PHYSICAL REVIEW VOLUME 176, NUMBER 2 10 DECEMBER 1968

Theory of a Two-Dimensional Ising Model with Random Impurities. I. Thermodynamics

BARRY M. MCCOY
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790
AND
TAI TSUN WU*
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 9 July 1968)

Recent experiments demonstrate that at the Curie temperature the specific heat may be a smooth function of the temperature. We propose that this effect can be due to random impurities and substantiate our proposal by a study of an Ising model containing such impurities. We modify the usual rectangular lattice by allowing each row of vertical bonds to vary randomly from row to row with a prescribed probability function. In the case that this probability is a particular distribution with a narrow width, we find that the logarithmic singularity of Onsager's lattice is smoothed out into a function which at T_c is infinitely differentiable but not analytic. This function is expressible in terms of an integral involving Bessel functions and is computed numerically.

1. INTRODUCTION

THE free energy of a two-dimensional Ising model without a magnetic field was first obtained by Onsager[1] in 1944. One of the most striking features of Onsager's result is that the specific heat has a logarithmic singularity at the Curie temperature T_c. The spontaneous magnetization for this lattice was determined by Yang[2] in 1952 and shown to approach zero as $(T_c-T)^{1/8}$ as $T\to T_c^-$. In spite of the fact that these calculations are for a two-dimensional system, they form the basis of much of our understanding of ferromagnetic phase transitions. The precise form of the singularity in the specific heat of various magnetic systems is a matter of considerable debate, but the existence of the singularity is rarely questioned. Similarly, while measurements show that the $\frac{1}{8}$ power singularity of the spontaneous magnetization of the Ising model is too small to fit the experimental data, experiments have not yet revealed a necessity to assume anything other than a power-law singularity at T_c.

However, recent precise measurements of the specific heats of EuS [3] and Ni [4] indicate that in these systems, among others, if measurements are made close enough to T_c, the specific heat is seen not to diverge to infinity. In fact it does not even have a discontinuous or infinite first derivative. To a high degree of accuracy these specific heats are smooth functions of the temperature. This smooth behavior is markedly different from Onsager's result.

Perhaps the most interesting feature of this discrepancy between precise experiments and the intuition gained from Onsager's calculation is not that there is a discrepancy but that the discrepancy occurs when T is very close to T_c and has resisted detection for so long. Onsager's calculation was carried out for a perfect lattice, one in which *all* the vertical interactions and *all* the horizontal interactions are respectively the same. Virtually any difference between the real system and the idealized Ising model with nearest-neighbor interactions can be invoked to "explain" the experimental data. Some are more reasonable than others; one possible example is the finite size of the actual sample. It is the purpose here to study one aspect for the various differences, namely the presence of random impurities. By impurities we refer not only to the presence of foreign material but also to any physical property that makes the various lattice sites different from each other. An example is the presence of various isotopes in a sample, e.g., nickel contains roughly 68% of Ni^{58}, 26% of Ni^{60}, 1% of Ni^{61}, 4% of Ni^{62}, and 1% of Ni^{64}. The presence of impurities in this sense seems unavoidable in most actual magnetic systems. If these impurities distribute themselves through the system in a regular ordered fashion, then while the symmetry of the lattice would be reduced it would not be destroyed. With sufficient labor such an ordered sort of impurity can be studied in the Ising model. However, such an ordering of impurities does not always take place. Therefore, if we want to realistically study the effects of impurities in magnetic systems we may have to allow the impurities to be distributed at random throughout the lattice. The regularity of the system now has been not merely reduced but totally destroyed. A phase transition is a cooperative phenomena in which the entire system takes part. It is therefore not at all obvious that the highly regular Onsager lattice should possess a phase transition behavior that is in any way related to such an impure system.

In order to gain any insight into the possibility for

* Supported in part by National Science Foundation Grant No. GP-8019.
[1] L. Onsager, Phys. Rev. **65**, 117 (1944).
[2] C. N. Yang, Phys. Rev. **85**, 808 (1952).
[3] B. J. C. van der Hoeven, Jr., D. T. Teaney, and V. L. Moruzzi, Phys. Rev. Letters **20**, 719 (1968).
[4] P. Handler, D. Mapother, and M. Rayl, Phys. Rev. Letters **19**, 356 (1967).

the random impurities to change qualitatively the nature of the phase transition, it is necessary to study a simple model in detail. Such a model will be described shortly. At least in this specific model, and very probably under quite general circumstances, the logarithmic singularity in the specific heat is no longer present when there are random impurities. Instead, the specific heat is an infinitely differentiable function of the temperature. Thus, in the usual language, the phase transition is of infinite order. In particular, in the presence of random impurities, the critical exponents no longer have a precise meaning.

In this series of papers we consider the following modification of the two-dimensional Ising model. We retain the features of Onsager's rectangular lattice to the extent that all horizontal interactions are the same and that the vertical interaction between any site in the jth row and its nearest neighbor in the $j+1$th row is the same no matter what column these sites are in. However, $E_2(j)$ is allowed to randomly vary from row to row. More specifically we assume that, for $j\neq j'$, $E_2(j)$ and $E_2(j')$ are independent random variables with identical probability distributions $P(E_2)dE_2$.

