Coulomb Scattering in the Three-Body Problem*

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Using the Faddeev formalism for the three-body problem with two charged particles, with separable nuclear interactions and an approximate form for the Coulomb wave function and Coulomb Green's function, a simplified model is set up for *p-d* scattering. The differential scattering cross section is calculated numerically for different incident proton energies from 2.08 to 14.0 MeV. The calculations are repeated with the Coulomb potential turned off, and the results are compared with the Coulomb scattering.

I. INTRODUCTION

We consider a system of three spinless particles of equal mass M, of which two are charged and the third is uncharged. Initially, we consider the two charged particles to be distinguishable bosons, and then symmetrize. Particles 1 and 2 are charged, particle 3 is uncharged. The aim of this paper is to set up a model for elastic proton-deuteron scattering with separable short-range interactions between any pair of particles, plus the Coulomb interaction of particles 1 and 2. We use the word "model" in the sense that a realistic *physical* model, with spin included, can be based on the present analytical treatment. It is not implied that in the absence of the Coulomb potential our present treatment would be a model for *n-d* scattering. For convenience and brevity, we shall refer to our particles as protons and neutrons although this is not strictly true, since we are treating them as bosons.

Our approach to the problem consists of using separable short-range potentials in the Faddeev equations, which are modified to incorporate the Coulomb potential between the protons. The use of nonlocal separable potentials enables us to reduce the integral equations for the three-body scattering amplitudes to equations in one vector variable. A partial-wave analysis will then reduce these equations to one-dimensional linear integral equations, which can be solved numerically. For the three-body problem without Coulomb forces, the Faddeev formalism with separable potentials has been considered by a number of authors.[1–3] In our case, the Faddeev equations can be solved, provided we can find a suitable representation for the Coulomb Green's function. In this paper, we make use of an approximate form for the Coulomb wave function in momentum space, suggested by Schulman.[4] The resulting representation for the Green's function, which of course is only an approximation, can be used to obtain a set of Coulomb-modified Faddeev equations.[5] These approximations were used in an earlier paper[6] to obtain the binding energy of He^3. The Schulman approximation for the Coulomb Green's function gave too much Coulomb repulsion in the He^3 case; an "improved" version of this approximation was found to give a better result. In this paper, we shall make use of the former approximation only. It is of course possible to do the scattering problem with the improved version also; but in view of the exploratory nature of this paper, and the extensive calculations involved, we have used Schulman's approximation only.

Even with this simplification, the *p-d* scattering problem is a hard one to solve. The main difficulty in our treatment of the problem arises from the asymmetrical charge distribution of the deuteron. It therefore becomes necessary to treat the Coulomb-distorted wave function, which describes the *p-d* scattering, in some reasonable approximation scheme. This is considered in detail in Sec. VI.

In this paper, our main interest lies in showing that it *is* possible to solve the *p-d* scattering problem approximately, within the framework of the Faddeev formalism and separable interactions. In principle, the method is straightforward. In practice, the solution of the Faddeev equations is extremely complicated, since the Coulomb-distorted wave functions occur in the inhomogeneous terms of the integral equations. Using a more realistic model with spin and improved nuclear potentials, and perhaps with a better approximation for the Coulomb-distorted wave function, it may be possible to improve upon the results obtained here. We wish to emphasize that the problem, although laborious and complicated, is a straightforward one.

In this paper, the short-range interaction between any pair of particles is taken to be of the separable Yamaguchi type

$$\langle q' \mid V \mid q \rangle = \lambda v(q') v(q), \tag{1}$$

where

$$v(q) = (\beta^2 + q^2)^{-1}.$$

We use this simple form for convenience, since our main interest lies in seeing how we can include the Coulomb force within the framework of Faddeev equations plus separable interactions. The parameter β is chosen to be 1.33 F^{-1}, which is approximately the mean value of the singlet and triplet values[7] $\beta_s = 1.16$ F^{-1} and $\beta_t = 1.44$ F^{-1}. The parameter λ, which is the coupling constant, is determined by the condition that the two-body T

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[1] C. A. Lovelace, Phys. Rev. **135**, B1225 (1964).
[2] J. Hetherington and L. Schick, Phys. Rev. **156**, 1647 (1967).
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[4] L. Schulman, Phys. Rev. **156**, 1129 (1967).
[5] J. Noble, Phys. Rev. **161**, 945 (1967).
[6] S. Adya, Phys. Rev. **166**, 991 (1968).
[7] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954). We use the low-energy *n-p* scattering data quoted in this reference.

matrix $\hat{T}(E)$ have a pole at $E=-\epsilon_d$, where ϵ_d is the deuteron binding energy. This gives

$$M\lambda=8\pi\beta[\beta+(M\mid\epsilon_d\mid)^{1/2}]^2. \quad (2)$$

Having chosen these parameters, we proceed as follows:

(1) The symmetrized wave function for both the incoming and outgoing channel is

$$\sqrt{2}^{-1}[\psi(1,2,3)+\psi(2,1,3)]. \quad (3)$$

The total symmetrized scattering amplitude for the process can be written in the form

$$Y=Y_{11}+Y_{21}. \quad (4)$$

Y_{11} is the amplitude for the reaction 1, (23)→1, (23), where we have taken particle 1 to be free initially; Y_{21} is the amplitude for the process 1, (23)→2, (13), in which the two charged particles exchange places.

(2) Y_{11} can be written as the sum of two parts, one involving the Coulomb potential only (and which includes the Rutherford scattering amplitude), and the other, T_D, involving the short-range potentials. The Coulomb potential manifests itself in T_D also through it's presence in the Green's-function operator.

Y_{21} can also be expressed as the sum of two terms. In this case, however, the term involving U^c explicitly, is of an order of magnitude which is vanishingly small as compared to the short-range term. Y_{21} is therefore simply T_R, where the operator T_R is the nuclear part of the exchange amplitude Y_{21}. T_R in Y_{21} corresponds to T_D in Y_{11}.

(3) We introduce three auxiliary operators Z_1, Z_2, Z_3. It is possible to express T_D and T_R in terms of the Z's, and the short-range potential V_{23}. The Z's satisfy a set of three simultaneous equations which are of the Faddeev type.[8] These equations can be further simplified by expressing the matrix elements of the Z_i in terms of functions h_i. The h_i are functions of a single vector variable, and may be called the "reduced amplitudes."

(4) A partial-wave analysis is made of the set of simultaneous equations satisfied by the h_i. The partial-wave components $h_i^{(l)}$ ($l=0, 1, 2, \cdots$) are now functions of a single scalar variable. This enables us to numerically solve the equations. Since this work deals with low-energy scattering, it was considered sufficient to take the first three partial waves only.

(5) A suitable representation must be found for the Coulomb-distorted wave function. This is one of the major difficulties of the problem, since we require that the form chosen for the wave function should make the problem amenable to a numerical solution. In the present case, we expand the exact wave function in terms of the complete set of two-particle states $u_n(\mathbf{r})$ which satisfy

$$[-\nabla_r^2+\hat{V}]u_n(\mathbf{r})=\epsilon_n u_n(\mathbf{r}). \quad (5)$$

Here, as elsewhere, carets denote that the operators are in the two-particle space.

Although the coefficients of the $u_n(\mathbf{r})$ can be written down formally, it is not possible to solve the problem numerically without making some approximations. Some of these approximations are unavoidable; others were made to reduce the complexity of the problem. The details are left for Sec. VI.

After inverting the integral equations numerically on a computer, we calculate the differential scattering cross section for different incident proton energies.

Finally, the calculations are repeated for the case where the Coulomb force is taken to be absent, although particles 1 and 2 are still treated as similar, and different from particle 3.

In Sec. II, we briefly consider the approximation which is made for the Coulomb Green's function and Coulomb wave function. Assuming a Hulthén wave function, we write down the relation between λ and β. The kinematics of the problem are also discussed.

