Inverse Scattering Problem for Separable Potentials*

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(Received 22 July 1968)

A method for constructing a separable potential directly from the two-nucleon phase shifts is presented. The solution of the inverse scattering problem is extended to include both a long-range attraction and the important strong, short-range repulsion. The occurrence of both an attraction and a strong repulsion in a single separable potential is shown to require that the phase shifts satisfy a modified Levinson theorem. Construction of a noncentral separable interaction directly from known eigenphase shifts and coupling parameters is also discussed. Using the construction procedure described in this paper, the task of repeated phase-shift fitting is avoided, and one can more easily test the sensitivity of nuclear structure calculations to various aspects of the two-nucleon interaction.

I. INTRODUCTION

IN nuclear structure theory it is assumed that the properties of nuclei can be understood and derived from the basic nucleon-nucleon interaction. This interaction is characterized by the eigenphase shifts and coupling parameters, which are now well known up to 350 MeV.[1] To solve the many-nucleon problem the customary first step is to construct a nonrelativistic potential, whose parameters are adjusted to fit these phase shifts. Although it is possible to incorporate the more dependable predictions of meson theory into the potential, one cannot give a unique form because the interaction is necessarily quite complicated. Indeed, several potentials are available that represent the two-nucleon data with various forms[2] and with different precision.[3] Each potential includes not only a phase shift fit, but also assumptions about the two-body wave function (i.e., the off-energy-shell behavior of the interaction). Since one cannot deduce a unique potential from two-nucleon experiments, it is difficult to know whether the predictions of a given nuclear structure calculation follow from the nature of the interaction or from the calculational method. Thus we confront the dilemma of simultaneously testing the nuclear force and the calculation methods.

Recently, great progress has been made in developing valid methods for calculating the binding energy and spectra of nuclei with several kinds of potentials.[4] However, it has not yet been determined how sensitive these predictions are to the two-body force. The main difficulty in determining this sensitivity is that one must face the task of repeatedly fitting the two-nucleon data for a wide variety of potentials.

In this paper, a method for constructing a simple separable potential directly from the phase shifts is discussed. One advantage of solving the inverse scattering problem and using the phase shifts themselves to construct a potential is that phase shift fitting is avoided. Another advantage is that one can test the sensitivity of nuclear calculations to the phase shifts for $E<350$ MeV and to the assumed phase shift behavior for higher energies, $E>350$ MeV. The phase shifts for $E<350$ MeV are mostly well determined, but there are uncertainties; for example, the $^1P_1$ phase shifts are poorly known. It is therefore important to determine the influence of such uncertainties upon calculational results. One way of determining the importance of phase shift uncertainties is to use a variety of potentials that are constructed directly from the phase shifts including their errors.

At higher energies, above the meson production threshold, the phase shifts are essentially unknown. Solving the inverse scattering problem is again advantageous in that one can use a variety of high-energy

* Research supported in part by the National Science Foundation.

[1] G. Breit, Rev. Mod. Phys. **39**, 560 (1967); R. E. Seamon, K. A. Friedman, G. Breit, R. D. Haracz, J. M. Holt, and A. Prakash, Phys. Rev. **165**, 1579 (1968); M. H. MacGregor, R. A. Arndt, and R. M. Wright, *ibid.* **169**, 1128 (1968).

[2] A selected list of various types of potentials is: J. A. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957); T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962); K. E. Lassila, M. H. Hull, H. M. Ruppel, R. A. McDonald, and G. Breit, Phys. Rev. **126**, 882 (1962); E. Lomon and H. Feshbach, Rev. Mod. Phys. **39**, 611 (1967); R. V. Reid, Ph.D. Thesis, Cornell University, 1967 (unpublished); A. Scotti and D. Y. Wong, Phys. Rev. **138**, B145 (1965); R. A. Bryan and B. L. Scott, *ibid.* **164**, 1215 (1967); S. Ogawa, S. Sawada, T. Ueda, W. Watari, and M. Yonezawa, Progr. Theoret. Phys. (Kyoto) Suppl. **39**, 140 (1967); G. Darewych and A. E. S. Green, Phys. Rev. **164**, 1324 (1967); A. M. Green, Nucl. Phys. **33**, 218 (1962); F. Tabakin and K. T. R. Davies, Phys. Rev. **150**, 793 (1966); and C. W. Nestor *et al.*, Nucl. Phys. **A113**, 14 (1968). See Ref. 6 for separable nucleon-nucleon potentials. Recent developments are presented in Rev. Mod. Phys. **39**, No. 3 (1967); and in Progr. Theoret. Phys. (Kyoto) Suppl **39**, (1967).

[3] H. P. Noyes, P. Signell, N. R. Yoder, and R. M. Wright, Phys. Rev. **159**, 789 (1967).

[4] M. Baranger, 1967 Varenna Lectures (to be published); M. MacFarlane, 1967 Varenna Lectures (to be published); R. Rajaraman and H. A. Bethe, Rev. Mod. Phys. **39**, 745 (1967); B. H. Brandow, *ibid.* **39**, 771 (1967); D. W. L. Sprung and P. C. Bhargava, Ann. Phys. (N.Y.) **42**, 222 (1967); C. W. Wong, Nucl. Phys. **A104**, 417 (1967); R. J. McCarthy and H. S. Köhler, *ibid.* **A106**, 313 (1967); T. T. S. Kuo and G. E. Brown, *ibid.* **85**, 40 (1966); **A114**, 241 (1968); D. M. Clement and E. U. Baranger, *ibid.* **89**, 145 (1966); B. Barrett, Phys. Rev. **154**, 955 (1967); A. K. Kerman and M. K. Pal, *ibid.* **162**, 970 (1967); W. H. Bassichis, B. A. Pohl, and A. K. Kerman, Nucl. Phys. **A112**, 360 (1968); K. T. R. Davies, S. J. Krieger, and M. Baranger, *ibid.* **84**, 545 (1966); A. K. Kerman, J. P. Svenne, and F. M. H. Villars, Phys. Rev. **147**, 710 (1966); M. Gmitro, J. Hendekovic, and J. Sawicki, *ibid.* **169**, 983 (1968).

phase shifts to alter and control the magnitude of potential matrix elements off-the-energy-shell. The effect of varying these off-energy-shell matrix elements upon the calculation of nuclear binding energies and spectra can then be studied.

The construction of a separable potential directly from the phase shifts is not a new idea.[5] The emphasis here is that the idea is a useful one, especially when generalized to include tensor forces and the effects of a strong, short-range repulsion, which are very important for nuclear saturation. It is important to remember, however, that the internucleon potential is not a separable interaction, since it is known to approach the local, one-pion-exchange potential at larger distances. One of the major, and unproved, assumptions of this paper is that by including the effects of a strong, short-range repulsion and the tensor force one can closely simulate the real interaction using separable forms. If this assumption is true, then the use of separable potentials constructed from the phase shifts offers many practical advantages.

