rameters are about the same magnitude as those that we have found, but are not in close agreement, either with our scattering data or with each other.

It is not known whether the comparison between effective nuclear forces from the bound-state problem with those in the scattering problem is really meaningful. Certainly, they arise from similar effects. It is bothersome that the charge-spin-independent term in both the Gillet–Vinh Mau and the Elliott-Flowers interactions is zero or near-zero, since the spin-charge-independent term found by Satchler is much stronger than the charge-exchange force.

We have concentrated on the direct-reaction mechanism as an explanation of the data because it is a reasonably well developed and reliable theory. In doing so, we have not given adequate consideration to the lower-energy data, which show strong fluctuations as a function of energy. Much information about reaction mechanism and nuclear structure is contained in the lower-energy data and could be obtained by analysis with an adequate resonance model. A promising attempt in this direction has been made by Hanna and Nagarajan.[45]

[45] J. S. Hanna and M. A. Nagarajan (unpublished).

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Nucleon-Nucleon Scattering from One-Boson-Exchange Potentials. III. S Waves Included*

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The nucleon-nucleon interaction is described over the laboratory scattering energy range 0–350 MeV by a potential used in conjunction with the Schrödinger equation. In momentum space the potential is a superposition of Born terms obtained from single exchanges of ω, ρ, π, η, σ_0, and σ_1 mesons, where the σ_0 and σ_1 are hypothetical scalar mesons with isotopic spin 0 and 1, respectively. Rather than taking the usual static limit, all terms of order p^2/M^2 are retained. The inclusion of S waves requires the introduction of a cutoff factor. The meson coupling constants, the masses of the σ_0 and σ_1, and a cutoff parameter are adjusted to fit the experimentally determined phase parameters. A comparison with experimental phase-shift analysis shows a good qualitative fit, on the average.

I. INTRODUCTION

THIS is the third in a series of articles[1] whose purpose is to represent the nucleon-nucleon interaction from 0 to 350 MeV in terms of a sum of pole contributions of the ω, ρ, π, η, σ_0, and σ_1 mesons in the cross channel. The requirements of unitarity are satisfied by using the Fourier transform of these pole, or Born, terms as a potential in the nonrelativistic Schrödinger equation. The resulting phase parameters can be compared with experimentally derived phases or used to compute the quantities (cross section, polarization, etc.) which can be directly compared with experiment. The parameters of the theory which are adjusted to fit the data are the coupling constants of all the mesons and the masses of the hypothetical $\sigma_0(T, J^\pi=0, 0^+)$ and the $\sigma_1(1,0^+)$ scalar mesons. We will return to the question of these scalar mesons later in the paper.

In I, a satisfactory fit to the phase parameters was obtained for states with relative orbital angular momenta $l\geq 1$. In that work two important approximations were made. When one transforms the pole terms from momentum to configuration space, the resulting potential is not local (static). An expansion in powers of $\mathbf{p}^2/M^2$, with M the nucleon mass and $\mathbf{p}$ any nucleon momentum, can be made. In I, all terms save one of order $\mathbf{p}^2/M^2$ were kept and the others neglected in order to obtain a local potential. The second important approximation was the introduction of a zero cutoff in configuration space to eliminate the $1/r^3$ divergence in the potential which would have otherwise occurred. The presence of this cutoff restricted the application of the model to P waves and higher.

In II, the aforementioned neglected $\mathbf{p}^2/M^2$ term was now included in the potential, thanks to Green's method[2] for dealing with the resulting $\nabla^2\phi(r)+\phi(r)\nabla^2$ term in configuration space. The inclusion of this $\mathbf{p}^2/M^2$ term

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[1] Ronald A. Bryan and Bruce L. Scott, Phys. Rev. **135**, B434 (1964); **164**, 1215 (1967), hereafter referred to as I and II, respectively.

[2] A. M. Green, Nucl. Phys. **33**, 218 (1962).

had the very desirable result that a cutoff was no longer needed for P or higher waves, so that both major deficiencies of I were removed at once. The resulting model gave a quantitative fit to the experimental data when suitably augmented by reasonable S-wave parameters.

Although the previous work gave some insight into the nucleon-nucleon interaction, the potentials involved have only limited usefulness because of the exclusion of S waves. There are many problems to which the single-boson-exchange mechanism can be applied unambiguously if S waves can also be described by that mechanism. Such problems include nucleon-nucleon bremsstrahlung, nucleon-antinucleon interactions, nuclear matter, hyperon-nucleon scattering [invoking $SU(3)$ or some other symmetry group], etc. This paper is an extension of II to include these S waves.

II. THE POTENTIAL

In order to avoid repetition, we merely exhibit the one-boson exchange potentials.[1,3] These potentials are correct through order p^2/M^2, where p is the magnitude of the three-momentum of any nucleon in the c.m. system, and M is the nucleon mass. Terms of order p^4/M^4 and higher have been neglected.