Section III deals with the total scattering amplitude, which is the sum of a direct and an exchange part.

The operators Z_i ($i=1, 2, 3$) and the coupled equations which they satisfy, are considered in Sec. IV.

Section V deals with the reduced amplitudes h_i. We also consider how these equations are solved numerically; in particular, the avoidance of singularities. The Coulomb-distorted wave function is treated in detail in Sec. VI.

Finally, the results and conclusions are summed up in Sec. VII.

II. COULOMB WAVE FUNCTION AND COULOMB GREEN'S FUNCTION

The Schulman[4] approximation for the Coulomb wave function uses the fact that p-p scattering is strongly peaked in the forward direction. In momentum space, the Coulomb wave function is approximately (see Ref. 4)

$$\phi_k(\mathbf{p})\simeq C(k)\delta(\mathbf{k}-\mathbf{p}), \quad (6)$$

where

$$C(k)=\exp(-\tfrac{1}{2}\pi\rho)\,\Gamma(1+i\rho)$$
$$=[\psi_k^c(\mathbf{r})]_{r=0}, \qquad \rho=\mu e^2/k$$

and

$$|C(k)|=\left[\frac{2\pi\mu e^2/k}{\exp(2\pi\mu e^2/k)-1}\right]^{1/2},$$

where $\psi_k^c(\mathbf{r})$ is the wave function in configuration space. This approximation is valid in the sense that if $f(\mathbf{p})$ is a smooth function of $\mathbf{p}$

$$\int\phi_k(\mathbf{p})f(\mathbf{p})\,d^3p\simeq C(k)f(\mathbf{k}).$$

[8] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Company, Inc., New York, 1966), Chap. 17, Sec. 4.3.

The Coulomb Green's function is then approximately given by

$$\langle \mathbf{p}' \mid \hat{G}_0{}^c(E) \mid \mathbf{p}\rangle = \frac{\delta(\mathbf{p}'-\mathbf{p}) \mid C(p) \mid^2}{E-(p^2/2\mu)}, \qquad (7)$$

where

$$\hat{G}_0{}^c(E) = (E-\hat{H}_0-\hat{U}^c)^{-1}.$$

This approximation to the Coulomb Green's function was used in the solution of the bound state problem, i.e., the binding energy of He^3, in Ref. 6.

Let $\mathbf{p}_i$ $(i=1, 2, 3)$ denote the momenta of the three particles in the c.m. system. We denote the relative momenta of any pair of particles by $\mathbf{q}_i$, so that $\mathbf{q}_{23}=\frac{1}{2}(\mathbf{p}_3-\mathbf{p}_2)$. It is customary to use the cyclic notation $\mathbf{q}_{23}\equiv\mathbf{q}_1$, etc. The kinetic energy part of the Hamiltonian is

$$H_0=p_i{}^2/2\eta+q_i{}^2/2\mu,$$

where we can use any of the pairs $(\mathbf{p}_1, \mathbf{q}_1)$, $(\mathbf{p}_2, \mathbf{q}_2)$, or $(\mathbf{p}_3, \mathbf{q}_3)$.

Here

$$\eta=\tfrac{2}{3}M, \qquad \mu=\tfrac{1}{2}M.$$

The short-range two-body forces are denoted by V_{ij}, or, using the cyclic notation, by V_k where i, j, k form a cyclic permutation.

We assume here that the two-body interactions are of the s-wave separable type

$$\langle \mathbf{q}_i' \mid \hat{V}_i \mid \mathbf{q}_i\rangle=\lambda v(q_i')v(q_i),$$

where $v(q)=(\beta^2+q^2)^{-1}$.

In configuration space, the two-particle bound state wave function corresponding to this potential is the Hulthén function

$$u(r)\sim\frac{e^{-\beta r}-e^{-\alpha r}}{r}, \qquad \alpha^2=2\mu \mid \epsilon_d \mid.$$

The two-body T matrix satisfies

$$\hat{T}_i=\hat{V}_i+\hat{V}_i\hat{G}_0\hat{T}_i,$$

$$\hat{G}_0(E)=(E-\hat{H}_0)^{-1}. \qquad (8a)$$

Using (1), the solution of (8a) can be written

$$\langle \mathbf{q}_i' \mid T_i \mid \mathbf{q}_i\rangle=v(q_i')v(q_i)\tau(E),$$

with

$$\tau(E)=\left[\lambda^{-1}-(2\pi)^{-3}\int\frac{v^2(q)\,d^3q}{E-(q^2/2\mu)+i\epsilon}\right]^{-1}. \qquad (8b)$$

Since $\hat{T}_i$ has a pole at $E=-\epsilon_d$, the deuteron binding energy, Eq. (8b) gives the desired relation between λ and β.

$$M\lambda=8\pi\beta[\beta+(M \mid \epsilon_d \mid)^{1/2}]^2.$$

For the three-body system, the total Hamiltonian is therefore

$$H=H_0+U^c+\sum_{i=1}^{3}V_j,$$

where U^c is the Coulomb potential acting between particles 1 and 2. The Green's function $G_0{}^c(W)\equiv(W-H_0-U^c)^{-1}$ can be represented by the approximate form

$$\langle \mathbf{p}_3'\mathbf{q}_3' \mid G_0{}^c(W) \mid \mathbf{p}_3\mathbf{q}_3\rangle\rightarrow\frac{\delta(\mathbf{p}_3'-\mathbf{p}_3)\delta(\mathbf{q}_3'-\mathbf{q}_3) \mid C(q_3) \mid^2}{W-(p_3{}^2/2\eta)-(q_3{}^2/2\mu)}, \qquad (9)$$

where the arrow means "can be approximately replaced by."

III. AMPLITUDE FOR ELASTIC p-d SCATTERING

Using the symmetrical wave function

$$(1/\sqrt{2})[\psi(1, 2, 3)+\psi(2, 1, 3)]$$

for the in and out channels, it can be shown that the total amplitude for the "p-d" elastic scattering can be written

$$Y=Y_{11}+Y_{21},$$

where Y_{11} is the amplitude for the process 1, (23)$\rightarrow$ 1, (23) and Y_{21} for 1, (23)$\rightarrow$2, (1, 3), since particles 1 and 2 are similar.

Consider first the direct amplitude Y_{11}. The total Hamiltonian is

$$H=H_0+U^c+V_1+V_2+V_3.$$

Let $\mid \phi_i\rangle$ be the eigenstates of H_0+V_1, and $\mid \chi_i\rangle$ those of $H_0+U^c+V_1$. Then Y_{11} is given by

$$\langle f \mid Y_{11} \mid i\rangle = \langle \phi_f \mid U^c+\tilde{V}+(U^c+\tilde{V})G^c(W)(U^c+\tilde{V}) \mid \phi_i\rangle, \qquad (10)$$

where

$$G^c(W)=(W-H)^{-1}, \qquad \tilde{V}=V_2+V_3.$$

Let

$$G_1{}^c(W)=(W-H_0-U^c-V_1)^{-1} \qquad (11)$$

and define the operator T_D by

$$G^c(W)=G_1{}^c(W)+G_1{}^c(W)\,T_D\,G_1{}^c(W). \qquad (12)$$

From (11) and (12), we have an integral equation for the operator T_D:

$$T_D=\tilde{V}+\tilde{V}G_1{}^cT_D. \qquad (13)$$

Equation (10) can be simplified to give

$$\langle f \mid Y_{11} \mid i\rangle=\langle \phi_f \mid U^c \mid \chi_i{}^{(+)}\rangle+\langle \chi_f{}^{(-)} \mid T_D \mid \chi_i{}^{(+)}\rangle. \qquad (14)$$

The first term $\langle \phi_f \mid U^c \mid \chi_i\rangle$ includes the Rutherford scattering amplitudes plus a correction term which takes into account the fact that the deuteron charge is not at the center.

The second term $\langle \chi_f{}^{(-)} \mid T_D \mid \chi_i{}^{(+)}\rangle$ would reduce, as $U^c\rightarrow 0$, to the amplitude for the chargeless problem.