Let us try to describe the model in greater detail. We are considering a collection of Ising lattices, each of which is specified by a particular set of interactions $\{E_2(j)\}$. We are interested in the thermodynamic limit where the size of these lattices becomes infinite. If, in the thermodynamic limit, the free energy of each lattice in our collection varied wildly from lattice to lattice our model would be useless. In that case the free energy of our random lattices would depend in detail on the arrangement of interactions. Fortunately, this is physically unreasonable and is indeed not the case. In the thermodynamic limit the free energy per site of each lattice does approach, with probability 1, the same value. Therefore, with probability 1 the Curie temperatures of any two lattices from this collection are the same. Furthermore, we expect the spontaneous magnetization of any two lattices to be the same with probability 1, because the spontaneous magnetization, like the free energy, is an average property of the entire lattice. However, not all quantities of interest have distributions which are so sharply peaked. For example, the spin-spin correlation function of neighboring spins *does* depend in detail on the local value of the interaction energies. For such quantities one needs more than an average value to characterize the result of a measurement made at an arbitrary position in the impure lattice.

The complete investigation of all aspects of this random Ising model is clearly beyond the scope of any one paper. In this paper we begin the investigation by considering the free energy in the absence of a magnetic field. In Sec. 2 we will formulate the mathematical problem to be solved and find a general formula for the critical temperature in terms of $P(E_2)$. In Sec. 3 we will derive several general properties of the integral equation found in Sec. 2. There is a great deal that can be said about the equations we derive because they depend on the arbitrary function $P(E_2)$. We wish in this paper to emphasize the physical consequences of random impurities. Therefore we specialize our treatment in Sec. 4 to a particular distribution $P(E_2)$ which has a narrow width (of order N^{-1}) that is particularly mathematically tractable. This distribution is in no way physically distinguished and our results are expected to be typical of a large class of narrow distributions. We show that while the critical temperature is shifted by a temperature on the order of N^{-1} the specific heat deviates appreciably (to order 1 as $N\rightarrow\infty$) from Onsager's specific heat only for $T-T_c\sim O(N^{-2})$. We furthermore find that at T_c the term of order 1 in the specific heat is *not* logarithmically divergent but is an *infinitely differentiable* function of T though it is *not* analytic. Finally we conclude in Sec. 5 with a discussion of the technical aspects of the calculation and outline several ways in which our calculations may be generalized.

2. FORMULATION OF THE PROBLEM

In this section we formulate the mathematical problem corresponding to the physical model described in the *Introduction.* We do this in two stages. The first stage is to relate the free energy to the solution of a pair of coupled recursion relations. This procedure has been discussed recently by us[5] for Onsager's lattice. That derivation of the recursion relation still holds for the class of lattices under consideration. We therefore only repeat as little of that derivation as is necessary to establish notation and refer the reader to the previous work for a detailed discussion. The final results are given in (2.9) and (2.11). The rest of this section is devoted to the study of the mathematical problem formed by (2.9) and (2.11) and is self-contained. We advise the interested, as opposed to the dedicated, reader[6] to omit the first stage except for the definitions.

We are interested in studying a particular class of rectangular two-dimensional Ising models with $\mathfrak{M}$ rows and $2\mathfrak{N}$ columns. We impose cyclic boundary conditions in the horizontal direction only. This class of systems is characterized by the Hamiltonians

$$\mathcal{E}=-E_1\sum_{j=1}^{\mathfrak{M}}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{j,k}\sigma_{j,k+1}-\sum_{j=1}^{\mathfrak{M}-1}E_2(j)\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{j,k}\sigma_{j+1,k},\quad(2.1)$$

where $\sigma_{j,k}$ is equal to $+1$ or -1, j and k label, respec-

[5] B. M. McCoy and T. T. Wu, Phys. Rev. **162**, 436 (1967). This paper is hereafter referred to as IV. The present notation differs from IV in a few trivial ways $\mathfrak{a}=-ia$, $\mathfrak{D}_n=iD_n$, and all other German letters are replaced by their Latin equivalents.

[6] The distinction seems to have been first clearly made by G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

tively, the row and column of each lattice site, and $k=\mathfrak{N}+1$ is identified with $k=-\mathfrak{N}+1$. As far as thermodynamics is concerned we may restrict E_1 and E_2 to be positive without loss of generality. We are interested in the limit $\mathfrak{N}\to\infty$ and $\mathfrak{M}\to\infty$; only in this limit will a phase transition occur. We complete the characterization by requiring that $E_2(j)$ be independent random variables with the probability density function $P(E_2)$.