In Sec. II it is shown how to construct a central separable potential directly from the phase shifts for a purely attractive or purely repulsive force. This simplified solution of the inverse scattering problem is then generalized to include both an attraction and the effects of a strong, short-range repulsion in a single separable potential (Sec. III). The method used to introduce the strong repulsion requires that the phase shifts obey a modified Levinson theorem, which is derived in Sec. IV. The extension of the solution to include noncentral interactions is discussed in Sec. V.

II. CENTRAL SEPARABLE POTENTIAL

The method of constructing a separable potential directly from the phase shifts is best understood by considering the case of a central potential. In momentum space, the separable potential[6] for two particles with relative orbital angular momentum L is

$$v_L(\mathbf{k}\mid\mathbf{k}')=\sigma_L(\hbar^2/M)\,(1/2\pi^2)\times(2L+1)\,g_L(k)\,g_L(k')\,P_L(\hat{k}\cdot\hat{k}'),\quad(1)$$

where M is the nucleon mass, $2\hbar k=\hbar\,|\,\mathbf{k}_1-\mathbf{k}_2\,|$ and $2\hbar k'=\hbar\,|\,\mathbf{k}_1'-\mathbf{k}_2'\,|$ are relative momenta, and P_L is a Lengendre polynomial. The sign factor $\sigma_L=-1$ for an attraction and $\sigma_L=+1$ for a repulsion. The potential function $g_L(k)$ is to be determined by the phase shift $\delta_L(k)$, which is assumed known for all energies $E(\mathrm{Rel})=\frac{1}{2}E(\mathrm{Lab})=(\hbar^2/M)k^2$. For this potential, one can solve the Schrödinger equation for the outgoing scattering wave of relative motion

$$\psi_L(r)=j_L(kr)-\frac{2}{\pi}\int_0^\infty\frac{d\kappa\,\kappa^2 j_L(\kappa r)}{\kappa^2-k^2-i\epsilon}\,T_L(\kappa\mid k),\quad(2a)$$

where the off-energy-shell T matrix is

$$T_L(k\mid k')=\sigma_L[g_L(k)\,g_L(k')/D_L^{(+)}(k')].\quad(2b)$$

The asymptotic wave function and the phase shifts are determined by the on-energy-shell T matrix

$$kT_L(k\mid k)=-\exp[i\delta_L(k)]\sin\delta_L(k)=\sigma_L k[g_L^2(k)/D_L^{(+)}(k)].\quad(3)$$

Here $j_L(kr)$ is a spherical Bessel function and $D_L(z)$ is defined by

$$D_L(z)=1+\sigma_L G_L(z)=1+\sigma_L\frac{2}{\pi}P\int_0^\infty\frac{d\kappa\,\kappa^2}{\kappa^2-z^2}\,g_L^2(\kappa).\quad(4a)$$

The function $D_L^{(\pm)}(k)$ is then given by

$$D_L^{(\pm)}(k)=D_L(k\pm i\epsilon)=D_L^{(\mp)}(-k).\quad(4b)$$

One can therefore write $G_L^{\pm}(k)$ as

$$G_L^{\pm}(k)=\frac{2}{\pi}\int_0^\infty\frac{d\kappa\,\kappa^2 g_L^2(\kappa)}{\kappa^2-k^2\mp 2i\epsilon k}=G_L(k)\pm ikg_L^2(k),\quad(5)$$

where $G_L(k)$ is the corresponding principal value integral. We use z to denote a complex variable and k to denote a real variable. Knowledge of the analytic properties of $D_L(z)$ will provide us with a solution of the inversion problem for Eq. (3), i.e., to find $g_L(k)$ from the phase shifts $\delta_L(k)$. Therefore, let us consider D_L as a function of the complex energy variable $\omega=z^2$:

$$D_L(\omega)=1+\frac{2}{\pi}\sigma_L\int_0^\infty\frac{d\kappa\,\kappa^2 g_L^2(\kappa)}{\kappa^2-\omega}.\quad(6)$$

This function has simple and useful analytic properties in the complex ω plane. For example, the function $D_L(\omega)$ has the property

$$D_L(k^2+i\epsilon)-D_L(k^2-i\epsilon)=2ik\sigma_L g_L^2(k),\quad(7)$$

which shows that $D_L(\omega)$ has a branch cut on the real ω axis. For a single separable interaction, Eq. (7) relates the discontinuity across this cut to the potential

[5] M. Bolsterli and J. MacKenzie, Physics **2**, 141 (1965). After completion of this manuscript the following important references were called to the author's attention: M. Gourdin and A. Martin, Nuovo Cimento **6**, 757 (1957); **8**, 699 (1958); A. Martin, *ibid.* **7**, 607 (1958); K. Chadan, *ibid.* **10**, 892 (1958); **47A**, 510 (1967). These papers contain complete, and quite mathematical, treatments of the inverse scattering problems for separable potentials. General discussions of the inverse scattering problem are also found in Ref. 7 and in R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952), and V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949).

[6] Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954); A. M. Mitra, Nucl. Phys. **32**, 529 (1962); Phys. Rev. **123**, 1892 (1961); F. Tabakin, Ann. Phys. (N.Y.) **30**, 51 (1964); J. H. Naqvi, Nucl. Phys. **58**, 289 (1964); **A103**, 565 (1967); Th. Hammann, G. Oberlechner, G. Trapp, and J. Yoccoz, J. Phys. (Paris) **28**, 755 (1967); T. R. Mongan, Phys. Rev. **175**, 1260 (1968).

function

$$\sigma_L k g_L^2(k) = \mathrm{Im} D_L^{(+)}(k). \tag{8}$$

This equation will be very useful later.

Let us consider the additional properties of $D_L(\omega)$. For an attraction ($\sigma_L = -1$), a separable potential gives a bound state if

$$\frac{2}{\pi}\int_0^\infty \frac{d\kappa\, \kappa^2 g_L^2(\kappa)}{\kappa^2 + k_B^2} = 1. \tag{9}$$

This bound-state condition follows immediately from the Schrödinger equation and shows that a bound state will occur at the energy $\omega = -k_B^2$, provided $D_L(\omega)$ has a zero at this negative energy. Since the left-hand side of Eq. (9) is a monotonically decreasing function of k_B, only *one* bound-state solution may occur for a single separable interaction. Thus, $D_L(\omega)$ has at most a single, simple zero at a negative, real value of ω.