A. Vector Meson (V)

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}\bar{\Psi}[g\gamma^\mu\phi_\mu{}^{(V)}+(f/4M)\sigma^{\mu\nu}(\partial_\nu\phi_\mu{}^{(V)}-\partial_\mu\phi_\nu{}^{(V)})]\Psi\,,$$

$$\begin{aligned}V^{(V)}=&g^2\frac{e^{-mr}}{r}-g^2\frac{1}{2M^2}\left(\nabla^2\frac{e^{-mr}}{r}+\frac{e^{-mr}}{r}\nabla^2\right)\\&+(g^2+gf)\frac{1}{2M^2}\left[m^2\frac{e^{-mr}}{r}-4\pi\delta^{(3)}(\mathbf{r})\right]\\&+(g+f)^2\frac{1}{6M^2}\left[m^2\frac{e^{-mr}}{r}-4\pi\delta^{(3)}(\mathbf{r})\right]\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\\&-(g+f)^2\frac{1}{4M^2}\left(\frac{1}{r^2}+\frac{m}{r}+\frac{m^2}{3}\right)\frac{e^{-mr}}{r}S_{12}\\&+(3g^2+4fg)\frac{1}{2M^2}\frac{1}{r}\frac{d}{dr}\left(\frac{e^{-mr}}{r}\right)\mathbf{L}\cdot\mathbf{S}.\end{aligned}\tag{1}$$

B. Scalar Meson (S)

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}g\bar{\Psi}\Psi\phi^{(S)}\,,$$

$$\begin{aligned}V^{(S)}=g^2\Bigg\{&-\frac{e^{-mr}}{r}-\frac{1}{2M^2}\left(\nabla^2\frac{e^{-mr}}{r}+\frac{e^{-mr}}{r}\nabla^2\right)\\&+\frac{1}{4M^2}\left[m^2\frac{e^{-mr}}{r}-4\pi\delta^{(3)}(\mathbf{r})\right]\\&+\frac{1}{2M^2}\frac{1}{r}\frac{d}{dr}\left(\frac{e^{-mr}}{r}\right)\mathbf{L}\cdot\mathbf{S}\Bigg\}.\end{aligned}\tag{2}$$

C. Pseudoscalar Meson (PS)

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}g\bar{\Psi}\gamma^5\Psi\phi^{(\text{PS})}\,,$$

$$\begin{aligned}V^{(\text{PS})}=g^2\Bigg\{&\frac{1}{12M^2}\left[m^2\frac{e^{-mr}}{r}-4\pi\delta^{(3)}(\mathbf{r})\right]\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\\&+\frac{1}{4M^2}\left(\frac{1}{r^2}+\frac{m}{r}+\frac{m^2}{3}\right)\frac{e^{-mr}}{r}S_{12}\Bigg\}.\end{aligned}\tag{3}$$

In each equation, m is the mass of the meson which is exchanged. We have chosen units such that $\hbar=c=1$. For $T=1$ mesons (π,ρ,σ_1) the meson field ϕ is replaced by $\boldsymbol{\tau}\cdot\boldsymbol{\phi}$; the result is that g^2 is replaced by $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2g^2$.

Predictions subject to experimental test are made by inserting the potential

$$V=\sum_\nu V^{(\nu)}\quad(\nu=\pi,\,\eta,\,\rho,\,\omega,\,\sigma_1,\,\sigma_0)$$

into the Schrödinger equation and solving it to get the phase shifts. Because the ∇^2 terms in the scalar and vector meson one-boson-exchange potential (OBEP) make the Schrödinger equation more complicated, we solve not for $u_l(r)$, the ordinary radial wave function for orbital angular momentum l, but rather for

$$v_l(r)=[1+2\phi(r)]^{1/2}u_l(r)\,,$$

where $\phi(r)$ is defined by setting the potential

$$V(r)=V_0(r)-M^{-1}[\nabla^2\phi(r)+\phi(r)\nabla^2].$$

$v_l(r)$ satisfies an ordinary radial wave equation

$$v_l''-[l(l+1)/r^2]v_l+k^2v_l=MW_lv_l\,,$$

but with

$$W_l(r)=\frac{V_{0,l}}{1+2\phi}-\left(\frac{d\phi/dr}{1+2\phi}\right)^2\frac{1}{M}+\frac{2\phi}{1+2\phi}\frac{k^2}{M},\tag{4}$$

where $V_{0,l}$ is the potential evaluated for the angular momentum state in question. For simplicity, we have suppressed mechanical spin and total angular momentum indices, as well as tensor coupling. The foregoing is described more fully in I and II.

The potential of Eq. (4) is no more difficult to deal with than any other static potential. However, in the case of OBEP, it does have singularities at $r=0$ which rule out S-state solutions. In particular, $V_{0,l}$ includes terms which go like $\delta^{(3)}(\mathbf{r})$, and W_l has the term $-(d\phi/dr)^2/(1+2\phi)^2M$ which goes like $-1/r^2$ at the origin. Accordingly we introduce a cutoff to reduce the degree of singularity to something which will permit an S-state solution. We choose to multiply the momentum-space version of the potential by $\Lambda^2/(\Lambda^2-q_\mu{}^2)$, where Λ is a mass, q_μ is the momentum transfer, and $q_\mu{}^2=q_0{}^2-\mathbf{q}^2$. If p_μ' is the energy-momentum vector of one of the nucleons in the final state, and p_μ is the energy-momentum vector of this nucleon in the initial state, then $q_\mu=p_\mu'-p_\mu$.

[3] D. Y. Wong, Nucl. Phys. **55**, 212 (1964). Note that our definition of the potential differs from his Eq. (2) by an additional factor of M/E.