Denote by $Z_{\{E_2\}}$ the partition function for the system described by (2.1) where the collection of bounds $\{E_2(j)\}$ is chosen at random according to the probability density function $P(E_2)$. We are interested in F_r, the free energy per site of the system in the thermodynamic limit. Under the assumption that with probability 1 this limit exists, F_r is defined as

$$F_r=-\beta^{-1}\lim_{\mathfrak{M}\to\infty,\mathfrak{N}\to\infty}(2\mathfrak{M}\mathfrak{N})^{-1}\ln Z_{\{E_2\}}. \quad (2.2)$$

Our class of lattices shares with Onsager's lattice the property of translational invariance in the horizontal direction. Therefore the calculation of Sec. 3 of IV may be taken over word for word to show that for any set of energies $E_2(j)$

$$Z_{\{E_2\}}^2=(2\cosh\beta E_1)^{4\mathfrak{M}\mathfrak{N}}\prod_{j=1}^{\mathfrak{M}-1}[\cosh\beta E_2(j)]^{4\mathfrak{N}} \times\prod_{\theta}|1+z_1e^{i\theta}|^{2\mathfrak{M}}\det C_{\{E_2\}}(\theta), \quad (2.3)$$

where $\prod_\theta$ is the product over the $2\mathfrak{N}$ roots of -1:

$$\theta=\prod(2n-1)/2\mathfrak{N}, \qquad n=1,2,\cdots 2\mathfrak{N}, \quad (2.4)$$

$$z_1=\tanh\beta E_1, \quad (2.5a)$$

$$z_2(j)=\tanh\beta E_2(j), \quad (2.5b)$$

and $\beta=(kT)^{-1}$. The $2\mathfrak{M}\times 2\mathfrak{M}$ matrix $C_{\{E_2\}}(\theta)$ is defined by

$$C_{j,j}=\begin{bmatrix} ia & b \\ -b & -ia\end{bmatrix}, \quad (2.6a)$$

$$C_{j,j+1}=-C_{j+1,j}{}^T=\begin{bmatrix} 0 & 0 \\ z_2(j) & 0\end{bmatrix} \quad (2.6b)$$

[compare with (3.9) of IV] and all other matrix elements are zero. Here we use

$$a(\theta)=-2z_1\sin\theta\,|1+z_1e^{i\theta}|^{-2}, \quad (2.7a)$$

$$b(\theta)=(1-z_1^2)\,|1+z_1e^{i\theta}|^{-2}. \quad (2.7b)$$

More explicitly $C_{\{E_2\}}(\theta)$ may be written as

$$\begin{bmatrix}
ia & b & & & & & & \\
-b & -ia & z_2(1) & & & & & \\
 & -z_2(1) & ia & b & & & & \\
 & & -b & -ia & z_2(2) & & & \\
 & & & -z_2(2) & \ddots & & \ddots & \\
 & & & & \ddots & \ddots & z_2(\mathfrak{M}-1) & \\
 & & & & & -z_2(\mathfrak{M}-1) & ia & b \\
 & & & & & & -b & -ia
\end{bmatrix}. \quad (2.8)$$

We may further follow the procedure of IV and define C_n to be the determinant of the $2n\times 2n$ random matrix of the form (2.8) and iD_n to be the corresponding $(2n-1)\times(2n-1)$ random determinant with the last row and column removed. Then $\det C_{\{E_2\}}(\theta)=C_{\mathfrak{M}}(\theta)$ and we obtain the recurrence relation for $n\geq 0$,

$$\begin{bmatrix} C_{n+1} & (\theta) \\ D_{n+1} & (\theta)\end{bmatrix}=\begin{bmatrix} a^2+b^2 & a \\ a & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z_2(n)^2\end{bmatrix}\begin{bmatrix} C_n & (\theta) \\ D_n & (\theta)\end{bmatrix} \quad (2.9)$$

together with the boundary condition

$$C_0(\theta)=1, \qquad D_0(\theta)=0, \quad (2.10)$$

where $z_2(0)=0$ by definition. Therefore, the free energy F_r is given as

$$F_r=-\beta^{-1}\left\{\ln(2\cosh\beta E_1)+\int_0^\infty dE_2P(E_2)\ln\cosh\beta E_2 +(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\ln|1+z_1e^{i\theta}|^2 +(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\ln CA(\theta)\right\}. \quad (2.11)$$

The object $C_n(\theta)$ is the first component of the vector we obtain by applying n random matrices of the form

$$\begin{bmatrix} a^2+b^2 & a\lambda \\ a & \lambda \end{bmatrix} \tag{2.12}$$

to the initial vector $[{}^1_0]$. Here $\lambda \equiv z_2^2$ is a random variable with the normalized probability distribution function $\mu(\lambda)$ given by

$$\mu(\lambda)\,d\lambda = P(E_2)\,dE_2. \tag{2.13}$$

The general theory of such random matrix products, in particular the existence with probability 1 of the limit of the right-hand side of (2.11), has been presented recently in a rigorous manner by Furstenberg.[7]

We begin our evaluation of (2.11) by noting that

$$\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\ln C_{\mathfrak{M}}(\theta) = \lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\sum_{n=0}^{\mathfrak{M}-1}\ln\frac{C_{n+1}}{C_n}. \tag{2.14}$$

Using the recursion relation (2.9) this may be rewritten as

$$\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\sum_{n=0}^{\mathfrak{M}-1}\ln\frac{C_{n+1}}{C_n}$$
$$= \lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\sum_{n=0}^{\mathfrak{M}-1}\ln[a^2+b^2+a\lambda(n)D_n/C_n]. \tag{2.15}$$