The exchange amplitude Y_{21} in which particle 1 is incident and particle 2 emerges is

$$\langle f \mid Y_{21} \mid i\rangle = \langle \tilde{\phi}_f \mid U+\tilde{W}+(U+\tilde{W})G^c(W)(U+\hat{V}) \mid \phi_i\rangle, \tag{15}$$

where $\mid \tilde{\phi}\rangle$ is the eigenstate of H_0+V_2 and

$$\tilde{W}=V_1+V_3. \tag{16}$$

In the exchange amplitude, the direct Coulomb amplitude is of an order which is vanishingly small compared to the direct nuclear term. To see this, let $\mid \tilde{\Psi}\rangle$ be the eigenstate of the total Hamiltonian H, and define T_R by

$$T_R=\tilde{W}+\tilde{W}G_1^cT_D. \tag{17}$$

Let $\mid \tilde{\chi}\rangle$ be the eigenstate of H_0+U+V_2. From (15)

$$\begin{aligned}\langle f \mid Y_{21} \mid i\rangle &= \langle \tilde{\phi}_f \mid U^c+\tilde{W} \mid \tilde{\Psi}_i\rangle \\ &= \langle \tilde{\chi}_f \mid U^c+\tilde{W} \mid \tilde{\Psi}_i\rangle - \langle \tilde{\chi}_f \mid U^cG_2(W)(U^c+\tilde{W}) \mid \tilde{\Psi}_i\rangle,\end{aligned}$$

where we have used

$$\langle \tilde{\phi}_f \mid = \langle \tilde{\chi}_f \mid - \langle \tilde{\chi}_f \mid U^cG_2$$

with

$$G_2(W)=(W-H_0-V_2)^{-1},$$

$$\begin{aligned}\langle f \mid Y_{21} \mid i\rangle &= \langle \tilde{\chi}_f \mid \tilde{W} \mid \tilde{\Psi}_i\rangle \\ &\quad + \langle \tilde{\chi}_f \mid U[1-G_2(W)(U+\tilde{W})] \mid \tilde{\Psi}_i\rangle \\ &= \langle \tilde{\chi}_f \mid \tilde{W}(1+G_1^cT_D) \mid \chi_i\rangle \\ &\quad + i\epsilon\langle \tilde{\chi}_f \mid UG_2 \mid \tilde{\Psi}_i\rangle\end{aligned}$$

using

$$\mid \tilde{\Psi}_i\rangle = \mid \chi_i\rangle + G_1^cT_D \mid \chi_i\rangle.$$

Finally,

$$\langle f \mid Y_{21} \mid i\rangle = \langle \tilde{\chi}_f^{(-)} \mid T_R \mid \chi_i^{(+)}\rangle + i\epsilon\langle \tilde{\chi}_f \mid UG_2 \mid \tilde{\Psi}_i\rangle.$$

The second term can be neglected, and we get

$$\langle f \mid Y_{21} \mid i\rangle = \langle \tilde{\chi}_f^{(-)} \mid T_R \mid \chi_i^{(+)}\rangle. \tag{18}$$

The total amplitude is therefore

$$\begin{aligned}\langle f \mid Y \mid i\rangle &= \langle \phi_f \mid U^c \mid \chi_i^{(+)}\rangle + \langle \chi_f^{(-)} \mid T_D \mid \chi,^{(+)}\rangle \\ &\quad + \langle \tilde{\chi}_f^{(-)} \mid T_R \mid \chi_i^{(+)}\rangle.\end{aligned} \tag{19}$$

The operators T_D and T_R can be decomposed into component operators Z_i, which satisfy a set of coupled equations. We consider this in Sec. IV.

IV. FADDEEV EQUATIONS FOR THE NUCLEAR AMPLITUDES

We now return to Eqs. (13) and (17) of Sec. III.

$$\begin{aligned}T_D &= \tilde{V}+\tilde{V}G_1^cT_D, \qquad \tilde{V}=V_2+V_3, \\ T_R &= \tilde{W}+\tilde{W}G_1^cT_D, \qquad \tilde{W}=V_1+V_3,\end{aligned} \tag{20}$$

where it is to be understood that the matrix elements of T_D are to be taken between the eigenstates $\mid \chi_f\rangle$ and $\mid \chi_f\rangle$, and those of T_R between the eigenstates $\mid \tilde{\chi}_f\rangle$ and $\mid \chi_i\rangle$.

Defining the auxiliary operators Z_2 and Z_3 by

$$Z_2=V_2+V_2G_1^cT_D, \qquad Z_3=V_3+V_3G_1^cT_D \tag{21}$$

we note from (20) that T_D can be written as

$$T_D=Z_2+Z_3. \tag{22}$$

We introduce a third operator $Z_1=V_1G_1^cT_D$ and obtain the set of equations

$$\begin{aligned}Z_1 &= V_1G_1^cT_D, \\ Z_j &= V_j+V_jG_1^cT_D, \qquad j=2, 3.\end{aligned} \tag{23}$$

In terms of the Z_i, we can write T_R as

$$T_R=V_1+Z_1+Z_3. \tag{24}$$

Equations (23) can be transformed so that they contain the two-body T matrices rather than the two body potentials V_i. The method is analogous to the usual procedure of obtaining the Faddeev equations. Using the relations

$$\begin{aligned}T_i &= V_i+V_iG_i^cV_i, \\ G_i^c &= G_0^c+G_0^cV_iG_i^c,\end{aligned} \tag{25}$$

where

$$G_i^c=(W-H_0-U-V_i)^{-1}, \qquad G_0^c=(W-H_0-U)^{-1},$$
$$U\equiv U^c$$

we can transform Eqs. (23) into the form

$$\begin{aligned}Z_1 &= T_iG_0^c(Z_2+Z_3), \\ Z_i &= T_i+T_iG_0^c(Z_j+Z_k), \qquad i=2, 3;\ i\neq j\neq k.\end{aligned} \tag{26}$$

We now have a set of three coupled integral equations for the operators Z_i $(i=1, 2, 3)$.

In terms of the Z's, the term $\langle \chi_f \mid T_D \mid \chi_i\rangle$ which occurs in the scattering amplitude [Eq. (19)] can be written explicitly as

$$\begin{aligned}&\int \langle \chi_f^{(-)} \mid \mathbf{p}_2'\mathbf{q}_2'\rangle\langle \mathbf{p}_2'\mathbf{q}_2' \mid Z_2 \mid \mathbf{p}_2\mathbf{q}_2\rangle\langle \mathbf{p}_2\mathbf{q}_2 \mid \chi_i^{(+)}\rangle \\ &\qquad \times d^3p_2'd^3q_2'd^3p_2d^3q_2 \\ &+\int \langle \chi_f^{(-)} \mid \mathbf{p}_3'\mathbf{q}_3'\rangle\langle \mathbf{p}_3'\mathbf{q}_3' \mid Z_3 \mid \mathbf{p}_3\mathbf{q}_3\rangle\langle \mathbf{p}_3\mathbf{q}_3 \mid \chi_i^{(+)}\rangle \\ &\qquad \times d^3p_3'd^3q_3'd^3p_3d_3q_3,\end{aligned} \tag{27}$$

with an analogous expression for the term

$$\langle \tilde{\chi}_f^{(-)} \mid T_R \mid \chi_i^{(+)}\rangle.$$

V. SOLUTION OF THE INTEGRAL EQUATIONS

To solve Eqs. (26), we use our approximate form for the Coulomb Green's function G_0^c, and the known forms

for the two-body T matrices in the three-body space,

$$\langle \mathbf{p}_i'\mathbf{q}_i' \mid T_i(W) \mid \mathbf{p}_i\mathbf{q}_i\rangle = v(q_i')\tau_i^c(p_i', W)\delta(\mathbf{p}_i'-\mathbf{p}_i)v(q_i),$$

where

$$\tau_3^c(p, W) \simeq \left[\lambda^{-1} - (2\pi)^{-3}\int v^2(q)\left(W - \frac{p^2}{2\eta} - \frac{q^2}{2\mu}\right)^{-1} \mid C(q) \mid^2 d^3q\right]^{-1} \tag{28}$$

and

$$\tau_{1,2}^c(p, W) \simeq \left[\lambda^{-1} - (2\pi)^{-3}\int v^2(q)\left(W - \frac{p^2}{2\eta} - \frac{q^2}{2\mu}\right)^{-1} \mid C(\mid \tfrac{3}{4}\mathbf{p} - \tfrac{1}{2}\mathbf{q} \mid) \mid^2 d^3q\right]^{-1}.$$