The effect of the cutoff factor is to transform each OBEP $V^{(\nu)}(r,m_\nu)$ to

$$[V^{(\nu)}(r,m_\nu)-V^{(\nu)}(r,\Lambda)]\Lambda^2/(\Lambda^2-m_\nu^2);$$

that is to say, one subtracts from each OBEP the same potential but with Λ replacing the meson mass; a constant factor then multiplies the difference. To show this, consider the momentum-space definition of the OBEP. Let us call it $\mathcal{V}^{(\nu)}(q,m_\nu)$. [For simplicity, we shall disregard the dependence of $\mathcal{V}^{(\nu)}$ on $\mathbf{P}=\frac{1}{2}(\mathbf{p}+\mathbf{p}')$, where $\mathbf{p}$ and $\mathbf{p}'$ are three-momenta corresponding to p_μ and p_μ', just as we shall disregard the dependence of the configuration space potential $V^{(\nu)}$ on ∇.]

$$\mathcal{V}^{(\nu)}(\mathbf{q},m_\nu)=\frac{1}{2\pi^2}g^2\bar{U}(\mathbf{p}')\Gamma_1 U(\mathbf{p})\frac{1}{\mathbf{q}^2+m_\nu^2}\times\bar{U}(-\mathbf{p}')\Gamma_2 U(-\mathbf{p}),$$

where Γ_1 and Γ_2 are 4×4 Dirac matrices appropriate to the meson exchanged, and U and $\bar{U}$ are Dirac spinors. Because of cutoff, $\mathcal{V}^{(\nu)}(\mathbf{q},m_\nu)\rightarrow\mathcal{V}^{(\nu)}(\mathbf{q},m_\nu)\Lambda^2(\Lambda^2-q_\mu^2)$, but q_0 vanishes since the scattering is elastic. Thus

$$\mathcal{V}_{\text{cut}}^{(\nu)}=\mathcal{V}^{(\nu)}(\mathbf{q},m_\nu)\Lambda^2/(\Lambda^2+\mathbf{q}^2).$$

Since

$$\frac{1}{\mathbf{q}^2+m_\nu^2}\frac{\Lambda^2}{\mathbf{q}^2+\Lambda^2}=\frac{\Lambda^2}{\Lambda^2-m_\nu^2}\left(\frac{1}{\mathbf{q}^2+m_\nu^2}-\frac{1}{\mathbf{q}^2+\Lambda^2}\right),$$

we may write

$$\mathcal{V}_{\text{cut}}^{(\nu)}=\frac{G^2}{2\pi^2}(\bar{U}\Gamma_1U)\frac{1}{\mathbf{q}^2+m_\nu^2}(\bar{U}\Gamma_2U)-\frac{G^2}{2\pi^2}(\bar{U}\Gamma_1U)\frac{1}{\mathbf{q}^2+\Lambda^2}(\bar{U}\Gamma_2U),\quad(5)$$

with $G^2=g^2\Lambda^2/(\Lambda^2-m_\nu^2)$. We now observe that the second term in Eq. (5) is identical to the first term in every respect save that Λ has replaced m_ν in the propagator. [The exchanged meson mass does not appear in the numerator; this is true for vector meson exchange, with $\Gamma_1=(g+f)\gamma_\mu-(f/2M)(p_\mu'+p_\mu)$ as well as for scalar exchange with $\Gamma_1=1$, or pseudoscalar exchange with $\Gamma_1=\gamma_5$.] Thus we may write

$$\mathcal{V}_{\text{cut}}^{(\nu)}=[\mathcal{V}^{(\nu)}(\mathbf{q},m_\nu)-\mathcal{V}^{(\nu)}(\mathbf{q},\Lambda)]\Lambda^2/(\Lambda^2-m_\nu^2).$$

In configuration space, then,

$$V_{\text{cut}}^{(\nu)}(\mathbf{r})=[V^{(\nu)}(\mathbf{r},m_\nu)-V^{(\nu)}(\mathbf{r},\Lambda)]\Lambda^2/(\Lambda^2-m_\nu^2).$$

With cutoff, the singularity $\delta^{(3)}(\mathbf{r})$ which appeared in the unmodified OBEP now vanishes,[4] leaving just terms which go as $1/r$ or a constant.

The singularity $1/r^2$ which appeared in $-[(d\phi/dr)/(1+2\phi)]^2/M$ is reduced to a constant. Thus the Schrödinger equation is now solvable for S states. It is also of interest to note that the $1/r^3$ singularities which appear in the S_{12} and $\mathbf{L}\cdot\mathbf{S}$ potentials are reduced through cutoff to order $1/r$.

The over-all phenomenological potential is thus

$$V(\mathbf{r})=\sum_\nu V_{\text{cut}}^{(\nu)}(\mathbf{r})\quad(\nu=\rho,\omega,\pi,\eta,\sigma_1,\sigma_0)$$
$$=\sum_\nu[V^{(\nu)}(\mathbf{r},m_\nu)-V^{(\nu)}(\mathbf{r},\Lambda)]\frac{\Lambda^2}{\Lambda^2-m_\nu^2}.\quad(6)$$

This potential is inserted in the Schrödinger equation,

$$-\frac{1}{M}\nabla^2\psi(\mathbf{r},t)+V(\mathbf{r})\psi(\mathbf{r},t)=\frac{i\partial\psi(\mathbf{r},t)}{\partial t},\quad(7)$$

and the phase shifts are solved for.

III. CALCULATION

The Schrödinger equation with the potential of Eq. (6) was solved numerically on a Honeywell 800 medium-speed computer to obtain values of the phase parameters. As before, the meson coupling constants and the masses of the σ_0 and σ_1 particles were adjusted in order to obtain a best fit to the experimental phase shifts, but, in addition, we now also adjust the cutoff mass Λ.

