for all symmetry types. (20)

In calculations of ^{40}Ca and ^{208}Pb, we found that $e_s^{(0)}$, $e_s^{(+)}$, $e_s^{(-)}$, and $e_s^{(w)}$ (with a reasonable choice of w) all gave nearly identical results for the total energy, the rms radii, and the SP levels. For ^{208}Pb, the most extreme case, the $0s_{1/2}$ neutron and proton levels only changed by about 0.6 MeV in going from prescription (18) to (19); the other SP levels changed even less and the total energy and radii changed by less than 1%.

TABLE III. Single-particle energies for the unoccupied symmetry types in ^{16}O, ^{40}Ca, and ^{48}Ca. The table shows the sensitivity to ϵ_x'. All other parameters are the same as for Tables I and II; ϵ_x' and the SP energies are in MeV.

^{16}O	$\epsilon_x'=-20$	$\epsilon_x'=-60$
$0d_{5/2}(n)$	0.43	2.26
$0d_{3/2}(n)$	4.59	5.78
$0d_{5/2}(p)$	3.81	5.49
$0d_{3/2}(p)$	7.58	8.63
^{40}Ca	$\epsilon_x'=-20$	$\epsilon_x'=-80$
$0f_{7/2}(n)$	−3.19	1.44
$0f_{5/2}(n)$	3.41	6.49
$0f_{7/2}(p)$	4.52	8.54
$0f_{5/2}(p)$	10.2	12.5
^{48}Ca	$\epsilon_x'=-20$	$\epsilon_x'=-80$
$0f_{5/2}(n)$	−0.37	4.02
$0f_{7/2}(p)$	−1.28	4.73
$0f_{5/2}(p)$	2.99	8.00

Once the BHF calculation has been done, it is also possible to calculate single-particle energies for completely unfilled symmetry types. This is done with Eq. (3) in which, however, we need a prescription for ϵ_x'. This prescription can make an appreciable difference, as shown in Table III. The lower ϵ_x', i.e., the more off-the-energy-shell one goes, the higher the excited state.

Finally, it is possible to study the effect on the BHF results of translating the entire particle spectrum up or down by a fixed amount in the solution of the Bethe-Goldstone equation.[11] Ideally, this equation should be solved with the particle spectrum that comes out of the BHF calculation. This has not yet been done, however, first because it is more difficult, but mostly because of the still somewhat unsettled controversy[25] regarding the particle spectrum. The present G matrix[23] uses free-particle intermediate states, with no allowance made for the attraction exerted by the nucleus on the low-lying particle states. Although an over-all downward translation of the spectrum is certainly not the best thing to do, it does allow for this attraction to some extent and it can be done easily.[31] All that is

[31] An alternative and better way to allow for this attraction would be to calculate G with a Pauli operator excluding a range of "valence states" in the vicinity of the Fermi surface. (See Ref. 5.) Then, after the BHF calculation, the effects of these valence states could be included by doing a shell-model calculation. The drawback of this method is that it makes the entire calculation considerably longer. It seems preferable, for a start, to get as close as possible to the complete answer with the BHF calculation alone. Note, however, that our G matrix, which uses free particle intermediate states and therefore large energy denominators, has many of the characteristics to be expected from excluding such a valence shell; thus such a shell-model calculation may be needed in any case.

needed is to define a new G matrix

$$G'(\omega)=G(\omega-2c), \tag{21}$$

where c is the amount of the constant translation (negative if downward). This lowering of the spectrum makes G more attractive and results in a lowering of the total energy, the SP levels, and a further decrease in radius. Köhler[32] and Wong[13] have given the rough rule that lowering the particle spectrum by 10 MeV increases the binding per nucleon by 1 MeV. By chance' this is exactly the figure that comes out of our calculations also.

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Nonmesonic Decay of Hydrogen Hyperfragments

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Approximately 35 000 K^--capture stars in an emulsion stack were examined for hyperfragments. Using a mass selection criterion for all events, a sample of 13 π^--mesonic decays and one nonmesonic decay of hydrogen hyperfragments was identified for the purpose of estimating the relative rates of decay for the π^--mesonic and nonmesonic modes.

I. INTRODUCTION

THE study of nonmesonic decays of hyperfragments has been limited by the difficulty in identifying the decaying hyperfragments. This is particularly the case for ${}_\Lambda$H, where first the charge of the particle must be determined by profile measurements to separate the $Z=1$ events from those with $Z\geq 2$. In order to separate the ${}_\Lambda$H events from lighter $Z=1$ particles, including a large number of Σ^- capture events, gap or blob measurements must be performed. Although a number of investigations have been carried out on the nonmesonic decays of various hypernuclei, no direct measurement of the nonmesonic decay of ${}_\Lambda$H has been done previously.

In this work, an attempt has been made to directly identify nonmesonic decays of ${}_\Lambda$H and estimate the nonmesonic-to-π^--mesonic decay ratio for ${}_\Lambda$H[4].

II. EXPERIMENTAL PROCEDURE

A. Scanning and Emulsion Stack Calibration

The scanning technique and the calibration of the emulsion stack have been described in a previous paper.[1] In both the previous and the present work, only the 45 interior pellicles from a stack of 100 KTB-5 emulsion pellicles (15 cm×10 cm×0.07 cm) were area-scanned. Approximately 35 000 K^--capture stars (or primary stars) were observed and recorded.

The same range-energy calibration correction has again been used on all measurements made in the present work. Also, the range and angle measurements and the kinematic analysis of the mesonic decays of ${}_\Lambda$H are the result of the same techniques as described previously.

B. Mesonic Decays of ${}_\Lambda$H

Of the approximately 300 mesonic hyperfragment decays found in the emulsion plates scanned, 72 mesonic decays of ${}_\Lambda$H[3,4] were identified. From these a sample of 13 hydrogen hyperfragments was selected whose hyperfragment prong had a projected range $R_{HF}\geq 350\,\mu$, and a dip angle $|\alpha|\leq 30°$. Since gap-interval measurements were used in the identification of the nonmesonic

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