In the integrals, the limit $\epsilon \to 0$ is understood. We now define the reduced amplitudes h_i. We first define

$$\langle \mathbf{p}_i'\mathbf{q}_i' \mid Z_i(W) \mid \mathbf{p}_1\mathbf{q}_1\rangle = v(\mathbf{q}_i')f_i(\mathbf{p}_i', \mathbf{p}_1, \mathbf{q}_1, W) \tag{29}$$

to obtain a set of equations for the f_i:

$$f_1(\mathbf{p}_1', \mathbf{p}_1, \mathbf{q}_1, W) = \tau_1^c(p_1', W)\left[0 - (2\pi)^{-3}\int d^3p_2''\, R(\mathbf{p}_2'', \mathbf{p}_1', W)f_2(\mathbf{p}_2'', \mathbf{p}_1, \mathbf{q}_1, W)\right.$$
$$\left. - (2\pi)^{-3}\int d^3p_3''\, S(\mathbf{p}_3'', \mathbf{p}_1', W)f_3(\mathbf{p}_3'', \mathbf{p}_1, \mathbf{q}_1, W)\right],$$

$$f_2(\mathbf{p}_2', \mathbf{p}_1, \mathbf{q}_1, W) = \tau_2^c(p_2', W)\left[v(-\tfrac{3}{4}\mathbf{p}_1 - \tfrac{1}{2}\mathbf{q}_1)\delta\{\mathbf{p}_2' - (-\tfrac{1}{2}\mathbf{p}_1 + \mathbf{q}_1)\}\right.$$
$$\left. - (2\pi)^{-3}\int d^3p_1''\, R(\mathbf{p}_1'', \mathbf{p}_2', W)f_1(\mathbf{p}_1'', \mathbf{p}_1, \mathbf{q}_1, W) - (2\pi)^{-3}\int d^3p_3''\, S(\mathbf{p}_3'', \mathbf{p}_2', W)f_3(\mathbf{p}_3'', \mathbf{p}_1, \mathbf{q}_1, W)\right], \tag{30}$$

$$f_3(\mathbf{p}_3', \mathbf{p}_1, \mathbf{q}_1, W) = \tau_3^c(p_3', W)\left[v(\tfrac{3}{4}\mathbf{p}_1 - \tfrac{1}{2}\mathbf{q}_1)\delta\{\mathbf{p}_3' - (-\tfrac{1}{2}\mathbf{p}_1 - \mathbf{q}_1)\}\right.$$
$$\left. - (2\pi)^{-3}\int d^3p_1''\, T(\mathbf{p}_1'', \mathbf{p}_3', W)f_1(\mathbf{p}_1'', \mathbf{p}_1, \mathbf{q}_1, W) - (2\pi)^{-3}\int d^3p_2''\, T(\mathbf{p}_2'', \mathbf{p}_3', W)f_2(\mathbf{p}_2'', \mathbf{p}_1, \mathbf{q}_1, W)\right].$$

The lth partial waves of kernels R, S, and T are given (with the propagators τ^c included) in Eq. (34). We multpliy each of these equations by $\chi_k(\mathbf{p}_1, \mathbf{q}_1)$, since the initial wave function for both T_D and T_R happens to be the same, $\chi_k(\mathbf{p}_1, \mathbf{q}_1)$, and integrate over $\mathbf{p}_1$ and $\mathbf{q}_1$. This gives

$$h_1^{(k)}(\mathbf{p}_1', W) = \tau_1^c(p_1', W)\left[0 - \Omega\int d^3p_2''\, R(\mathbf{p}_2'', \mathbf{p}_1', W)h_2^{(k)}(\mathbf{p}_2'', W) - \Omega\int d^3p_3''\, S(\mathbf{p}_3'', \mathbf{p}_1', W)h_3^{(k)}(\mathbf{p}_3'', W)\right],$$
$$h_2^{(k)}(\mathbf{p}_2', W) = \tau_2^c(p_2', W)[I_k^{(+)}(\mathbf{p}_2') - \Omega R h_1 - \Omega S h_3], \tag{31}$$
$$h_3^{(k)}(\mathbf{p}_3', W) = \tau_3^c(p_3', W)[I_k^{(-)}(\mathbf{p}_3') - \Omega T h_1 - \Omega T h_2],$$

where

$$\Omega = (2\pi)^{-3}, \qquad h_i^{(k)}(\mathbf{p}_i', W) = \int d^3p_1 d^3q_1 f_i(\mathbf{p}_i', p_1, q_1, W)\chi_k(\mathbf{p}_1, \mathbf{q}_1),$$

$$I_k^{(\pm)}(\mathbf{p}') = \int d^3p\, v(\mathbf{p} + \tfrac{1}{2}\mathbf{p}')\chi_k\{\mathbf{p}, \pm(\mathbf{p}' + \tfrac{1}{2}\mathbf{p})\}$$

and the notation Rh_1, Sh_3, etc., is a condensed form for integrals of the type appearing in the first of Eqs. (31).

In terms of the reduced amplitudes h_i we can write

$$\langle \chi_{k'}^{(-)} \mid T_D \mid \chi_k^{(+)}\rangle = \int \langle \chi_{k'}^{(-)} \mid \mathbf{p}_2'\mathbf{q}_2'\rangle v(\mathbf{q}_2')h_2^{(k)}(\mathbf{p}_2')d^3p_2'd^3q_2' + \int \langle \chi_{k'}^{(-)} \mid \mathbf{p}_3'\mathbf{q}_3'\rangle v(\mathbf{q}_3')h_3^{(k)}(\mathbf{p}_3')d^3p_3'd_3q_3', \tag{32a}$$

$$\langle \tilde{\chi}_{k'}^{(-)} \mid T_R \mid \chi_k^{(+)}\rangle = \int \langle \chi_{k'}^{(-)} \mid \mathbf{p}_1'\mathbf{q}_1'\rangle v(\mathbf{q}_1')h_1^{(k)}(\mathbf{p}_1')d^3p_1'd^3q_1'$$
$$+ \langle \tilde{\chi}_{k'}^{(-)} \mid V_1 \mid \chi_k^{(+)}\rangle + \int \langle \tilde{\chi}_{k'}^{(-)} \mid \mathbf{p}_3'\mathbf{q}_3'\rangle v(\mathbf{q}_3')h_3^{(k)}(\mathbf{p}_3')d^3p_3'd_3q_3'. \tag{32b}$$

The natural coordinates of $|\chi_f^{(-)}\rangle$ are $(\mathbf{p}_1, \mathbf{q}_1)$ since particle 1 is free in this final state; those of $|\tilde{\chi}_f^{(-)}\rangle$ are $(\mathbf{p}_2, \mathbf{q}_2)$ since particle 2 is free in this final state. Thus, for instance, $\langle\chi_f^{(-)}|\mathbf{p}_2'\mathbf{q}_2'\rangle$ means $\langle\chi_f^{(-)}|-\frac{1}{2}\mathbf{p}_1'+\mathbf{q}_1', -\frac{3}{4}\mathbf{p}_1'-\frac{1}{2}\mathbf{q}_1'\rangle$ using the kinematical results given in Appendix A.