The values of the phase shifts obtained from the Schrödinger equation were fitted to the phase shifts obtained by Arndt and MacGregor[5] from the experimental data. This search-fitting was done at 50, 142, and 310 MeV and included all S, P, and D waves. In addition the effective range and scattering lengths were also included in the search.

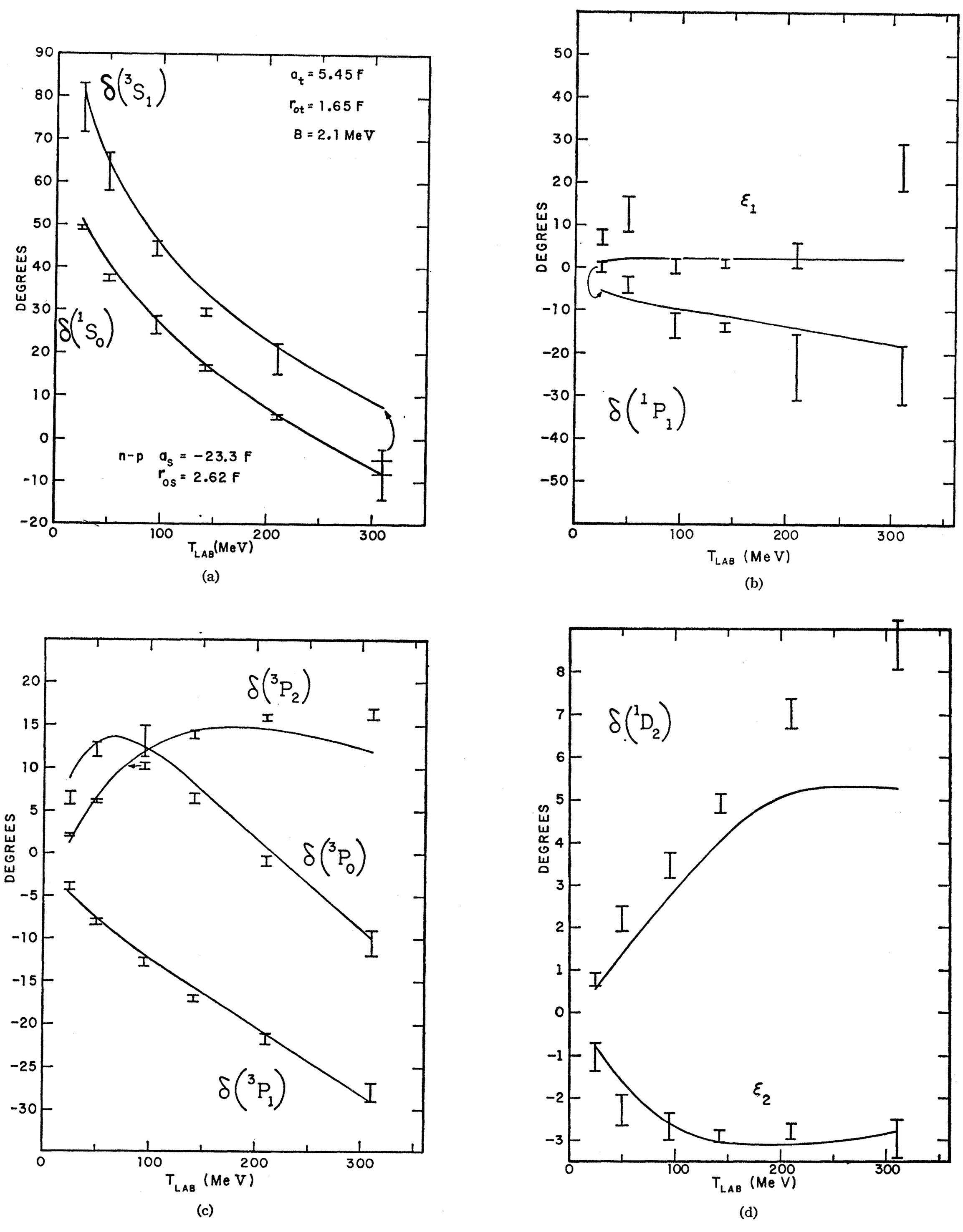
In Table I under the heading "This work" we list the values of the coupling constants which gave the best fit. The phase shifts given by these parameters are presented in Table II and plotted in Fig. 1. The error bars in Fig. 1 are the uncorrelated uncertainties in the

TABLE I. Meson coupling constants and masses which yield the results given in Table II and Fig. 1, listed under the heading "This work." The quantities within parentheses were not searched, but rather were fixed beforehand. The cutoff mass was $\Lambda=1500$ MeV. The results of paper II are also presented (see Ref. 1).

		This work			From II		
Meson	T, J^P	Mass (MeV)	g^2	f/g	Mass (MeV)	g^2	f/g
π	1, 0^-	(138.7)	12.55	···		12.5	···
η	0, 0^-	(548.7)	2.60	···		10.6	···
σ_1	1, 0^+	600	1.65	···	770	5.8	···
σ_0	0, 0^+	550	8.19	···	590	9.9	···
ρ	1, 1^-	(763)	1.81	1.13		1.36	3.82
ω	0, 1^-	(782.8)	17.26	0.0		19.1	0.0

[4] More accurately, the cutoff smears out the δ function. The expression in which every $\delta^{(3)}(\mathbf{r})$ appears, $m^2(e^{-mr})/r-4\pi\delta^{(3)}(\mathbf{r})$, is replaced by $m^2(e^{mr})/r-\Lambda^2(e^{-\Lambda r})/r$.