This can be interpreted as an average over n. The only drawback is that while the $\lambda(n)$ are independent random variables the D_n/C_n are not at all independent. We now remark that because the matrix (2.12) is real and acts on a two-dimensional vector space,

$$x_n = C_n/D_n \tag{2.16}$$

may be thought of as the tangent of the angle which the vector $[{}^{C_n}_{D_n}]$ makes with the D axis. From (2.9) we find

$$x_{n+1} = [(a^2+b^2)x_n + a\lambda(n)]/[ax_n+\lambda(n)]. \tag{2.17}$$

Because the space of angular dependence of these vectors is compact, as n becomes large the variable x_n will approach a limiting stationary distribution $\nu(x)$ that is independent of the initial vector.[7] This stationary distribution is characterized by the property that, if we apply a random matrix (2.12) to it and average the resulting distribution over $\mu(\lambda)$ of (2.13), $\nu(x)$ will transform into itself. Therefore, $\nu(x)$ satisfies the equation

$$\nu(x) = \int_{-\infty}^{\infty} dx' \int_0^1 d\lambda\,\delta\left(x - \frac{(a^2+b^2)x'+a\lambda}{ax'+\lambda}\right)\mu(\lambda)\nu(x'). \tag{2.18}$$

We may perform the λ integration to obtain

$$\nu(x) = \frac{b^2}{(x-a)^2}\int_{-\infty}^{\infty} dx'\,x'\mu\left[x'\,\frac{a^2+b^2-ax}{x-a}\right]\nu(x'). \tag{2.19}$$

The work of Furstenberg[7] guarantees that a solution to this equation exists and is unique.

Once we possess a stationary distribution $\nu(x)$ we may replace the average over n in (2.15) by an average over λ and over x. We thus arrive at the final result that with probability 1

$$\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\ln C_{\mathfrak{M}}(\theta)$$
$$= \int_{-\infty}^{\infty} dx\,\nu(x)\int_0^1 d\lambda\,\mu(\lambda)\,\ln(a^2+b^2+a\lambda x^{-1}). \tag{2.20}$$

An identical analysis may be carried out for the quantity $D_{\mathfrak{M}}$ and we find that with probability 1

$$\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\ln D_{\mathfrak{M}}(\theta)$$
$$= \int_{-\infty}^{\infty} dx\,\nu(x)\int_0^1 d\lambda\,\mu(\lambda)\,\ln(\lambda+ax). \tag{2.21}$$

Since $C_{\mathfrak{M}}$ and $D_{\mathfrak{M}}$ are the components of the same vector, their average rate of growth must each be separately equal to the average rate of growth of the vector itself. Therefore the right-hand sides of (2.20) and (2.21) must be equal so that

$$\int_{-\infty}^{\infty} dx\,\nu(x)\int_0^1 d\lambda\,\mu(\lambda)\,\ln(a^2+b^2+a\lambda x^{-1})$$
$$= \int_{-\infty}^{\infty} dx\,\nu(x)\int_0^1 d\lambda\,\mu(\lambda)\,\ln(\lambda+ax). \tag{2.22}$$

To prove this directly consider the difference d between the two sides of Eq. (2.22)

$$d = \int_{-\infty}^{\infty} dx\,\nu(x)\int_0^1 d\lambda\,\mu(\lambda)\,\ln\left(\frac{a^2+b^2+a\lambda x^{-1}}{\lambda+ax}\right). \tag{2.23}$$

This may be rewritten as

$$d = \int_{-\infty}^{\infty} dx\,\nu(x)\int_0^1 d\lambda\,\mu(\lambda)\,\ln\left(x\,\frac{a^2+b^2+a\lambda x^{-1}}{\lambda+ax}\right)$$
$$- \int_{-\infty}^{\infty} dx\,\nu(x)\,\ln x. \tag{2.24}$$

In the first integral we now replace the variable λ by

$$q = x(a^2+b^2+a\lambda x^{-1})/(\lambda+ax) \tag{2.25}$$

to obtain

$$d = \int_{-\infty}^{\infty} dq\int_{-\infty}^{\infty} dx\,\mu\left(x\,\frac{a^2+b^2-qa}{q-a}\right)\frac{xb^2}{(q-a)^2}\nu(x)\,\ln q$$
$$- \int_{-\infty}^{\infty} dx\,\nu(x)\,\ln x. \tag{2.26}$$

We now use the integral equation (2.19) on the first term of (2.26) to find that as expected $d=0$.