In addition to the bound-state zero and the right-hand cut, $D(\omega)$ can have additional "potential" singularities on the second, or unphysical, Riemann sheet of the cut ω plane. These singularities on the second sheet ($\mathrm{Im} z < 0$) can be quite complicated and depend sensitively on the interaction. To discuss these singularities it is more convenient to use the complex variable $z = \sqrt{\omega}$, which maps the first (physical) sheet of the cut ω plane into the upper ($\mathrm{Im} z > 0$) half-plane, while the second sheet is mapped into the lower ($\mathrm{Im} z < 0$) half-plane. *In this complex z plane, $D_L(z)$ is an analytic function in the upper plane with a possible bound-state zero at $z = ik_B$.*[7]

The "potential" singularities in the lower plane can be quite intricate. For example, a local Yukawa potential of range a^{-1} gives a potential cut from $-ia$ to $-i\infty$ and a strip $-ia < \mathrm{Im} z < 0$, where $D_L(z)$ is analytic. Also, a separable potential of the Yamuguchi form, $g(k) = \alpha/(k^2+a^2)$, gives a double pole at $-ia$, whereas $g(k) = \alpha/(k^2+a^2)^{1/2}$ produces a simple pole at $-ia$. Mitra[6] uses $g_0(k) = \alpha k^{-2} \ln[1+(k^2/a^2)]$, which gives a Yukawa-type cut from $-ia$ to $-i\infty$. (Another example of the singularities possible in the lower plane is generated by

$$g(k) = \{Bk^2/[(k-d)^2+b^2][(k+d)^2+b^2]\}, \tag{10}$$

which produces poles at $k = -ib \pm d$.[6]) These examples illustrate that for a general function $g(k)$ the singularities in the lower plane can be complicated. *Fortunately, we do not need to know these potential singularities in order to solve the inversion problem for separable potentials.* The only information required is that $D_L(z)$ is analytic in the upper half-plane, with a possible bound-state zero at $z = +ik_B$. General proofs of the analytic properties of $D_L(z)$, which are illustrated in Fig. 1, are given in Ref. 7. For convenience let us follow the usual[7] nomenclature and call $D_L(z)$ a Fredholm determinant.

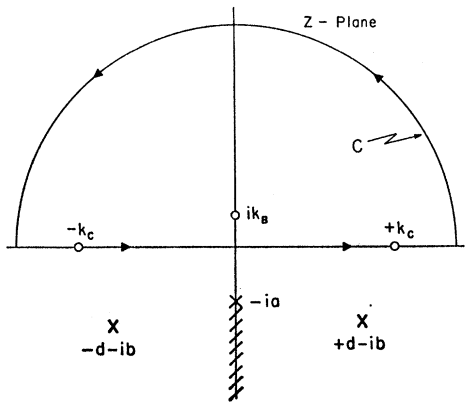

FIG. 1. The analytic properties of $D_L(z)$ in the complex z plane. The Fredholm determinant $D_L(z)$ has zeros at $z = +ik_B$ for a bound state and at $z = \pm k_c$ to produce a strong repulsion [Eq. (22)]. The complicated singularities in the lower half-plane ($\mathrm{Im} z < 0$) are illustrated by a potential cut of the Yukawa type from $-ia$ to $-i\infty$ and by the complex poles at $\pm d - ib$. The contour C encloses the upper half-plane, where $D_L(z)$ is analytic.

Our task is to use the known analytic properties of $D_L(z)$ to solve the inversion problem for separable potentials. For this purpose, consider the quantity

$$\ln \mathfrak{D}_L(z) = \ln\{[(z+ik_B)/(z-ik_B)] D_L(z)\}. \tag{11}$$

From our previous discussion, we know that $\ln \mathfrak{D}_L(z)$ is analytic in the upper half-plane, *provided the only zero of D_L is the bound-state zero.* Furthermore, we see from Eqs. (4) and (5) that $D_L(z)$ approaches 1 for large values of z; consequently, $\ln \mathfrak{D}_L(z)$ vanishes on the infinite semicircle for $\mathrm{Im} z > 0$ (see Fig. 1).

The above properties of $\ln \mathfrak{D}_L$ and the assumption of no additional $D_L(z)$ zeros suffice for the application of Cauchy's theorem

$$\ln \mathfrak{D}_L(z) = (2\pi i)^{-1} \int_C \frac{\ln \mathfrak{D}_L(z')\, dz'}{z'-z}$$

$$= (2\pi i)^{-1} \int_{-\infty}^{\infty} \frac{dk' \ln \mathfrak{D}_L^{(+)}(k')}{k'-z}, \tag{12}$$

where the contour C is given in Fig. 1. Taking the real part of this identity, one finds that for $z \to k + i\epsilon$ (k real)

$$\mathrm{Re} \ln \mathfrak{D}_L^{(+)}(k) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{dk'\, \mathrm{Im} \ln \mathfrak{D}_L^{(+)}(k')}{k'-k}. \tag{13}$$

To use Eq. (13), one must first note that the S matrix can be written as

$$S_L(k) = 1 - 2ikT_L(k) = \exp[2i\delta_L(k)]$$

$$= D_L^{(-)}(k)/D_L^{(+)}(k), \tag{14}$$

[7] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966). The modified Levinson theorem is also proved by A. Martin, Nuovo Cimento **7**, 607 (1958), and M. Bertero *et al.*, Nucl. Phys. **A113**, 625 (1968).

where Eqs. (3)–(7) were used. We can also write

$$\tan\delta_L(k) = -\mathrm{Im}D_L^{(+)}(k)/\mathrm{Re}D_L^{(+)}(k). \quad (15)$$

It follows that the phase of the Fredholm determinant is simply $-\delta_L(k)$, i.e.,

$$D_L^{(+)}(k) = |D_L| \exp[-i\delta_L(k)]. \quad (16)$$

Now using Eq. (16), we find

$$\ln\mathfrak{D}_L^{(+)}(k) = \ln|D_L| - i\delta_L(k) + \ln\left(\frac{k+ik_B}{k-ik_B}\right). \quad (17)$$

Combining Eqs. (13) and (17), we have

$$\ln|D_L| = -\Delta_L(k) + \frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-k}\ln\left(\frac{k'+ik_B}{k'-ik_B}\right), \quad (18)$$

where $\Delta_L(k)$ is the principal value integral

$$\Delta_L(k) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{dk'\delta_L(k')}{k'-k}. \quad (19)$$

The second principal value integration in Eq. (18) can be completed by the method described in the Appendix. The Fredholm determinant is found to be

$$D_L^{(+)}(k) = \left(\frac{k^2+k_B^2}{k^2}\right)\exp\{-[\Delta_L(k)+i\delta_L(k)]\}. \quad (20)$$

At this point Eq. (8) is very useful and we find that the potential function is

$$-\sigma_L g_L^2(k) = \left(\frac{k^2+k_B^2}{k^2}\right)^{(1-\sigma_L)/2}\frac{\sin\delta_L(k)}{k}\exp[-\Delta_L(k)], \quad (21)$$

with $\sigma_L=-1$ for an attraction and $\sigma_L=+1$ for a repulsion. Equation (21) defines a separable potential directly from its phase shifts and bound-state energy.