Making a partial-wave analysis of Eq. (31), we get for the lth partial wave, the set of equations

$$\begin{pmatrix} 1 & R^l & S^l \\ R^l & 1 & S^l \\ T^l & T^l & 1 \end{pmatrix}\begin{pmatrix} h_1^l \\ h_2^l \\ h_3^l \end{pmatrix} = \begin{pmatrix} 0 \\ \tau_2^c I^{(+),l} \\ \tau_3^c I^{(-),l} \end{pmatrix}, \tag{33}$$

where

$$R^l(p, p', W) = \frac{\mu p'^2}{2\pi^2}\frac{\tau_{1,2}^c(p, W)}{(pp')^3}\int_{-1}^{+1}\frac{dx\, P_l(x) M(p, p', x)[\exp\{M(p, p', x)\}-1]^{-1}}{[A(p, p')+x][B(p, p')+x][C(p, p')+x]},$$

with

$$M(p, p', x) = E_1(p, p')/[D_1(p, p', x)]^{1/2},$$

where

$$\begin{aligned} A(p, p') &= (\beta^2+p'^2+0.25p^2)/pp', \\ B(p, p') &= A(p', p), \qquad C(p, p') = (p^2+p'^2-WM)/pp', \\ E_1(p, p') &= 4\pi\mu e^2/(2pp')^{1/2}, \qquad D_1(p, p', x) = (p'^2+p^2)/2pp'-x. \end{aligned} \tag{34}$$

Let

$$E_2(p, p') \equiv E_3(p, p') = E_1(p, p')/\sqrt{2},$$

and

$$D_2(p, p', x) = [(4p^2+p'^2)/4pp']+x, \qquad D_3(p, p', x) = D_2(p', p', x).$$

For $S^l(p, p')$, $T^l(p, p')$, we replace $\tau_{1,2}^c$, $D_1(p, p', x)$, E_1 by $\tau_{1,2}^c$, D_2, E_2 and τ_3^c, D_3, E_3, respectively. A slight inaccuracy in the kernel, as it appeared in Eq. (32) of Ref. 6, is corrected by Eqs. (33) and (34) in this work. The numerical results of Ref. 6 are unchanged since the correct equations were used in the calculations.

Consider the kernel $R^l(u, v, W)$ [Eq. (34)]. It has branch points at $A(u, v) = \pm 1$, $B(u, v) = \pm 1$, $C(u, v) = \pm 1$. The first of these relations gives

$$\beta^2+v^2+\tfrac{1}{4}u^2 = \pm uv$$

or

$$-\beta^2 = (v\pm\tfrac{1}{2}u)^2,$$

a relation which cannot be satisfied for real values of v and u. The same is true for the case of $B(u, v)$.

The third relation $C(u, v) = \pm 1$, however, can be satisfied for real values of u and v. This is so because $C(u, v) = \pm 1$ can be rewritten in the form $u^2+v^2+(u+v)^2-2mW=0$, or $(u+v)^2-2mW<0$ for real u, v. This relation can be satisfied for real u, v; hence if we integrate Eqs. (33) along the real axis, we would expect to cross the singularities of the kernel.

In order to surmount this difficulty, we allow both the variables u and v to become complex.[9,10] Let

$$u \to ue^{-i\phi}, \qquad v \to ve^{-i\phi}, \tag{35}$$

when the condition on ϕ is discussed below. We now solve the integral equations, and obtain $h_i(ue^{-i\phi})$. In order to obtain $h_i(u)$, we let v alone become complex in the next step

$$u \to u, \qquad v \to ve^{-i\phi}.$$

The quantities $h_i(ve^{-i\phi})$ occurring inside the integral signs having already been obtained in the first step; we thus get the $h_i(u)$.

The transformation $u \to ue^{-i\phi}$, $v \to ve^{-i\phi}$, when substituted into the branch-point relations $A(u, v) = \pm 1$, etc., implies that

$$u_0^2+v^2 \pm u_0 v - WM\cos(2\phi) = 0, \tag{36a}$$

$$WM\sin(2\phi) = 0, \tag{36b}$$

and

$$u_0^2+v^2 \pm u_0 v + \beta^2\cos(2\phi) = 0, \tag{36c}$$

$$\beta^2\sin(2\phi) = 0, \tag{36d}$$

for a fixed value $u = u_0$. This gives, from (36a) and (36c),

$$\sin(2\phi) = 0 \quad \text{and} \quad (WM-\beta^2)\cos(2\phi) = 0,$$

and since

$$WM \neq \beta^2, \qquad \phi = 0 \quad \text{and} \quad \phi = \tfrac{1}{4}\pi.$$

Therefore, if we choose ϕ to lie between 0 and $\frac{1}{4}\pi$, we can avoid the singularities of the kernel.

In this discussion, we have assumed that the only singularities are those of the kernels R^l, S^l, T^l, and that the reduced amplitudes h_i themselves have no singularities.

Having deformed the contour as explained above, we now replace the integrals by sums, using Gaussian

[9] J. Hetherington and L. Schick, Phys. Rev. **137**, B935 (1965).
[10] R. Aaron and R. D. Amado, Phys. Rev. **150**, 857 (1966).

quadratures, and solve the equations numerically on a high-speed electronic computer.

VI. COULOMB-DISTORTED WAVE FUNCTION FOR p-d SCATTERING

In this section, we consider the Coulomb-distorted wave function $\psi_k(\mathbf{p}, \mathbf{q})$ which describes the proton-deuteron scattering process; we called it χ in Secs. III to V. It is simpler to consider the corresponding wave function $\Phi_k(\mathbf{R}, \mathbf{r})$ in configuration space, where r is the distance between the proton and neutron which form the deuteron, and R is the distance from the external proton (1) to the c.m. system of the deuteron. $\Phi_k(\mathbf{R}, \mathbf{r})$ satisfies the equation

$$\left[-\nabla_R^2-\nabla_r^2+\frac{e^2}{|\mathbf{r}_1-\mathbf{r}_2|}+V_{23}-(E_k-|\epsilon_d|)\right]\times\Phi_k(\mathbf{R}, \mathbf{r})=0. \quad (37)$$

$|\mathbf{r}_1-\mathbf{r}_2|$ is the distance between the two protons, and the total energy $E=E_K-|\epsilon_d|$. $|\epsilon_d|$ is the deuteron binding energy and

$$E_K=K^2/2\eta. \quad (38)$$

For brevity, we omit the mass factors associated with the ∇^2 operators.

Eqution (37) can be rewritten in the form

$$[-\nabla_R^2-\nabla_r^2+(e^2/R)+V_{23}-(E_k-|\epsilon_d|)]\Phi_k(\mathbf{R}, \mathbf{r})=\{(e^2/R)-[e^2/(|\mathbf{R}-\tfrac{1}{2}\mathbf{r}|)]\}\Phi_k(\mathbf{R}, \mathbf{r}). \quad (39a)$$

We seek a solution of the form[11]

$$\Phi_k(\mathbf{R}, \mathbf{r})=\sum_{n=0}^{\infty}a_{n,k}(\mathbf{R})u_n(\mathbf{r}), \quad (39b)$$

where the summation includes the integration over the continuous states. Here $u_n(\mathbf{r})$ are eigenfunctions of the operator $[-\nabla_r^2+V_{23}]$, and $u_0(\mathbf{r})$ is the deuteron wave function. Substituting the assumed form of $\Phi_k(\mathbf{R}, \mathbf{r})$ into (39), multiplying on the left by $u_0^*(\mathbf{r})$, and integrating over r, we get the following equation for $a_{0,k}(\mathbf{R})$:

$$[\nabla_R^2+E_k-(e^2/R)]a_{0,k}(\mathbf{R})=\int u_0^*(\mathbf{r})\left[-\frac{e^2}{R}+\frac{e^2}{|\mathbf{R}-\tfrac{1}{2}\mathbf{r}|}\right]\Phi_k(\mathbf{R}, \mathbf{r})\,d^3r. \quad (40)$$

Equation (40) is an exact equation. In order to solve it, we approximate $\Phi_k(\mathbf{R}, \mathbf{r})$ on the right-hand side by $u_0(\mathbf{r})a_{0,k}(\mathbf{R})$, since we expect the main contribution to Φ_k to come from the first term of the summation (39b). With this approximation, Eq. (40) becomes

$$\left[\nabla_R^2+E_k-\frac{e^2}{R}-\int d^3r\,|u_0(r)|^2\left(-\frac{e^2}{R}+\frac{e^2}{|\mathbf{R}-\tfrac{1}{2}\mathbf{r}|}\right)\right]\times a_{0,k}(\mathbf{R})\simeq 0. \quad (41)$$

[11] J. B. French and M. L. Goldberger, Phys. Rev. **87**, 89 (1950). A similar approach is used in this reference to study the scattering of a deuteron by a point charge.