We may readily derive the expression for the critical temperature T_c in terms of $P(E_2)$ if we note that at $a=0$ the matrix (2.12) is diagonal. The only possible

[7] H. Furstenberg, Trans. Am. Math. Soc. **108**, 377 (1963).

stationary distributions of vectors are those with the vectors parallel to the C or D axis. In the first case the vectors are multiplied by $b^2(0)$ under the action of (2.12) so we find

$$\lim_{\theta\to 0}\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\ln C_{\mathfrak{M}}=\ln b(0)^2 \quad (2.27)$$

and in the latter case the vectors are multiplied by λ so that

$$\lim_{\theta\to 0}\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\ln C_{\mathfrak{M}}=\int_0^1 d\lambda\,\mu(\lambda)\ln\lambda. \quad (2.28)$$

The free energy is related to the larger of these two expressions and the condition for $T=T_c$ is obtained by equating (2.27) and (2.28)

$$0=\int_0^1 d\lambda\,\mu(\lambda)\ln[\lambda b(0)^{-2}]$$

$$=\int_0^\infty dE_2P(E_2)\ln\{z_2^2[(1+z_1)/(1-z_1)]^2\} \quad (2.29\text{a})$$

or

$$2\beta E_1+\int_0^\infty dE_2P(E_2)\ln z_2=0. \quad (2.29\text{b})$$

If $P(E_2)=\delta(E_2-E_2^0)$ then (2.29) reads

$$z_2^0(1-z_1)/(1+z_1)=1, \quad (2.30)$$

which is the known result for $T=T_c$ in the ferromagnetic Onsager lattice. The correctness of this criteria for locating T_c will be seen more explicitly in Sec. 4 where we demonstrate for a specific $\mu(\lambda)$ that at the T_c given by (2.29) the free energy F_r is not an analytic function of the temperature.

We note that (2.29) has the property that T_c is zero if $P(E_2)=p\delta(E_2)+\cdots$, where $0<p\leq 1$. This is to be expected for our model. A δ function at $E_2=0$ means that in every random lattice there are, with probability 1, vertical bonds with zero strength a finite distance apart. But for our class of lattices the condition $E_2(j)=0$ cuts the lattice into two separate pieces. Therefore the term $p\delta(E_2)$ causes the lattice, with probability 1, to be cut up into an infinite number of strips of finite width and any two-dimensional Ising lattice that is not infinite in both dimensions has no critical temperature. However, we also note that if $\mu(\lambda)$ is bounded near $E_2=0$ and $E_1>0$ then T_c is greater than zero.

We finally remark that the argument leading to (2.29a) does not depend on the fact that E_2 rather than E_1 is random. We therefore see by a similar argument that if E_1 and E_2 are random with the joint probability density $P(E_1, E_2)$ there is a ferromagnetic phase transition at T_c determined from

$$\int_0^\infty dE_1\int_0^\infty dE_2P(E_1,E_2)\ln[z_2^{-1}(1-z_1)/(1+z_1)]=0. \quad (2.31)$$

3. INTEGRAL EQUATION FOR $\nu(x)$

In this section we study several general properties of the integral equation (2.19). The limits of (2.19) have formally been written as $-\infty$ to $+\infty$, but over much of this range $\nu(x)$ vanishes identically. To see this, first consider the stationary distribution corresponding to the Onsager lattice where

$$\mu(\lambda)=\delta(\lambda-\bar{\lambda}). \quad (3.1)$$

The stationary angular distribution of vectors will clearly be a δ function at that value of x which is unchanged by the application of the matrix (2.12) with $\lambda=\bar{\lambda}$. From (2.17) we see that the values of x which satisfy this eigenvector equation obey

$$x_0(\bar{\lambda})=[(a^2+b^2)x_0(\bar{\lambda})+a\bar{\lambda}]/[ax_0(\bar{\lambda})+\bar{\lambda}], \quad (3.2\text{a})$$

which is also usefully expressed as

$$\bar{\lambda}=x_0(\bar{\lambda})[a^2+b^2-ax_0(\bar{\lambda})]/[x_0(\bar{\lambda})-a]. \quad (3.2\text{b})$$

There are two solutions to (3.2) because the matrix (2.12) has two eigenvalues. To obtain the correct free energy we must choose that solution which has the larger eigenvalue. It may easily be seen from (2.12) or from (2.20) and (2.21) that the correct solution of (3.2) is

$$x_0(\bar{\lambda})=(2a)^{-1}\{a^2+b^2-\bar{\lambda}+[(a^2+b^2-\bar{\lambda})^2+4\bar{\lambda}a^2]^{1/2}\}, \quad (3.3\text{a})$$

with

$$x_0(\bar{\lambda})^{-1}=(2a\bar{\lambda})^{-1}\times\{-(a^2+b^2-\bar{\lambda})+[(a^2+b^2-\bar{\lambda})^2+4\bar{\lambda}a^2]^{1/2}\}. \quad (3.3\text{b})$$

If we use (3.1), (3.3), and (2.21) in (2.11) we find that the free energy for Onsager's lattice is[1]

$$-\beta F_0=\ln(2\cosh\beta E_1\cosh\beta\bar{E}_2)+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\ln[\tfrac{1}{2}\,|\,1+z_1e^{i\theta}\,|^2\{a^2+b^2+\bar{z}_2^2+[(a^2+b^2-\bar{z}_2^2)^2+4a^2\bar{z}_2^2]^{1/2}\}]. \quad (3.4)$$

From (3.3) we see that the eigenvector of the matrix (2.12) with the larger eigenvalue always lies in the range

$$ax_0(1)<ax<ax_0(0)=a^2+b^2. \quad (3.5)$$

Consider any vector not in this range. Pick any matrix of the form (2.12) with $\bar{\lambda}$ not equal to 0 or 1 such that the eigenvector with the smaller eigenvalue does not lie in the direction of this vector. If we apply this matrix to the vector a sufficient number of times, the resultant vector will lie inside (3.5). But this is true not only if all the matrices correspond to $\bar{\lambda}$ but also if the matrices correspond to λ's lying in some neighborhood of $\bar{\lambda}$. Therefore, there is a nonzero probability that, after the application of (2.12) a finite number of times on a given vector, the resultant vector will lie in the range (3.5). Furthermore, it is clear from (2.17) and (3.2) that if x_n satisfied (3.5) then x_{n+1} does also. Therefore we conclude that no distribution of vectors that does not vanish outside the range $ax_0(1) < ax < (a^2+b^2)$ can be stationary.