For a purely attractive potential we have $\sigma_L=-1$, and the phase shift is positive, $\delta_L>0$. One might conclude from Eq. (21) that it is impossible for a single separable potential to produce both an attraction and the short-range repulsion required by the $^1S_0$ two-nucleon phase shifts. However, in Sec. III, we show how to generalize Eq. (21) to include both an attraction and a repulsion in a single separable potential.

III. INVERSION PROBLEM WITH A REPULSION

The solution of the inversion problem presented in Sec. II applies to a purely attractive or purely repulsive separable potential. However, the two-nucleon interaction is known to have a long-range attraction and a short-range repulsion; for example, the $^1S_0$ and $^3P_0$ phase shifts become repulsive at high energies.[1] One must include this repulsion since it does appear in the interaction and because it plays a decisive role in determining nuclear binding energies and spectra. Let us now consider how one may introduce both an attraction and a repulsion with a single separable potential.

The essential assumption needed to introduce a short-range repulsion is that the Ferdholm determinant has additional, simple zeros and $z=\pm k_c$, where $E_c=2(\hbar^2/M)k_c^2$ is the laboratory energy at which $\tan\delta_L(k)$ changes sign. It is completely consistent with the general analytic properties of $D_L(z)$ to assume that

$$D_L^{(+)}(\pm k_c) = 0. \quad (22)$$

This assumption is equivalent to the requirements

$$g_L(k_c) = 0 \quad (23a)$$

and

$$G_L(k_c) = \frac{2P}{\pi}\int_0^{\infty}\frac{d\kappa\,\kappa^2 g_L^2(k)}{\kappa^2-k_c^2} = 1, \quad (23b)$$

which have been used successfully in a numerical example.[8] (We have now set $\sigma_L=-1$.)

In this numerical example[8] it was shown that the above condition [Eq. (22)] introduces an extra node in the scattering wave function at $k_c^{-1}\simeq 0.45$ fm. The extra node at short distances arises from the fact that Eq. (22) is the condition for the occurrence of a bound state at the *positive* energy E_c. The extra zero in the wave function has the effect of reducing the integrated probability of finding the nucleons at short distances. This average reduction in the wave function at short distances is the sense in which Eq. (22) introduces the effects of a strong short-range repulsion.

To see that Eq. (22) introduces a sign change in $\tan\delta_L$, consider Eqs. (5), (8), and (15), which give the phase shift

$$\tan\delta_L(k) = kg_L^2(k)/[1-G_L(k)] = kg_L^2(k)\Big/\left((k_c^2-k^2)\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{d\kappa\,\kappa^2 g_L^2(k)}{(\kappa^2-k_c^2)(\kappa^2-k^2)}\right). \quad (24)$$

Since $g_L(k)$ is assumed to have simple zeros at $k=\pm k_c$, we can write $g_L(k)=(k_c^2-k^2)h_L(k)$, and hence

$$\tan\delta_L(k) = (k_c^2-k^2) \times kh_L^2(k)\Big/\left(\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{d\kappa\,\kappa^2 g_L^2(\kappa)}{(\kappa^2-k_c^2)(\kappa^2-k^2)}\right). \quad (25)$$

The integral in Eq. (25) is positive at $k=k_c$ and we see that $\tan\delta_L(k)$ changes sign at the energy E_c; this sign change corresponds to a repulsion. Therefore, Eq. (22) provides a means of including both an attraction and a repulsion in a single separable potential. Of course, we could apply this condition [Eq. (22)] to a

[8] F. Tabakin, Phys. Rev. **174**, 1208 (1968).

set of N_L points $k_{c1}, k_{c2}, \cdots, k_{cN_L}$ and thereby generate N_L sign changes in $\tan\delta_L$.

The possibility of $D_L(z)$ having zeros on the real axis is usually rejected because it is thought that the wave function would become either unbounded or zero.[7] However, in the present case the ratio $g_L(k_c)/D_L^{(+)}(k_c)$ is nonzero and bounded; correspondingly, the wave function given by Eq. (2) is well behaved at all energies. Hence, it is permissible for $D_L(z)$ to have zeros on the real axis.

The introduction of additional zeros in $D_L(z)$ requires that we reconsider the previous solution of the inversion problem. Instead of $\mathfrak{D}_L(z)$ given by Eq. (11), we must now define

$$\mathfrak{D}_L(z)=\left(\frac{z^2+\Lambda^2}{z^2-k_c^2}\frac{z+i\Lambda}{z-i\Lambda}\right)\left(\frac{z+ik_B}{z-ik_B}\right)D_L(z), \quad (26)$$

where the possibility of a bound state is included, and Λ is an arbitrary real number. The function $\mathfrak{D}_L(z)$ is free of zeros both on the real axis and throughout the upper half-plane. Furthermore, $\mathfrak{D}_L(z)$ is analytic in the upper plane ($\mathrm{Im}z>0$) and approaches 1 [$\mathfrak{D}_L(z\to\infty)\to 1$] on the upper infinite semicircle. Hence, $\ln\mathfrak{D}$ can be used in Cauchy's residue theorem

$$\ln\mathfrak{D}_L(z)=(2\pi i)^{-1}\int_C\frac{dz'}{z'-z}\ln\mathfrak{D}_L(z'), \quad (27)$$

where the contour is the same as before (Fig. 1). From Eq. (27), it follows that, for $z\to k+i\epsilon$,

$$\mathrm{Re}\ln\mathfrak{D}_L^{(+)}(k)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa-k}\mathrm{Im}\ln\mathfrak{D}_L^{(+)}(\kappa). \quad (28)$$

Since $\ln\mathfrak{D}_L^{(+)}$ is given by

$$\ln\mathfrak{D}_L^{(+)}(k)=\ln\left(|D|\frac{k^2+\Lambda^2}{k^2-k_c^2}\right)-i\delta_L+\ln\left(\frac{k+i\Lambda}{k-i\Lambda}\right)+\ln\left(\frac{k+ik_B}{k-ik_B}\right), \quad (29)$$

the Fredholm determinant is now

$$D_L^{(+)}(k)=\left(\frac{k^2-k_c^2}{k^2}\right)\left(\frac{k^2+k_B^2}{k^2}\right)\times\exp[-i\delta_L(k)]\exp[-\Delta_L(k)]. \quad (30)$$

Here $\Delta_L(k)$ is again given by Eq. (19), and the contour integration which is presented in the Appendix has been used twice.