Consider the integral

$$V(R)=\int d^3r\,|u_0(r)|^2\left(-\frac{e^2}{R}+\frac{e^2}{|\mathbf{R}-\tfrac{1}{2}\mathbf{r}|}\right).$$

For $R>\frac{1}{2}r$, $V(R)\equiv 0$ since $u_0(r)$ is spherically symmetric. Using the expansion for $R<\frac{1}{2}r$

$$\frac{1}{|\mathbf{R}-\tfrac{1}{2}\mathbf{r}|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}^*(\Omega_R)Y_{lm}(\Omega_r)\frac{R^l}{(\frac{1}{2}r)^{l+1}}$$

and noting that only the $l=0$ term survives the integration, we finally get

$$V(R)=4\pi e^2\int_{2R}^{\infty}u_0^2(r)\left(-\frac{r^2}{R}+2r\right)dr. \quad (42)$$

We use the Hulthén wave function

$$u_0(r)=C_0\frac{e^{-\beta r}-e^{-\alpha r}}{r},$$

where

$$\alpha^2=2\mu|\epsilon_d|, \qquad C_0^2=\alpha\beta(\alpha+\beta)/2\pi(\alpha-\beta)^2.$$

$a_{0,k}(\mathbf{R})$ therefore satisfies the equation

$$[\nabla_R^2+E_k-(e^2/R)-V(R)]a_{0,k}(\mathbf{R})=0. \quad (43)$$

In momentum space

$$\left(E_k-\frac{p^2}{2\eta}\right)b_{0,k}(\mathbf{p})-\int d^3p'\,\hat{U}^c(\mathbf{p}, \mathbf{p}')b_{0,k}(\mathbf{p}')-\int d^3p'\,\tilde{V}(\mathbf{p}, \mathbf{p}')b_{0,k}(\mathbf{p}')=0, \quad (44)$$

where b, $\tilde{U}^c$, $\tilde{V}$ are the momentum space counterparts of a, U^c, V, respectively. We are going to treat the last term $\int d^3p'\,\tilde{V}(\mathbf{p}, \mathbf{p}')b_{0,k}(\mathbf{p}')$ as a perturbation. This approximation, together with the Schulman approximation for the Coulomb wave function and Green's function gives the following *approximate* solution of Eq. (44):

$$b_{0,k}(\mathbf{p})\simeq C(k)\delta(\mathbf{k}-\mathbf{p})+(2\pi)^{-3}\lim_{\epsilon\to 0}\frac{M_k(p)\,|C(p)|^2}{E-(p^2/2\eta)+i\epsilon}, \quad (45)$$

$$C(k)=\exp(-\tfrac{1}{2}\pi\eta e^2/k)\,\Gamma[1+(i\eta e^2/k)], \quad (46)$$

where

$$M_k(p)=2\pi C(k)\int_0^{\infty}dr\,V(r)\sin(pr)\sin(kr).$$

It must be emphasized that both terms on the right of Eq. (45) are approximate; this is due to our inability to write down exact and closed forms for the Green's function and the wave function, respectively.

Using (45), together with the momentum-space

deuteron wave function

$$\tilde{u}_0(p)=\frac{N}{-[|\epsilon_d|+(p^2/2\mu)](\beta^2+p^2)},$$

where

$$N^2=\alpha\beta(\alpha+\beta)^3/4\pi^2\mu^2$$

gives the first approximation to $\psi_k(\mathbf{p},\mathbf{q})$:

$$\psi_k(\mathbf{p},\mathbf{q})\simeq b_{0,k}(\mathbf{p})\tilde{u}_0(\mathbf{q}). \quad (47)$$

Consider the terms for $n\neq 0$ in Eq. (39b). The $u_n(\mathbf{r})$ are now the two-body scattering states, satisfying

$$[-\nabla_r^2+V_{23}(r)]u_n(\mathbf{r})=\epsilon_n u_n(\mathbf{r}). \quad (48)$$

Using Eq. (48), we derive an equation for each of the $a_n(\mathbf{R})$, just as we did for $a_0(\mathbf{R})$:

$$[-\nabla_R^2+(e^2/R)-E_R+(\epsilon_n+|\epsilon_d|)]a_{n,k}(\mathbf{R})$$
$$=\int d^3r\, u_n^*(\mathbf{r})\left[\frac{e^2}{R}-\frac{e^2}{|\mathbf{R}-\frac{1}{2}\mathbf{r}|}\right]\Phi_k(\mathbf{R},\mathbf{r}). \quad (49)$$

For $\epsilon_n=-|\epsilon_d|$ (deuteron case), we get back the $a_0(R)$ equation.

On the right-hand side of (49), we approximate $\Phi_k(\mathbf{R},\mathbf{r})$ by $a_0(\mathbf{R})u_0(\mathbf{r})$, and use the orthogonality of $u_0(\mathbf{r})$ and $u_n(\mathbf{r})$, to obtain the following approximate equation:

$$[\nabla_R^2-(e^2/R)+E_k-(\epsilon_n+|\epsilon_d|)]a_{n,k}(\mathbf{R})$$
$$\simeq a_{0,k}(\mathbf{R})e^2\int\frac{d^3r\, u_n^*(\mathbf{r})u_0(r)}{|\mathbf{R}-\frac{1}{2}\mathbf{r}|}. \quad (50)$$

To solve Eq. (50), we need to know the following: (1) $a_{0,k}(\mathbf{R})$ which occurs on the right, (2) $u_n^*(\mathbf{r})$, (3) the Green's function $G^c(\mathbf{R},\mathbf{R}',W)$, and

(4) $$I_n(\mathbf{R})\equiv\int\frac{u_n^*(\mathbf{r})e^2u_0(\mathbf{r})d^3r}{|\mathbf{R}-\frac{1}{2}\mathbf{r}|}.$$

(1) $a_{0,k}$ is given by Eq. (43). For the present purpose, it suffices to take the approximate equation

$$[\nabla_R^2+E_k-(e^2/R)]a_{0,k}(\mathbf{R})=0. \quad (51)$$

Therefore,

$$a_{0,k}(\mathbf{R})\simeq\psi_k^c(\mathbf{R}) \quad \text{Coulomb wave form}$$
$$\simeq C(k)\exp(i\mathbf{k}\cdot\mathbf{R}) \quad \text{Schulman approx.}$$

(2) $$u_n(\mathbf{r})=\sum_{l=0}^{\infty}u_n^{(l)}(r)P_l(\hat{n}\cdot\hat{r})(2l+1), \quad (52)$$

where $u_n^{(l)}(r)$ are given in Appendix C.