[5] R. A. Arndt and M. H. MacGregor (private communication). For a more recent version of these phase parameters, see R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966); **154**, 1549 (1967); **159**, 1422 (1967); **173**, 1272 (1968).



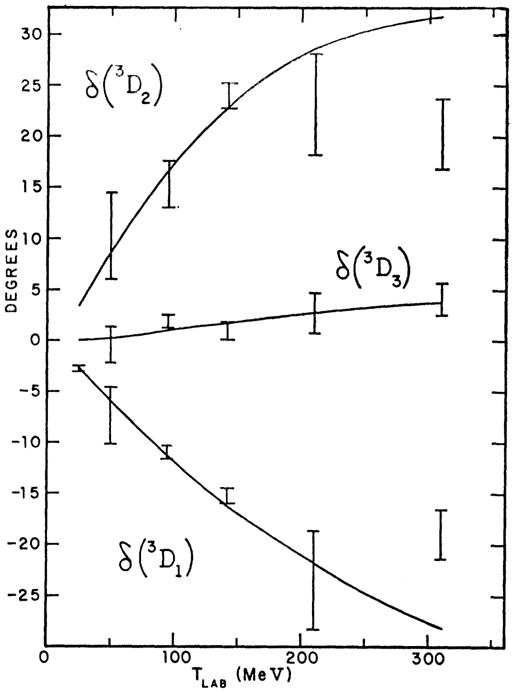
experimentally determined phase shifts as given by Arndt and MacGregor. In Table III the n-p scattering lengths and effective ranges predicted by this potential are given along with the experimental values which were employed in the search.[6]

IV. DISCUSSION

A. Fit to Experimental Phase Shifts

The phase shifts predicted by this model are in reasonable qualitative agreement with the experimental

[6] Richard Wilson, *The Nucleon-Nucleon Interaction* (Interscience Publishers, Inc., New York, 1963), p. 37. For a more recent estimate of the N-N scattering lengths and effective ranges, see H. Fiedeldey and H. P. Noyes, in *Three-Particle Scattering in Quantum Mechanics, Proceedings of the Texas A&M Conference* (W. A. Benjamin, Inc., New York, 1968), p. 195.



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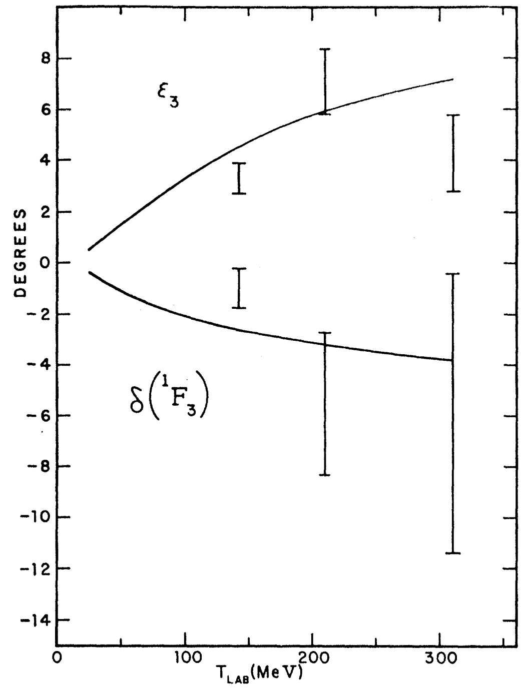

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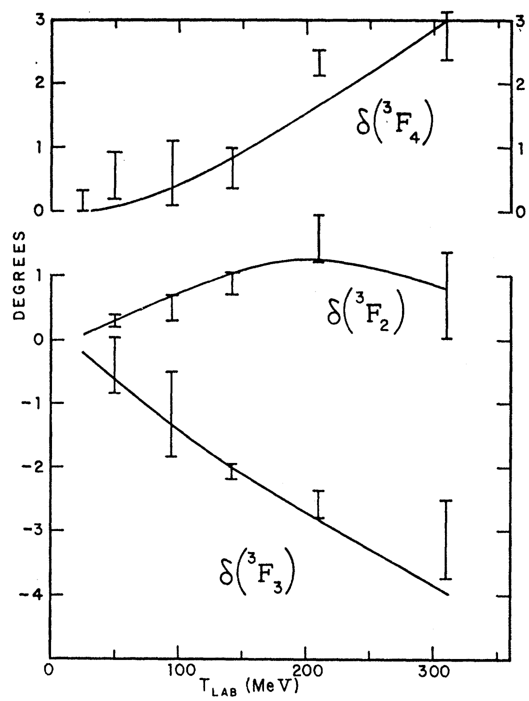

(g)

FIG. 1. The phase shifts calculated using the cutoff OBEP Eq. (6), in a nonrelativistic Schrödinger equation. The phase shifts are plotted as a function of lab energy. The error bars represent the phase-shift analysis of Arndt and MacGregor (Ref. 5).

phase shifts, although the fit is not quantitatively precise, as one can see from inspection of the graphs of Fig. 1. The goal of this research was to extend the OBEP model II to include the $^1S_0$ and $^3S_1$ states, and this was largely achieved, but at the expense of considerably worsening the fit to the $^1D_2$ phase shift, and to some extent, the $^3P_2$ phase shift. Model II predicts a $\delta(^1S_0)$ which is approximately 90° too positive over the range 25–310 MeV, and a $\delta(^3S_1)$ which is approximately 40° too negative over this same range. Therefore to fit experiment there must be introduced strong $^3S_1$-$^1S_0$ splitting. This was achieved through readjustment of the meson parameters and through adjustment of the cutoff parameter Λ, as follows. The σ_1 meson

TABLE II. Nuclear bar phase parameters computed by using the cutoff OBEP, Eq. (6), with the meson parameters listed in Table I in the nonrelativistic Schrödinger equation. The Coulomb potential has been included in all $T=1$ partial waves.