In fact the same argument shows that if

$$\mu(\lambda) \equiv 0 \quad \text{unless} \quad \lambda_0' \leq \lambda \leq \lambda_0 \tag{3.6}$$

$$\nu(x) \equiv 0 \quad \text{unless} \quad ax_0(\lambda_0) < ax < ax_0(\lambda'_0). \tag{3.7}$$

To see what restriction (3.7) makes on the limits of integration in (2.19), it is convenient to plot in Fig. 1 for $a>0$, (1) the region in the $x-x'$ plane where the kernal of (2.19) is not zero; (2) the contours along which $\mu[x'(a^2+b^2-a)/(x-a)]$ is constant; and (3) the region where the kernal of (2.19) does not vanish and $\nu(x)$ and $\nu'(x')$ are different from zero. We then readily see that (2.19) may be more explicitly written (when $a>0$) as

$$\nu(x) = \frac{b^2}{(x-a)^2} \int_{\max[x_0(\lambda_0),\lambda_0'(x-a)/(a^2+b^2-ax)]}^{\min[\lambda_0(x-a)/(a^2+b^2-ax),x_0(\lambda_0')]} dx'x'\nu(x')\mu\left(x'\frac{a^2+b^2-ax}{x-a}\right) \tag{3.8}$$

for $x_0(\lambda_0) < x < x_0(\lambda_0')$ and $\nu(x) \equiv 0$ otherwise. If $a<0$ we clearly have

$$\nu(x,a) = \nu(-x,-a). \tag{3.9}$$

In the sequel we concentrate on the special case $\lambda_0'=0$ where (3.8) specializes to

$$\nu(x) = \frac{b^2}{(x-a)^2} \int_{x_0(\lambda_0)}^{\min\{\lambda_0[(x-a)/(a^2+b^2-ax)],(a^2+b^2)/a\}} dx'x'\nu(x')\mu\left(x'\frac{a^2+b^2-ax}{x-a}\right). \tag{3.8$'$}$$

An important difference between the Onsager lattice and all random lattices may be seen from Fig. 1. If $\mu(\lambda) = \delta(\lambda - \bar{\lambda})$ then, as we have seen earlier, $\nu(x)$ is given by a δ function at that value of x that lies within the interval (3.5) where the curve $x' = \bar{\lambda}(x-a)(a^2+b^2-ax)^{-1}$ intersects the line $x=x'$. This intersection is at the point $x_0(\bar{\lambda})$ given by (3.3). There are three distinct cases:

(a) if $b(0)^2 > \bar{\lambda}(T>T_c)$ then as $\theta \to 0$

$$x_0(\bar{\lambda}) \sim [b(0)^2 - \bar{\lambda}]/a, \tag{3.10a}$$

(b) if $b(0)^2 < \bar{\lambda}(T<T_c)$ then as $\theta \to 0$

$$x_0(\bar{\lambda}) \sim \bar{\lambda}a/[\bar{\lambda} - b(0)^2], \tag{3.10b}$$

(c) and if $b(0)^2 = \bar{\lambda}(T=T_c)$ then as $\theta \to 0$

$$x_0(\bar{\lambda}) \to \pm \bar{\lambda}^{1/2}. \tag{3.10c}$$

We see from (3.10) that a δ-function distribution of λ leads to a δ-function stationary distribution. In the limit $\theta \to 0$, the δ-function stationary distribution remains at a finite value of x if $T=T_c$ but moves to zero (or infinity) if T is less than (or greater than) T_c.

Contrast the above case with the case of a very narrow $\mu(\lambda)$. As long as a is sufficiently far away from zero, a narrow $\mu(\lambda)$ must give use to a correspondingly narrow $\nu(x)$ because the projection on the x axis of the region of the $x=x'$ line where the kernel of (3.8) is appreciably different from zero is small. However, if θ gets close enough to zero and T is such that $\mu(\lambda)$ is different from zero in the region (however small) where $\lambda \sim b(0)^2$, then the above projection on the x axis becomes enormous. This dramatic broadening of $\nu(x)$ when $T \sim T_c$ and $\theta \sim 0$ will be exploited in the next section to obtain the dominant contribution to the specific heat for the particular narrow distribution

$$\mu(\lambda) = N\lambda_0^{-N}\lambda^{N-1} \tag{3.11}$$

for

$$0 \leq \lambda \leq \lambda_0 = \tanh^2\beta E_2^0$$

and

$$\mu(\lambda) = 0 \qquad \text{otherwise.}$$

The above discussion, however, shows that there is nothing particular about this form of $\mu(\lambda)$, and we expect that the physical properties which (3.11) leads to will be qualitatively the same for a wide class of narrow distributions.