Using Eqs. (8) and (30), we conclude that the potential function given by

$$g_L^2(k)=\left(\frac{k^2-k_c^2}{k^2}\right)\left(\frac{k^2+k_B^2}{k^2}\right)\frac{\sin\delta_L(k)}{k}\exp[-\Delta_L(k)] \quad (31)$$

will produce the phase shifts $\delta_L(k)$. Equation (31) is a solution of the inversion problem for $\sigma_L=-1$, including a bound state of energy $E_B=(\hbar^2/M)k_B^2$ and a repulsion which causes $\tan\delta_L(k)$ to change sign at the energy $E_c=2(\hbar^2/M)k_c^2$. Clearly, the solution given by Eq. (31) is valid only if the product $(k^2-k_c^2)\sin\delta_L(k)$ is positive. This product is indeed positive, provided one uses the modified Levinson theorem described in Sec. IV and also observes that both $\tan\delta_L(k)$ and $\sin\delta_L(k)$ change sign at the energy E_c. Another feature of Eq. (31) is that the arbitrary number Λ does not appear. The potential function $g_L^2(k)$ is thus well defined; it satisfies the condition $g(k_c)=0$ and can be calculated from the phase shifts $\delta_L(k)$ once the phase shifts are given absolute meaning by the modified Levinson theorem (Sec. IV). The potential function for a pure attraction [Eq. (21)] is obtained from Eq. (31) by setting $k_c\to 0$; the case of no bound state is obtained by setting $k_B\to 0$ in either Eqs. (21) or (31).

Our solution of the inversion problem can be extended to the case of $2N_L$ real zeros of $D_L(z)$, with the result that

$$g_L^2(k)=\prod_{i=1}^{N_L}\left(\frac{k^2-k_{ci}^2}{k^2}\right)\left(\frac{k^2+k_B^2}{k^2}\right)^{N_L{}^B}\frac{\sin\delta_L(k)}{k}\times\exp[-\Delta_L(k)]. \quad (32)$$

Recall that a single separable potential permits at most one bound state in each partial wave so that $N_L{}^B=0$ or 1. Equation (32) defines a separable potential that produces N_L sign changes in $\tan\delta_L(k)$ at the energies $E_{ci}=(2\hbar^2/M)k_{ci}^2$.

IV. MODIFIED LEVINSON THEOREM

The solution of the inversion problem for separable potentials, as expressed by Eqs. (31) and (32), is not complete until the phase shifts are given absolute meaning so that $\Delta_L(k)$ and $g_L(k)$ can be evaluated without ambiguity. Let us adopt the usual convention that the asymptotic phase shift is zero, $\delta_L(\infty)=0$. Then the phase shift at zero energy is fixed by the requirement that $\lim_{k\to 0}k^{-L}g_L(k)$ be finite. [Of course, if one is willing to deal with a singular potential, then $\lim_{k\to 0}k^{-L}g_L(k)$ may be unbounded and the zero-energy phase shift will not be determined by the following discussion.]

The potential function $k^{-L}g_L(k)$ might appear to be singular as $k\to 0$ because of the k^{-2} factors in Eq. (31). However, $\Delta_L(0)$ can have a logarithmic singularity which, for the proper choice of $\delta_L(0)$, keeps $k^{-L}g_L(k)$ bounded. The proper value for $\delta_L(0)$ is deduced simply if we consider the relation

$$\frac{|k_x^2-k^2|}{k^2}=\exp\left\{\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa-k}F(\kappa/k_x)\right\}, \quad (33)$$

where $F(k/k_x)$ is defined by

$$F(k/k_x) = -F(-k/k_x) = \pi \quad \text{for} \quad k \leq k_x$$
$$= 0 \quad \text{for} \quad k > k_x \qquad (34)$$

and k_x denotes either k_B or k_{ci}. Using Eq. (33), we can rewrite the potential function [Eq. (32)] for small k as

$$g_L^2(k) = (-1)^{N_L+1} k^{2L} \left(\frac{k_B^2+k^2}{k_B^2-k^2}\right)^{N_L{}^B} a_L(k) \cos\delta_L(k)$$
$$\times \exp\left\{-\frac{P}{\pi}\int_{-\infty}^{\infty} \frac{d\kappa}{\kappa - k} \tilde{\delta}_L(\kappa)\right\}, \qquad (35a)$$

where

$$a_L(k) = -\tan\delta_L(k)/k^{2L+1} \qquad (35b)$$

and

$$\tilde{\delta}_L(k) = \delta_L(k) - N_L{}^B F(k/k_B) - \sum_{i=1}^{N_L} F(k/k_{ci}). \qquad (36)$$

Since $a_L(0)$ is a finite constant (the scattering length), the choice of $\delta_L(0)$ which gives a finite positive limit for $k^{-2L} g_L^2(k)$ is $\delta_L(0) = 0$ or

$$\delta_L(0) - \delta_L(\infty) = (N_L{}^B + N_L)\pi. \qquad (37)$$

Here the number of bound states is $N_L{}^B = 0$ or 1, and the number of nodes in $g_L^2(k)$ is N_L. This choice for $\delta_L(0)$ gives

$$\lim_{k\to 0} k^{-2L} g_L^2(k) = (-1) N_L{}^{B+1} a_L$$
$$\times \exp\left\{-\frac{P}{\pi}\int_{-\infty}^{\infty} \frac{d\kappa}{\kappa} \tilde{\delta}_L(\kappa)\right\}, \qquad (38)$$

which implies that for $a_L > 0$ only one bound state occurs and for $a_L < 0$ there is no bound state. This property agrees with the known $^1S_0$ and $^3S_1$ two-nucleon scattering lengths.

Equation (37) is a modified form of Levinson's theorem, which can also be derived directly from the appearance of additional zeros of $D_L(z)$ on the real axis.[7] In our presentation, it is emphasized that Eq. (37) follows not only from these additional zeros, but also from the requirement that $k^{-L} g_L(k)$ be bounded. The standard Levinson theorem is obtained by setting $N_L = 0$ in Eq. (37), i.e., for a monotonic separable potential the standard Levinson theorem holds.

The modified Levinson theorem [Eq. (37)] is a restriction on the high-energy behavior of the phase shifts which can be handled by the construction procedure. This restriction follows from the manner in which a repulsion has been introduced. Nevertheless, there is still considerable lattitude in the models that one can assume for the high-energy phase shifts. For example, the specific dependence of the phase shifts on energy can be greatly altered within the restriction of the modified Levinson theorem. Also one can vary N_L and control the number of sign changes in $\tan\delta_L(k)$.

A great advantage of the construction procedure offered by Eqs. (1) and (32) is that the off-energy-shell behavior of the interaction is stipulated simply by making appropriate models for the high-energy ($E > 350$ MeV) phase shifts. In this way, the role of off-energy-shell behavior could be studied without the need for repeated phase shift fitting. In Sec. V, this desirable feature of solving the inversion problem is extended to the case of noncentral forces.

V. INVERSION PROBLEM–NONCENTRAL FORCES

The construction of a separable potential directly from the eigenphases and coupling parameters for noncentral forces is a direct generalization of our previous discussion. Let us follow the suggestion of Bolsterli and MacKenzie[5] and introduce a separable potential in the "diagonal representation" of the T matrix.