Phase parameter (deg) \ T_{lab} (MeV)	25	50	95	142	210	310
$\delta(^1S_0)$	51.29	41.15	27.83	17.37	5.48	−8.19
$\delta(^3S_1)$	81.52	64.05	46.33	34.19	21.41	7.56
ϵ_1	1.70	1.95	1.99	2.05	2.13	2.27
$\delta(^1P_1)$	−5.45	−7.65	−9.69	−11.42	−14.04	−18.20
$\delta(^3P_0)$	8.93	12.86	12.30	8.22	0.89	−9.87
$\delta(^3P_1)$	−4.46	−7.54	−11.76	−15.68	−21.14	−28.83
$\delta(^3P_2)$	2.81	6.66	11.81	14.29	14.59	11.89
ϵ_2	−0.74	−1.64	−2.62	−3.04	−3.12	−2.79
$\delta(^1D_2)$	0.58	1.38	2.78	4.03	5.16	5.28
$\delta(^3D_1)$	−2.63	−6.14	−11.60	−16.21	−21.69	−28.25
$\delta(^3D_2)$	3.45	8.48	16.66	22.99	28.60	31.81
$\delta(^3D_3)$	+0.00	0.14	0.82	1.67	2.84	3.73
ϵ_3	0.51	1.54	3.17	4.53	5.91	7.16
$\delta(^1F_3)$	−0.39	−1.04	−1.97	−2.01	−3.19	−3.78
$\delta(^3F_2)$	0.09	0.30	0.70	1.04	1.25	0.82
$\delta(^3F_3)$	−0.20	−0.62	−1.36	−2.61	−2.80	−3.97
$\delta(^3F_4)$	0.02	0.09	0.38	0.85	1.70	3.01
ϵ_4	−0.04	−0.17	−0.47	−0.76	−1.12	−1.53
$\delta(^3G_3)$	−0.05	−0.23	−0.81	−1.57	−2.77	−4.56
$\delta(^3H_4)$	+0.00	0.02	0.09	0.17	0.32	0.50

potential has the leading term $-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 g_1^2[\exp(-m_1r)]/r$; this term is attractive in the $^1S_0$ state and repulsive in the $^3S_1$ state; hence it acts the wrong way, as pointed out, incidently, by Müller and Dosch.[7] However, the σ_1 was introduced to bring about a fit to D waves, not S waves. Hence we reduce its effect on S states by keeping the potential's strength constant at 1.4 F, for the D wave fit, but lowering the mass and then suitably lowering the coupling constant. This yields a weaker potential within 1.4 F.

The ρ meson parameter can also be readjusted to improve the $^1S_0$-$^3S_1$ fit. The leading term in the ρ OBEP is $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 g_\rho^2[\exp(-m_\rho r)]/r$, which is attractive in the $^3S_1$ state and repulsive in the $^1S_0$ state. The increase in g_ρ^2 will therefore improve the fit to experiment. However, the quantity $(3g_\rho^2+4g_\rho f_\rho)$, which weights the spin-orbit term, should be kept constant in order not to destroy the P-wave fit. This is accomplished by simultaneously lowering the ratio f_ρ/g_ρ.

These measures improved the fit to the $^1S_0$ and $^3S_1$ experimental phase shifts. However much of the undesired $^1S_0$-$^3S_1$ splitting was also eliminated simply by varying Λ; this must have to do with the various δ functions of the several meson OBEP which are smeared out more or less as Λ is varied. A value for Λ of 1500 MeV together with optimization of the meson parameters resulted in a rather good $^3S_1$ and $^1S_0$-wave fit. We had expected that two phenomenological parameters would be needed to adjust the two S waves—one for each scattering length—assuming that the OBEP middle and outer portions would then be sufficient to give the correct effective range and higher momentum dependence (i.e., fit out to 310 MeV). In fact, one was sufficient.

TABLE III. Low-energy n-p parameters computed by using the cutoff OBEP, Eq. (6), with the meson parameters listed in Table I, in the nonrelativistic Schrödinger equation. The experimental values are given by Wilson (Ref. 6).

	Parameter	Expt.	Calc.
Singlet	a_s	−23.68 ±0.03 F	−23.3 F
n-p	r_{0s}	2.5 ±0.1 F	2.62 F
	a_t	5.40 ±0.01 F	5.45 F
Triplet	r_{0t}	1.73 ±0.01 F	1.65 F
	E_B	2.2245±0.0002 MeV	2.1 MeV

An unexpected difficulty was experienced in attempting to fit the experimental $^1D_2$ phase shift; this phase shift rises only to 5° at 310 MeV rather than 9°. This we find is due to the very strong $(\nabla^2\phi+\phi\nabla^2)$ part of the potential; whereas the dominant meson contributions tend to cancel in the case of the static part of the central isoscalar potential, $g_\omega^2[\exp(-m_\omega r)]/r-g_0^2\times[\exp(-m_0r)]/r$, these add in the case of the momenrum-dependent part of this central potential; $\phi=g_\omega^2\times[\exp(-m_\omega r)]/2Mr+g_0^2[\exp(-m_0r)]/2Mr$. Since the ω and σ_0 coupling constants are rather large in the Schrödinger equation pole model, this momentum-dependent term is quite strong. This term has the interesting feature of acting like an attraction at threshold and then, as the scattering energy increases, acting increasingly less attractive until it becomes repulsive. This may be seen from Eq. (4). If this term were perhaps twice as strong as it is, it could yield the physical $^1S_0$ phase shift from 0 to 300 MeV all by itself. However, in so doing, it would yield almost no $^1D_2$ phase shift since the attractive "well" $-(1/M)\times[\phi'/(1+2\phi)]^2$ is of very short range, and vanishes at the range of the 300-MeV D-wave impact parameter.