Before studying the power-law distribution (3.11) in detail, it is convenient to perform some manipulations on (3.8′) that are useful for any $\mu(\lambda)$ such that $\mu(\lambda)=0$ if $\lambda > \lambda_0$. We first note that it is possible to change variables so that the upper limit of (3.8′) is transformed from a curve and a straight line into two straight lines. To do this let

$$\eta = x_0(x-x_0)/(\lambda_0 + x_0 x), \tag{3.12}$$

where $x_0 \equiv x_0(\lambda_0)$. We also introduce

$$B^2 = \lambda_0(x_0-a)/x_0(\lambda_0+ax_0) = \frac{a^2+b^2+\lambda_0-[(a^2+b^2-\lambda_0)^2+4a^2\lambda_0]^{1/2}}{a^2+b^2+\lambda_0+[(a^2+b^2-\lambda_0)^2+4a^2\lambda_0]^{1/2}} \tag{3.13}$$

so that

$$0 \leq \eta \leq B^2 \leq 1. \tag{3.14}$$

Define

$$\nu(x) = X(\eta)(d\eta/dx) = X(\eta)[(1-\eta)^2 x_0/(\lambda_0+x_0^2)], \tag{3.15}$$

so that

$$\int_0^{B^2} X(\eta)\,d\eta=1. \tag{3.16}$$

Then (3.8′) may be rewritten as

$$X(\eta)=\frac{\lambda_0 B^2(\lambda_0+x_0^2)}{(B^2x_0^2+\lambda_0\eta)^2}\int_0^{\min[B^2,\eta B^{-2}]} d\eta' X(\eta')$$

$$\times\frac{\eta'\lambda_0+x_0^2}{1-\eta'}\,\mu\left[\frac{\eta'\lambda_0+x_0^2}{1-\eta'}\,\frac{B^2-\eta}{\lambda_0^{-1}B^2x_0^2+\eta}\right]. \tag{3.17}$$

This equation may be cast into a somewhat simpler form when $\eta\leq B^4$ if we define

$$Y(\eta)=\int_0^{\eta} X(\eta')\,d\eta' \tag{3.18}$$

and integrate (3.17) over η to obtain

$$Y(\eta)=\lambda_0 B^2(\lambda_0+x_0^2)\int_0^{\eta} d\eta_1\int_0^{\eta_1 B^{-2}} d\eta' X(\eta')$$

$$\times\frac{\eta'\lambda_0+x_0^2}{(1-\eta')\,(B^2x_0^2+\lambda_0\eta_1)^2}\,\mu\left[\frac{\eta'\lambda_0+x_0^2}{1-\eta'}\,\frac{B^2-\eta_1}{\lambda_0^{-1}B^2x_0^2+\eta_1}\right]. \tag{3.19}$$

Interchange the order of integration and make the change of variable from η_1 to

$$\zeta=\frac{\eta'\lambda_0+x_0^2}{1-\eta'}\,\frac{B^2-\eta_1}{\lambda_0^{-1}B^2x_0^2+\eta_1} \tag{3.20}$$

to obtain

$$Y(\eta)=\int_0^{B^{-2}\eta} d\eta' X(\eta')$$

$$\times\int_{[(\lambda_0\eta'+x_0^2)/(1-\eta')][(B^2-\eta)/(\lambda_0^{-1}B^2x_0^2+\eta)]}^{\lambda_0} d\zeta\,\mu(\zeta)$$

$$=Y(\eta B^{-2})-\int_0^{\eta B^{-2}} d\eta' X(\eta')$$

$$\times\int_0^{[(\lambda_0\eta'+x_0^2)/(1-\eta')][(B^2-\eta)/(\lambda_0^{-1}B^2x_0^2+\eta)]} d\zeta\,\mu(\zeta). \tag{3.21}$$

Using the definition of Y, we may rewrite this as

$$\int_\eta^{\eta B^{-2}} d\eta' X(\eta')=\int_0^{\eta B^{-2}} d\eta' X(\eta')$$

$$\times\int_0^{[(\lambda_0\eta'+x_0^2)/(1-\eta')][(B^2-\eta)/(\lambda_0^{-1}B^2x_0^2+\eta)]} d\zeta\,\mu(\zeta). \tag{3.22}$$

It remains only to write out the special case (3.11).