For noncentral forces, the T matrix is determined by the Lippmann-Schwinger equation for coupled channels

$$T_{LL'}{}^{\alpha}(k/k') = V_{LL'}{}^{\alpha}(k/k')$$
$$- \frac{2}{\pi}\sum_{l}\int_0^{\infty} \frac{d\kappa\, \kappa^2 V_{Ll}{}^{\alpha}(k/\kappa)}{\kappa^2 - k^2 - i\epsilon} T_{lL}{}^{\alpha}(\kappa/k'), \qquad (39)$$

where α denotes JST, the total angular momentum, spin, and isotopic spin of the two-nucleon system. The relative orbital angular momenta $L = J \pm 1$ are coupled in nucleon-nucleon scattering; for example, the $^3S_1 + {}^3D_1$ and $^3P_2 + {}^3F_2$ states are coupled. Matrix elements between states of different parity are, of course, equal to zero: $V_{J\pm 1,J} = 0$. Correspondingly, the $L = L' = J$ states (such as $^3D_2$ and $^3P_1$) are uncoupled and Eq. (39) reduces to Eq. (2). Equation (39) is obtained directly from the Schrödinger equation with outgoing wave boundary conditions and using the partial-wave decompositions[6]

$$\langle \mathbf{k} \mid V \mid \mathbf{k}' \rangle = (2/\pi)(\hbar^2/M) \sum_{(\alpha MLL')} i^{L'-L} V_{LL'}{}^{\alpha}(k \mid k')$$
$$\times \mathcal{Y}_{LS}{}^{JM}(\hat{k}) \mathcal{Y}_{L'S}{}^{JM+}(\hat{k}')\, P_T \qquad (40a)$$

and

$$\langle \mathbf{k} \mid T \mid \mathbf{k}' \rangle = (2/\pi)(\hbar^2/M) \sum_{(\alpha MLL')} i^{L'-L} T_{LL'}{}^{\alpha}(k \mid k')$$
$$\times \mathcal{Y}_{LS}{}^{JM}(\hat{k}) \mathcal{Y}_{L'S}{}^{JM+}(k')\, P_T, \qquad (40b)$$

where the sum extends only over the allowed two-nucleon states and P_T is an isospin projection operator.

For coupled states, the diagonal representation of the S matrix

$$S_{LL'}{}^{\alpha}(k) = \delta_{LL'} - 2ikT_{LL'}{}^{\alpha}(k \mid k)$$
$$= \sum_l U_{lL}{}^{\alpha}(k) \exp[2i\delta_l{}^{\alpha}(k)] U_{lL}{}^{\alpha}(k) \qquad (41)$$

is defined by a real orthogonal matrix U:

$$U_{LL'}^{\alpha}(k)=\begin{array}{c} \\ J-1 \\ J+1 \\ J \end{array}\begin{array}{c} \begin{array}{ccc} J-1 & J+1 & J \end{array} \\ \begin{bmatrix} \cos\epsilon_\alpha & +\sin\epsilon_\alpha & 0 \\ -\sin\epsilon_\alpha & \cos\epsilon_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}. \quad (42)$$

The matrix U makes the S matrix properly symmetric and unitary. Here $\delta_{J\pm1}^{\alpha}$ and ϵ_α are the Blatt-Biedenharn eigenphases and coupling parameter. It is the coupling parameter ϵ_α that especially characterizes the nature of the tensor force.

Now let us assume that the potential is also diagonalized by U and that it is a single separable term in the diagonal representation

$$V_{ll'}^{\alpha}(k\mid k')\equiv\sum_{LL'}U_{lL}^{\alpha}(k)\,V_{LL'}^{\alpha}(k\mid k')\,U_{i'L'}^{\alpha}(k')$$
$$=\sigma_l\delta_{ll'}g_l^{\alpha}(k)\,g_l^{\alpha}(k'). \quad (43)$$

The choice of over-all sign, $\sigma_l=\pm1$, is based on knowledge of the eigenphases. For example, the proper choice for the ${}^3S_1+{}^3D_1$ coupled channel is $\sigma_0=-1$ and $\sigma_2=+1$ (see below). For the uncoupled cases (${}^3D_2$, ${}^3P_1$), Eq. (43) reduces to

$$V_{JJ}^{\alpha}(k\mid k')=\sigma_J g_J^{\alpha}(k)\,g_J^{\alpha}(k') \quad (44)$$

and the results of our previous discussion [Eqs. (21), (31), or (32)] can be used to construct $g_J(k)$ from the phase shifts, $\delta_J{}^{JST}(k)$. Let us therefore concentrate on the coupled-channel part of Eqs. (39)–(43).

The assumption that the potential is diagonalized by the orthogonal matrix U is equivalent to expressing $V_{LL'}$ as a special, two-term separable potential. If we define $\tilde{g}_L$ and $\tilde{h}_L$ for coupled states by

$$\tilde{g}_L(k)=U_{J-1,L}(k)\,g_{J-1}(k),$$
$$\tilde{h}_L(k)=U_{J+1,L}(k)\,g_{J+1}(k), \quad (45)$$

then the potential consists of two separable terms:

$$V_{LL'}(k\mid k')=\sum_l U_{lL}(k)\,g_l(k)\,\sigma_l g_l(k')\,U_{lL'}(k')$$
$$=\sigma_{J-1}\tilde{g}_L(k)\,\tilde{g}_{L'}(k')+\sigma_{J+1}\tilde{h}_L(k)\,\tilde{h}_{L'}(k'). \quad (46)$$

Equations (45) and (46) have considerable practical significance which will be discussed later (Sec. VI).

It is well known that for a sum of separable potentials the Schrödinger equation can be solved exactly.[6] For a potential that is also separable in the diagonal representation, the solution of the Lippmann-Schwinger equation is simply

$$T_{LL'}^{\alpha}(k\mid k')=\sum_l U_{lL}^{\alpha}(k)\,t_l^{\alpha}(k\mid k')\,U_{lL'}^{\alpha}(k'), \quad (47)$$

where $t_l(k\mid k')$ is determined on and off the energy shell by the potential function $g_l(k)$

$$t_l^{\alpha}(k\mid k')=\sigma_l^{\alpha}g_l^{\alpha}(k)\,g_l^{\alpha}(k')/D_l^{(+)}(k'). \quad (48)$$

The function $D_l^{(+)}(k)$ is given by Eq. (4). The eigenphase shift $\delta_l(k)$ is obtained from the on-energy-shell t matrix

$$kt_l^{\alpha}(k\mid k')=-\exp[i\delta_l^{\alpha}(k)]\sin\delta_l^{\alpha}(k)$$
$$=\sigma_l k g_l^2(k)/D_l^{(+)}(k). \quad (49)$$

These expressions provide a means of solving the inversion problem for noncentral forces. Use of the diagonal representation assumption [Eq. (43)] has simplified our task to constructing $g_{J-1}^{\alpha}(k)$ directly from the eigenphase shift $\delta_{J-1}^{\alpha}(k)$ and separately finding $g_{J+1}^{\alpha}(k)$ from $\delta_{J+1}^{\alpha}(k)$. The great advantage of the diagonal representation assumption is that the construction of $g_J^{\alpha}(k)$ is completely analogous to the case of a central potential and independent of the coupling parameter $\epsilon_J^{\alpha}(k)$. It is only at the final stage of writing the full T matrix [Eq. (47)] that one must introduce the known coupling parameter.