[7] H. G. Dosch and V. F. Müller, Nuovo Cimento 39, 886 (1965). They point out that the σ_1 increases the spread between the $\delta(^1S_0)$ and $\delta(^3S_1)$ phase shifts, in contradiction to experiment.

One recalls that the conventional "hard core" plus attractive well potential model yields too positive a $^1D_2$ phase shift when adjusted to fit the $^1S_0$ phase shift over the range 0–300 MeV; the $\nabla^2\phi+\phi\nabla^2$ potential yields hardly an $^1D_2$ phase shift at all. The OBEP we describe here predicts a $^1D_2$ phase shift between these two extremes, but somewhat too close to the zero extreme. This difficulty might be alleviated by now including $\mathbf{p}^4$ and higher-order terms, such as the quadratic spin-orbit term, but not likely, since we have calculated that these corrections are small. It would seem more reasonable to reduce the strength of the ω and σ_0 coupling constants, but then the spin-orbit term would be too weak. This may be an indication against the use of the Schrödinger equation to unitarize the pole terms; g_ω and g_0 are much smaller when the pole terms are unitarized by means of a dispersion relation or geometric unitarization.[8]

B. Meson Parameters

In Table I, we have presented the values of the 10 adjustable paremeters in the OBEP along with those values obtained from II. There are many differences between the two sets of constants, as might be expected, but there are also striking similarities. It is also of interest to compare these values with those reported by other estimates, although it is to be expected that the coupling constants reported here should be somewhat distorted by forcing the OBE model with cutoff to accommodate S waves.

First, the pion coupling constant has stayed rather constant near 12. The value of 12.55 given in this work is nearer the value obtained in phase-shift analyses[5] (≈ 13) of nucleon-nucleon scattering than the value[9] of 14.7 generally accepted for this constant obtained from meson-nucleon scattering. However, note that in keeping only terms to order p^2/M^2 in the scattering amplitude we have essentially replaced $(g_\pi^2)_{\mathrm{REL}}(M^2/E^2)$ by $(g_\pi^2)_{\mathrm{NR}}$. Thus if $(g_\pi^2)_{\mathrm{REL}}=14.7$ then we might expect to find $(g_\pi^2)_{\mathrm{NR}}\approx 13$–14 over the range 0–320 MeV.

Another important feature is the large coupling constant for the ω meson. This feature has persisted through each of the papers and is needed to obtain the correct $\mathbf{L}\cdot\mathbf{S}$ splitting for the P waves (see II). Of interest also is the fact that in each paper the tensor coupling for the ω vanishes; $f_\omega/g_\omega=0$. This is in agreement with predictions from the nucleon electromagnetic form-factor data.

The ρ-meson parameters have shifted from the previous OBEP values of Ref. 2 in the expected direction; f_ρ/g_ρ is smaller and g_ρ^2 is larger. g_ρ^2 is now substantially larger than the value of 0.6 given recently by Sakurai[10] and the ratio of $f_\rho/g_\rho(1.13)$ is much smaller than the value of 4 which is favored by the nucleon electromagnetic form-factor data.

The ω and ρ coupling constants can be related by the use of higher symmetry arguments. Thus if one assumed pure F-type coupling of the vector meson octet to the baryon octet, $SU(3)$ predicts $g_{\omega_8}{}^2=3g_\rho{}^2$, where ω_8 signifies the isotopic singlet member of the unmixed octet. Our ratio is 9.9, which is far from the required value even noting that the physical ω is really a mixture of the ω_8 and ω_0 (unitary singlet) vector mesons. However, another estimate of $g_\omega{}^2/g_\rho{}^2$ has been suggested by Sugawara and von Hippel[11] based on the nonet scheme. In that work, they find that the ϕ can be decoupled from the nucleon, and, if one assumes mainly F-type coupling to the vector octet, then $g_\omega{}^2=9g_\rho{}^2$. This number is very nearly the ratio we find.

The η coupling constant has been reduced considerably down to 3.46, which is in fair agreement with the $SU(3)$ prediction of about 2 which one obtains if an F/D ratio of $\frac{1}{3}$ is assumed.

The σ_0 meson probably represents an average of the $T=0$, 2π S-wave contribution to N-N scattering over a considerable range of the π-π effective mass. Walker[12] reports that the $T=0$ S-wave π-π phase shift varies from 30° at $m_{\pi\pi}=300$ MeV to $\approx 90°$ at $m_{\pi\pi}=900$ MeV. Our searches employing a zero-width σ_0 show a best fit when the σ_0 mass is 550 MeV.

There may also be a resonance with the σ_1 quantum numbers. Rosenfeld *et al.*[13] report a meson with $T=1$, $J=0^+$, $G=-1$, with a mass of 1016 MeV and a width of 25 MeV. To fix the σ_1 mass at this value in our searches, however, we would have to introduce an additional parameter for the $^3S_1$ state, corresponding to the scattering length.