(3) Green's function

$$G^c(\mathbf{R},\mathbf{R}',W)$$
$$\simeq\frac{1}{2\pi^2}\int_0^\infty\frac{k^2|C(k)|^2}{W-(k^2/2\eta)+i\epsilon}\frac{\sin(k|\mathbf{R}-\mathbf{R}'|)}{(k|\mathbf{R}-\mathbf{R}'|)},$$

where we have assumed

$$\psi_k^c(\mathbf{R})\simeq C(k)\exp(i\mathbf{k}\cdot\mathbf{R}) \quad \text{Coulomb wave function (Schulman approx.).} \quad (53)$$

(4)

$$I_n(\mathbf{R})\equiv\int\frac{d^3r\, u_n^*(r)e^2u_0(r)}{|\mathbf{R}-\frac{1}{2}\mathbf{r}|}=e^2\sum_{l=0}^{\infty}D_n^{(l)}(R)P_l(\hat{n}\cdot R). \quad (54)$$

The $D_n^{(l)}(R)$ are given in Appendix B.

In our case, we take the $l=0$ term only in the summation in Eq. (54); this simplification is a result of our choice of the two-body short-range potential.

Substituting Eqs. (51)–(54) into (50), and using the relation

$$\frac{\sin k|\mathbf{R}-\mathbf{R}'|}{k|\mathbf{R}-\mathbf{R}'|}=\sum_{l=0}^{\infty}(2l+1)j_l(kR)j_l(kR')P_l(\hat{R}\cdot\hat{R}'),$$

we can write the formal solution (with $E_{k,n}=E_k-\epsilon_n-|\epsilon_d|$)

$$a_{n,k}(\mathbf{R})\simeq a_{n,k}^0(\mathbf{R})$$
$$+\int d^3R'\,G^c(\mathbf{R},\mathbf{R}',E_{k,n})C(k)\exp(i\mathbf{k}\cdot\mathbf{R}')D_n^{(0)}(R'), \quad (55)$$

where $a_{n,k}^0(\mathbf{R})$ is the solution of the homogeneous equation corresponding to Eq. (50). We choose $a_{n,k}^{(0)}(\mathbf{R})=0$, so that $\Phi_k(\mathbf{R},\mathbf{r})$ satisfies the correct boundary conditions, i.e., the incoming part of $\Phi_k(\mathbf{R},\mathbf{r})$ should contain only a term which represents a proton incident on a deuteron. The break-up terms should be in the outgoing part only.

The total wave function is then

$$\Phi_k(\mathbf{R},\mathbf{r})=a_{0,k}(\mathbf{R})u_0(r)+(2\pi)^{-3}\int d^3n\, a_{n,k}(\mathbf{R})u_n(r).$$

VII. RESULTS AND CONCLUSIONS

For incident proton energies of $E_{p,\text{lab}}=2.08$, 3.0, 5.2, 9.7, and 14.0 MeV we calculate the differential scattering cross section as a function of the c.m. angle. The results are shown by the solid lines in Figs. 1 to 5. Experimental results for 2.08-, 3.0-MeV protons are given in Ref. 12, for 5.2-MeV protons in Ref. 13, for 9.7-MeV protons in Ref. 14, and 13.93-MeV protons in Ref. 15.

The calculations are repeated for the case in which the Coulomb potential is "switched off." The results

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[13] Louis Rosen and J. C. Allred, Phys. Rev. **82**, 777 (1951).
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[15] S. Kikuchi, J. Sanada, S. Suwa, I. Hayashi, K. Nisimura, and S. Fukunaga, in *Nuclear Forces and the Few-Nucleon Problem*, edited by T. C. Griffith and E. A. Power (Pergamon Press, London, 1960), Vol. 2, p. 669.

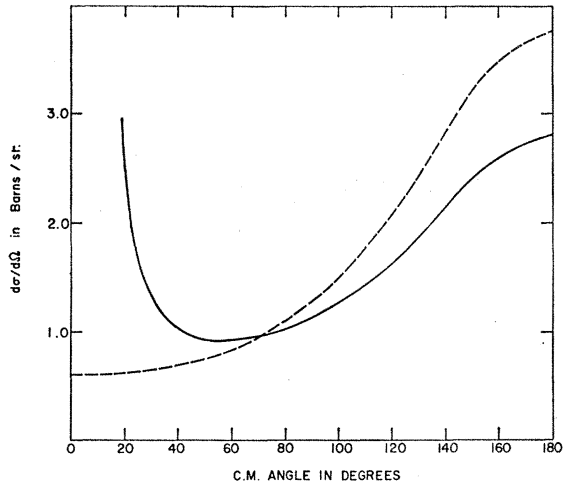

FIG. 1. Plot of differential scattering cross section in b/sr, against c.m. angle in degrees, for incident "proton" energy E_p(lab) = 2.08 MeV. Solid lines, Coulomb scattering; dashed lines, scattering without Coulomb potential.

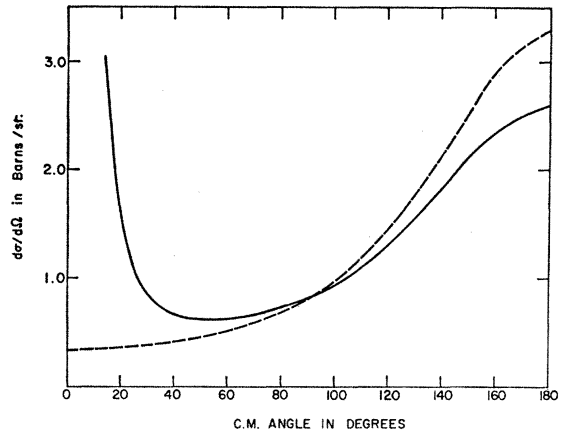

FIG. 2. Plot of differential scattering cross section in b/sr, against c.m. angle in degrees, for incident "proton" energy E_p(lab) = 3.0 MeV. Solid lines, Coulomb scattering; dashed lines, scattering without Coulomb potential.

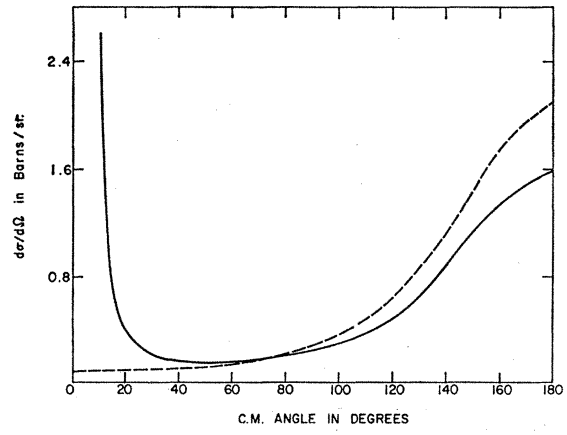

FIG. 3. Plot of differential scattering cross section in b/sr, against c.m. angle in degrees, for incident "proton" energy E_p(lab) = 5.2 MeV. Solid lines, Coulomb scattering; dashed lines, scattering without Coulomb potential.

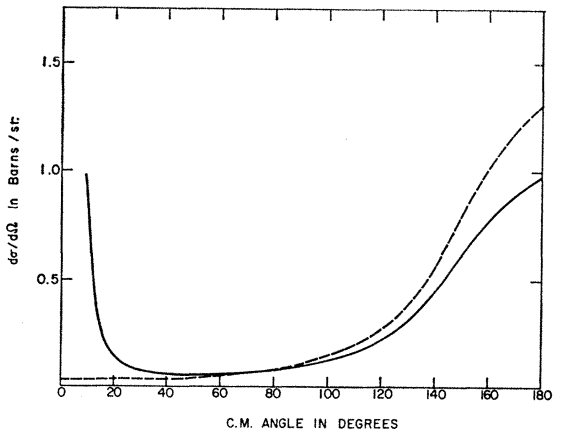

FIG. 4. Plot of differential scattering cross section in b/sr, against c.m. angle in degrees, for incident "proton" energy E_p(lab) = 9.7 MeV. Solid lines, Coulomb scattering; dashed lines, scattering without Coulomb potential.

are shown by the dotted curves in Figs. 1 to 5. As far as the p-d scattering is concerned, our results show qualitative agreement with experiment, so far as the order of magnitude of $d\sigma/d\Omega$, and the general shape of the curves are concerned. We expect no more, in view of the simplifications we have made, and we have therefore not drawn in the experimental curves. It is more instructive to compare the Coulomb scattering with the non-Coulomb scattering in our model. In the backward direction, our results show that $d\sigma/d\Omega$ for the Coulomb-less problem is consistently larger than for the Coulomb case; although the difference decreases with increasing energy. At a c.m. angle of 180°, the ratio of $d\sigma/d\Omega$ for the Coulomb and non-Coulomb problem is of the order of $|C(k)|^4$

$$\frac{\sim |C(k)|^4 [d\sigma/d\Omega]_{180^\circ \text{ Coulomb}}}{[d\sigma/d\Omega]_{180^\circ \text{ non-Coulomb}} \sim |C(k)|^4},$$

where $C(k)$ is given by Eq. (45).