If follows from the similarity of Eqs. (3) and (49) that the previously described construction procedure can be used to find the potential functions $g_{J\pm1}$ from the eigenphases $\delta_{J\pm1}$. To completely establish this similarity, one must examine the analytic properties of the function $D_l(z)$, which appears in Eq. (49). As before, $D_l(z)$ is analytic in the upper half z plane and can have complicated singularities in the lower plane ($\mathrm{Im}z<0$) (Fig. 1). If $\tan\delta_{J\pm1}^{\alpha}$ is known to change sign at an energy E_c, one must again assume that $D_l(z)$ has real zeros at $z=+k_c$ [Eqs. (22)–(31)]. If there is a bound state (the ${}^3S_1+{}^3D_1$ state of the deuteron), then $D_l(z)$ may have a zero at $z=+ik_B$. For the case of coupled states this bound-state zero requires further discussion.

Consider the Schrödinger equation in momentum space for a bound state of energy $E_B=-(\hbar^2/M)k_B^2$

$$(k^2+k_B^2)\,W_L^{\alpha}(k)=-\frac{2}{\pi}\sum_{L'}\int_0^{\infty}d\kappa\,\kappa^2V_{LL'}^{\alpha}(k\mid\kappa)W_{L'}(\kappa). \quad (50)$$

For two nucleons a bound state occurs only when $\alpha=(JST)=(110)$ and $L=0$, 2, corresponding to the ${}^3S_1+{}^3D_1$ state of the deuteron. Introducing the diagonal representation of $V_{LL'}$ and the wave function $\phi_l=\sum_L U_{lL}W_L$, we find that the $l=0$ and $l=2$ equations are decoupled:

$$(k^2+k_B^2)\phi_l=-\sigma_l g_l(k)\,C_l. \quad (51)$$

Here C_l depends only on k_B and is given by

$$C_l=\frac{2}{\pi}\int_0^{\infty}d\kappa\,\kappa^2g_l(\kappa)\phi_l(\kappa) \quad (52)$$

(no sum on l). It follows from Eqs. (51) and (52) that

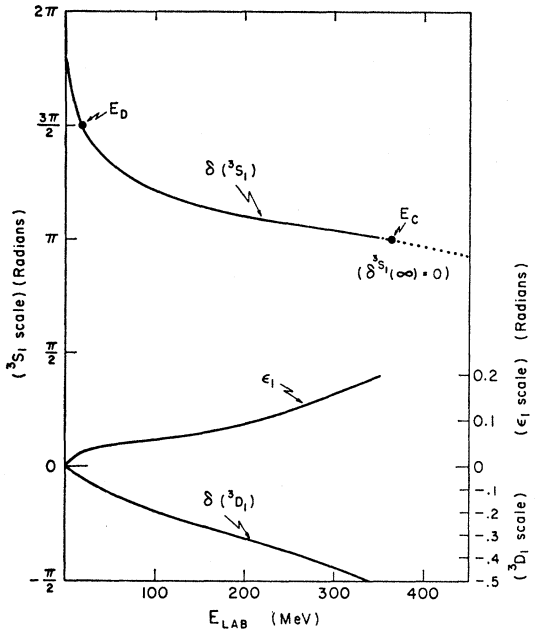

FIG. 2. The $^3S_1$ and $^3D_1$ Blatt-Biedenharn eigenphase shifts and coupling parameter $\epsilon_1(k)$ (Ref. 1). This case has a bound state ($N^B=1$) and an S-wave repulsion ($N_0=1$, $N_2=0$); see Eq. (57). Correspondingly, $\delta(^3S_1)$ becomes repulsive at $E_c\sim 360$ MeV, and $t_l(k\mid k)$ has a pole at $E_D\simeq 17$ MeV, where $\tan\delta_0(E_D)=\infty$. The low-energy behavior of $\epsilon_1(k)$ is related to the deuteron's quadrupole moment Q and the triplet effective range r_t. An S-wave repulsion is indicated by the dotted curve.

the condition for a bound state is

$$C_l D_l(ik_B)=0. \tag{53}$$

Now one must consider the actual behavior of the $^3S_1$ and $^3D_1$ eigenphases $\delta_0{}^\alpha(k)$ and $\delta_2{}^\alpha(k)$ (Fig. 2). It is known that the $^3S_1$ eigenphase has a positive scattering length and an effective range, which are related to k_B by $k_B{}^{-1}=a^{-1}+\frac{1}{2}k_B{}^2r_0$. Correspondingly, $\tan\delta(^3S_1)$ becomes infinite at the collision energy of $E_D\simeq 17$ MeV and $\delta(^3S_1)$ reveals a bound-state pole in $t_l(k\mid k)$. It follows that $D_0(ik_B)=0$ is the proper way to satisfy the bound-state condition for $l=J-1=0$ [Eq. (53)]. However, for the $^3D_1$ eigenphase $\tan\delta(^3D_1)$ is known to be finite at all energies below 350 MeV (Fig. 2). In fact, $\delta(^3D_1)$ is relatively small and negative in the range $0<E<350$ MeV (Fig. 2). Thus the bound-state condition for $l=J+1=2$ must be satisfied by having $D_2(ik_B)\neq 0$ and $C_2=0$. Also, to produce the negative $\delta(^3D_1)$ eigenphase the sign σ_2 must be positive ($\sigma_2=+1$), whereas the $^3S_1$ bound-state zero implies that $\sigma_0=-1$.

The bound-state wave function is now determined:

$$W_L(k)=C_0U_{L0}(k)g_0(k)/[k^2+k_B{}^2], \tag{54}$$

and C_0 is fixed by normalization to be

$$C_0=\left(\int_0^\infty \frac{g_0{}^2(\kappa)\kappa^2 d\kappa}{(\kappa^2+k_B{}^2)^2}\right)^{-1/2}. \tag{55}$$

An interesting consequence of the condition $C_2=0$ is that the D-state probability is determined from ϵ_1 and δ_0 by

$$P_D=\int_0^\infty d\kappa\,\kappa^2W_2{}^2(\kappa)=C_0{}^2\int_0^\infty \frac{d\kappa\,\kappa^2(\sin^2\epsilon_1(\kappa))\,g_0{}^2(\kappa)}{(\kappa^2+k_B{}^2)^2}. \tag{56}$$

Therefore, the bound-state properties (the binding energy, quadrupole moment, D-state probability, and wave function) are completely specified by this construction procedure.