One might question whether the σ_1 should be introduced at this stage. We introduce it to fit the P, D, and higher waves even though it makes it a little harder to fit the S waves. Babikov and Kiselev[14] have a OBEP model for the P and higher waves which does not use a σ_1 at all; they use just the σ_0, ρ, π, ω, and η. They obtain a reasonably good fit, but not a close fit [i.e., at 330

[8] For a review of various pole models in N-N scattering, see R. A. Bryan, in *Proceedings of the International Conference on Nuclear Physics, Gatlinburg, Tennessee, 1966*, edited by R. L. Becker (Academic Press Inc., New York, 1967), p. 603.

[9] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[10] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966); also P. Signell and J. W. Durso, *ibid.* **18**, 185 (1967). Note that the definition of g_ρ^2 in these two papers is a factor of 4 larger than ours.

[11] H. Sugawara and Frank von Hippel, Phys. Rev. **145**, 1331 (1966).

[12] W. D. Walker, Rev. Mod. Phys. **39**, 695 (1967); see also E. Malamud and P. E. Schlein, Phys. Rev. Letters **19**, 1056 (1967), who analyze the reaction $\pi^-+p\rightarrow\pi^++\pi^-+n$ for the $T=0$, S-wave π-π phase shift; they find that it passes through 90° at a dipion mass of 730 MeV, implying the existence of a σ_0 meson with this mass. The width is not uniquely determined, but one estimate is $\Gamma=150$ MeV.

[13] A. H. Rosenfeld, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, P. Soding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. **39**, 1 (1967).

[14] V. V. Babikov and V. S. Kiselev, in *Proceedings of the International Conference on Nuclear Structure, Tokyo, Japan, 1967*, edited by J. Sanada (supplement to the Journal of the Phys. Soc. Japan **24**, 1968) p. 618.

MeV, their $\delta(^1P_1)$, and $\delta(^3D_1)$, $\delta(^3D_2)$, and $\delta(^3D_3)$ are about 10° too high, on the average]. We believe that for a close fit, something in addition to the five OBEP above has to be used. (Of course this still does not prove that something physically is there; it might be a result of using the Schrödinger equation to unitarize the Born terms.)

Kiang, Preston, and Yip[15] have an OBEP model which employs only the π, ω, and σ_0 mesons, not even the ρ or η. However they employ a different value for g_ω^2 in the singlet odd state than in the other states, so a $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ dependence is implied, as well as a $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ and a $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ dependence. To replace their state-dependent OBE potentials with ordinary OBE potentials, three or more would be required.

Green and Sawada[16] find, as do Kiang *et al.*, that much of the nucleon-nucleon information can be fit with just the π, ω, and σ_0 potentials. However, when trying for a close fit to the phase shifts, they arrive at the same six OBE potentials that we do, with much the same coupling constants, but they include the σ_1 for a different reason. It is dictated from higher symmetry considerations. They consider the σ_1 to be linked to the ρ in a 5-vector in the same way they consider the σ_0 is linked to the ω. It is interesting that this scheme automatically provides the necessary cancellation of the leading term in the vector-meson-exchange term whether it be the ω or the ρ.

V. SUMMARY

In this work we have improved the approximations of our previous OBEP by consistently keeping all contributions to the scattering amplitude in momentum space through order $\mathbf{p}^2/M$. Taking the Fourier transform of this amplitude as a potential to be used in the nonrelativistic Schrödinger equation we found a $1/r^2$ divergence which was eliminated by means of a smooth cutoff measured by the parameter Λ. The 10 adjustable parameters of the potential were then varied to find a best fit to the experimentally determined phase shifts from 0–310 MeV including low energy S-wave parameters. The fit was qualitatively good over the entire range and at least semiquantitative over the range 0–150 MeV.

A useful aspect of this potential is that it has a basis in field theory, albeit simple, so that it can be related to other problems, such as nucleon-antinucleon scattering[17] or binding, using the G-parity rule for the coupling constants, or to hyperon-nucleon scattering, using $SU(3)$ to relate the coupling constants. The form of cutoff that we use is convenient, too; it is just carried over unchanged. If we had used a hard core, say, there would be some question as to what it transforms into in the nucleon-antinucleon system (e.g., if the hard core is thought to be due to vector meson exchange, it would transform into an infinite attraction).

The particular momentum-dependent nature of the potential can be tested in treating problems where S-wave attraction plays a dominant role, as, for example, in the nuclear-matter problem where the off-energy-shell matrix elements are so important. These are completely unprobed by elastic scattering. The off-shell elements are also important in the p-p bremsstrahlung calculations. Brown has carried out calculations using this and other models.[18]

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[15] D. Kiang, M. A. Preston, and P. Yip, Phys. Rev. **170**, 907 (1968).

[16] A. E. S. Green and T. Sawada, Rev. Mod. Phys. **39**, 594 (1967).

[17] R. A. Bryan and R. J. N. Phillips have calculated $\bar{p}$-p and $\bar{n}$-p cross sections and polarizations using the OBE N-N potentials (I, II, and III) modified by the G-parity rule, plus an additional imaginary well of the Woods-Saxon type. They find good agreement with experiment. R. J. N. Phillips, Rev. Mod. Phys. **39**, 681 (1967); R. A. Bryan and R. J. N. Phillips, Nucl. Phys. **B5**, 201 (1968) and *erratum* (to be published).

[18] Virginia Brown, Phys. Letters **25B**, 506 (1967).