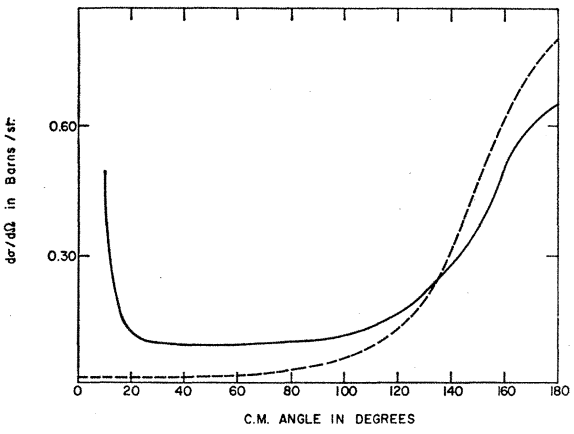

FIG. 5. Plot of differential scattering cross section in b/sr, against c.m. angle in degrees, for incident "proton" energy E_p(lab) = 14.0 MeV. Solid lines, Coulomb scattering; dashed lines, scattering without Coulomb potential.

Now the term $\langle\chi_{k'}^{-} \mid V_1 \mid \chi_k^{+}\rangle$, which is the exchange term, predominates in the backward scattering as compared to all other terms which occur in the scattering amplitude.

To a first approximation,

$$\frac{\langle\chi_{k'}^{-} \mid V_1 \mid \chi_k\rangle_{\text{Coulomb}}}{\langle\chi_{k'}^{-} \mid V_1 \mid \chi_k\rangle_{\text{non-Coulomb}}} \simeq \mid C(k) \mid^2, \qquad \text{for } k'=k$$

the factor $\mid C(k) \mid^2$ arising, of course, from the Schulman approximation for the Coulomb-modified wave functions χ, which occur in the numerator on the left.

If we refer back to Eqs. (32a) and (32b), Sec. V, we see that the following approximations were made: (i) The use of an approximate Coulomb Green's function in solving the integral equations for the h_i; the Green's function appears in the kernels. (ii) The use of an approximate Coulomb wave function in the initial and final states.

Our results suggest that the anomaly in the backward scattering can be traced largely to approximation (ii) above. In other words, G^c with Schulman's approximation is probably good enough for solving the integral equations for the h_i's; the main difficulty lies with the wave function, since the ratio is $\sim \mid C(k) \mid^4$.

In principle, at least, step (ii) could be improved upon by a better approximation than Schulman's, perhaps even by using the exact Coulomb wave. This might reduce the disparity in the backward direction. Our calculations were carried out with the Schulman approximation only.

In the forward direction, the Coulomb-less problem shows no peak, even at 14.0 MeV. However, the results of our calculations verify that unitarity is satisfied. Our Coulomb-less problem is not of course a model for n-d scattering since we have symmetrized between particles 1 and 2 only.

ACKNOWLEDGMENTS

The author is grateful to Dr. R. Amado, and to Dr. J. Noble for their advice, encouragement and many valuable suggestions.

APPENDIX A

If p_i $(i=1, 2, 3)$ are the momenta of the 3 particles in their c.m. system, and q_i $(i=1, 2, 3)$ are the relative momenta, using the notation of Sec. II, then

$$\begin{aligned}
\mathbf{p}_1 &= -\tfrac{1}{2}\mathbf{p}_2 - \mathbf{q}_2 = -\tfrac{1}{2}\mathbf{p}_3 + \mathbf{q}_3,\\
\mathbf{q}_1 &= -\tfrac{3}{4}\mathbf{p}_3 - \tfrac{1}{2}\mathbf{q}_3 = \tfrac{3}{4}\mathbf{p}_2 - \tfrac{1}{2}\mathbf{q}_2,\\
\mathbf{p}_2 &= -\tfrac{1}{2}\mathbf{p}_3 - \mathbf{q}_3 = -\tfrac{1}{2}\mathbf{p}_1 + \mathbf{q}_1,\\
\mathbf{q}_2 &= -\tfrac{3}{4}\mathbf{p}_1 - \tfrac{1}{2}\mathbf{q}_1 = \tfrac{3}{4}\mathbf{p}_3 - \tfrac{1}{2}\mathbf{q}_3,\\
\mathbf{p}_3 &= -\tfrac{1}{2}\mathbf{p}_1 - \mathbf{q}_1 = -\tfrac{1}{2}\mathbf{p}_2 + \mathbf{q}_2,\\
\mathbf{q}_3 &= -\tfrac{3}{4}\mathbf{p}_2 - \tfrac{1}{2}\mathbf{q}_2 = \tfrac{3}{4}\mathbf{p}_1 - \tfrac{1}{2}\mathbf{q}_1.
\end{aligned}$$

APPENDIX B

The $D_n^{(l)}(R)$ which occur in Eq. (54), Sec. VI, are given by

$$D_n^{(l)}(R) = 4\pi i^l \left[2^{-l} R^{-l-1} \int_0^{2R} dr\, r^{l+2} u_n^{(l)}(r)\, u_0(r) + (-)^l 2^{l-1} R^l \int_{2R}^{\infty} dr\, r^{-l+1} u_n^{(l)}(r)\, u_0(r) \right].$$

APPENDIX C

The $\mid u_n\rangle$ satisfy the equation

$$[\hat{H}_0 + \hat{V}] \mid u_n\rangle = \epsilon_n \mid u_n\rangle \tag{C1}$$

or

$$[\hat{\nabla}_n^2 + \epsilon_n] u_n(\mathbf{r}) = \int \hat{V}(\mathbf{r}, \mathbf{r}')\, u_n(\mathbf{r}')\, d^3r'.$$

A formal solution is, using

$$\tilde{V}(\mathbf{p}, \mathbf{p}') = \lambda v(p) v(p')$$

and

$$V(\mathbf{r}, \mathbf{r}') = (2\pi)^{-3} \int \langle \mathbf{r} \mid \mathbf{p}\rangle \tilde{V}(\mathbf{p}, \mathbf{p}') \langle \mathbf{p}' \mid \mathbf{r}'\rangle d^3p\, d^3p',$$

$$u_n(\mathbf{r}) = \exp(i\mathbf{n}\cdot\mathbf{r}) + (4\pi)^{-1} \iint d^3r'\, d^3r'' \frac{\exp(in \mid \mathbf{r}-\mathbf{r}' \mid)}{\mid \mathbf{r}-\mathbf{r}' \mid} \hat{V}(\mathbf{r}', \mathbf{r}'') u_n(\mathbf{r}''). \tag{C2}$$

In our case, we require the solution to consist of the outgoing part only. A partial-wave expansion gives an integral equation for $u_n^0(r)$, of the form

$$\begin{aligned}
u_n^{(0)}(r) &= \int_0^{\infty} K(r, r')\, u_n^{(0)}(r')\, dr',\\
u_n^{(l)}(r) &= 0 \qquad (l \neq 0).
\end{aligned} \tag{C3}$$

This simplification results from our s-wave Yamaguchi potential. The eigenfunction equation (C3) was solved numerically, to obtain the eigenfunctions.