We have therefore established that $D_0(z)$ does have a bound-state zero at $z=ik_B$, whereas $D_2(z)$ does not. The analytic properties of $D_l(z)$ are now completely revealed and the results of the previous discussion (Secs. II–IV) can be applied. The potential functions $g_L{}^\alpha$ are determined from the corresponding eigenphases $\delta_{J\pm1}{}^\alpha$ using either Eqs. (21), (31), or (32). For example, the $^3S_1$ eigenphase determines $g_0(k)$ using Eq. (31), which includes the bound state and a possible repulsion at high energies. Equation (21) is useful for determining $-\sigma_L g_L{}^2(k)$ from the $^3D_1$ eigenphase shift.

A modified Levinson theorem also applies to the eigenphases. If one requires that $k^{-l}g_l(k)$ be bounded, then the proper choice (Sec. IV) for the eigenphases is given by

$$\delta_l{}^\alpha(0)-\delta_l{}^\alpha(\infty)=(N^B+N_l)\pi. \tag{57}$$

It is only for the $^3S_1$ eigenphase that $N^B=1$. The number of sign changes in $\tan\delta_L$ and, correspondingly, the number of nodes in $g_l{}^\alpha(k)$ is given by N_l. [We always assume that $\delta_l{}^\alpha(\infty)=0$]. For $N_l=0$, Eq. (57) reduces to the standard Levinson theorem with eigenphases $\delta_{J-1}{}^\alpha(0)+\delta_{J+1}{}^\alpha(0)=N^B\pi$.[7] The modified Levinson theorem again arises from the manner in which a repulsion has been introduced [Eq. (22)]. The applicability of the construction procedure presented in this paper is therefore restricted by Eq. (57).

Once the potential functions $g_{J\pm1}{}^\alpha$ have been constructed from the eigenphases, the complete potential [Eq. (46)], T matrix [Eq. (47)], and wave function can be given. This last step requires that we know the coupling parameter $\epsilon_\alpha(k)$.

VI. CONCLUSION

A flexible and convenient solution of the inversion problem for separable potentials has been presented. For each allowed two-nucleon state, only one principal-value integration is needed to construct a separable potential from the phase shifts. This integral can be found numerically once the phase shifts are given. Of course, the phase shifts are only known up to 350 MeV and one must assume the phase-shift behavior above

this energy. An important feature of separable interactions is that the higher-energy phase shifts automatically stipulate the off-energy-shell structure of the interaction. Therefore, the construction procedure provides a convenient means of studying the role of the assumed phase shifts, and the corresponding off-energy-shell matrix elements, in various nuclear structure calculations (Sec. I).

The construction of a separable potential has been extended to include the effects of a strong, short-range repulsion.[8] This essential repulsion is incorporated by permitting the Fredholm determinant to have zeros on the real axis [Eq. (22)]. As a consequence of introducing these zeros, the phase shift behavior is restricted by a modified Levinson theorem[7] [Eq. (37)]. Only when this theorem is satisfied by the phase shifts can one apply the construction procedure described in this paper. However, the modified Levinson theorem can be satisfied formally at extremely high energies. For example, the phase shifts of a local potential up to, say, 5 BeV could be used to construct a separable interaction provided the phase shifts above 5 BeV were forced to conform to the modified Levinson theorem. Therefore, from a practical viewpoint, the construction procedure is quite flexible.

It has also been shown in Sec. V that a separable potential can be constructed from the eigenphases and coupling parameter in the case of coupled two-nucleon states. This treatment of noncentral forces is made possible by the use of the "diagonal representation" suggested by Bolsterli and MacKenzie.[5] As a result of using the diagonal representation, constructing a separable potential from the eigenphases requires no more work than for a central potential. Therefore, by using this construction procedure and various models for the high-energy eigenphases and coupling parameter, one can conveniently test the sensitivity of nuclear-structure predictions to the noncentral aspects of the two-nucleon interaction.

There are other practical advantages to using a separable potential in the diagonal representation [Eq. (43)]. Several authors have used two-term separable potentials to fit the nucleon-nucleon scattering data.[6] The *four* potential functions $\tilde{g}_{J\pm1}$ and $\tilde{h}_{J\pm1}$ are then parametrized as convenient analytic functions and the parameters are adjusted to fit the eigenphases and coupling parameters. This task is greatly simplified by using the diagonal representation, since one needs to parametrize only *three* functions $g_{J\pm}$ and ϵ_J to fit the known quantities $\delta_{J\pm1}{}^{\alpha}$ and $\epsilon_\alpha(\mathrm{Exp})$. Also, the diagonal-representation assumption decouples the curve-fitting problem into three distinct and quite simple tasks [see Eq. (49)].[9]

The main point of this paper is that one can easily construct a separable potential directly from the phase shifts, including a strong repulsion and noncentral forces. The task of fitting phase shifts is then avoided and tests of nuclear structure predictions can be made with relative ease.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor M. Baranger and Professor N. Austern for helpful remarks. Use of the University of Pittsburgh computer under Grant No. G-11309 is also acknowledged.

APPENDIX

The solution of the inversion problem involves evaluating the integral

$$\frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa-k}\ln\left(\frac{\kappa+ik_B}{\kappa-ik_B}\right). \tag{A1}$$

The integrand has a branch cut from $+ik_B$ to $-ik_B$. Although it is possible to use a contour around this cut, an easier yet correct way of integrating (A1) is to use the limit of

$$\lim_{S\to0}\frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa-k}\ln\left(\frac{\kappa+ik_B}{\kappa+iS}\frac{\kappa-iS}{\kappa-ik_B}\right)$$

$$=\lim_{S\to0}\left\{\frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa-k}\ln\left(\frac{\kappa+ik_B}{\kappa+iS}\right)\right.$$

$$\left.-\frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa-k}\ln\left(\frac{\kappa-ik_B}{\kappa-iS}\right)\right\}. \tag{A2}$$

The first integrand in (A2) is now analytic in the upper half-plane; the second integrand is analytic in the lower plane. Thus we can close the corresponding contours with the upper/lower infinite semicircles and use Cauchy's theorem. The principal value integral (A1) is therefore

$$\lim_{S\to0}\left[\ln\left(\frac{k+ik_B}{k+iS}\right)+\ln\left(\frac{k-ik_B}{k-iS}\right)\right]=\ln\left(\frac{k^2+k_B{}^2}{k^2}\right). \tag{A3}$$

[9] The repulsion introduced by Eq. (22) is a very strong interaction. In Ref. 8 it was shown that perturbative methods cannot be used and, therefore, Brueckner theory is required for this strong potential. However, a smooth potential with a weak repulsion can be generated by appropriate choices for $\tilde{g}_{J\pm1}$ and $\tilde{h}_{J\pm1}$ in Eq. (46) (Ref. 6). The diagonal representation greatly simplifies the curve-fitting problem for these smooth interactions.