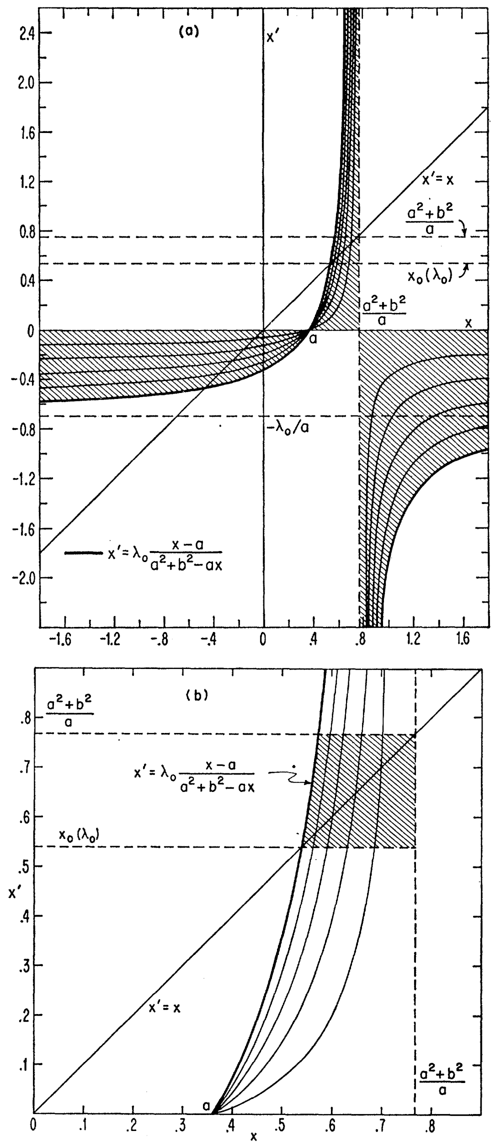

FIG. 1. (a) Contours along which the function

$$\mu(x'(a^2+b^2-ax)/(x-a))$$

is constant. The kernal of the integral equation for $\nu(x)$ is different from zero only in the shaded region. (b) Enlargement of 1(a). The shaded region is the only region in which the kernal of the integral equation does not vanish and $\nu(x)$ and $\nu(x')$ are different from zero.

We work the ζ integral in (3.22) and obtain

$$\int_\eta^{\eta B^{-2}} d\eta' X(\eta')=\int_0^{\eta B^{-2}} d\eta' X(\eta')N^{-1}$$

$$\times\left(\frac{\lambda_0\eta'+x_0^2}{1-\eta'}\,\frac{B^2-\eta}{\lambda_0^{-1}B^2x_0^2+\eta}\right)\mu\left(\frac{\lambda_0\eta'+x_0^2}{1-\eta'}\,\frac{B^2-\eta}{\lambda_0^{-1}B^2x_0^2+\eta}\right). \tag{3.23}$$

The right-hand side of this equation may be further simplified using the original integral equation (3.17). We obtain

$$\int_{\eta}^{\eta B^{-2}} d\eta' X(\eta') = \frac{(B^2-\eta)(B^2x_0^2+\eta\lambda_0)}{NB^2(\lambda_0+x_0^2)} X(\eta). \quad (3.24)$$

Finally, it is useful to make the exponential change of variable

$$\eta = e^{-\tau}, \quad (3.25)$$

where $-\ln B^2 < \tau < \infty$. Call

$$U(\tau) = \eta X(\eta), \quad (3.26)$$

so that

$$\int_{-\ln B^2}^{\infty} d\tau\, U(\tau) = 1. \quad (3.27)$$

Then we obtain

$$\int_{\tau}^{\tau+\ln B^2} d\tau' U(\tau') = -\frac{(B^2-e^{-\tau})(B^2x_0^2+\lambda_0 e^{-\tau})}{NB^2(\lambda_0+x_0^2)} e^{\tau} U(\tau). \quad (3.28)$$

4. POWER-LAW DISTRIBUTION

In this section we determine the dominant contribution to the specific heat of the lattice characterized by the power-law distribution (3.11). We consider only the case of large N and ignore all contributions to the specific heat which vanish as $N\to\infty$. Then the discussion of Sec. 3 shows that when a is away from zero $\nu(x)$ is given by a narrow distribution which will differ but little from the $\nu(x)$ of an Onsager problem. We are not interested in these small deviations and concentrate our attention on the opposite extreme when a is close to zero. From (2.29a) we find that $T=T_c$ if

$$\ln[b(0)^2\lambda_0^{-1}] = \ln B^2(0) = -N^{-1}. \quad (4.1)$$

When B^2 is close to this value and $a\sim 0$, the discussion of Sec. 3 shows that $\nu(x)$ is very broad. Therefore when x is of order 1, $\nu(x)$ [and hence $U(\tau)$] may be treated as slowly varying. Furthermore, (4.1) shows that the region of integration of (3.28) is of the order of N^{-1}. Hence to leading order in N^{-1} we may expand

$$U(\tau') \sim U(\tau) + (\tau'-\tau) U'(\tau) \quad (4.2)$$

and do the integral in (3.28) to obtain the approximate differential equation

$$\tfrac{1}{2}(\ln B^2)^2 (d/d\tau) U(\tau) + U(\tau) \ln B^2$$
$$= [-(B^2-e^{-\tau})(B^2x_0^2e^{\tau}+\lambda_0)/NB^2(\lambda_0+x_0^2)] U(\tau). \quad (4.3)$$

This is readily solved and we find

$$U(\tau) = C_N(B^2, x_0) \exp\{s\tau - te^{-\tau} - ue^{\tau}\} \quad (4.4)$$

where, by (3.3),

$$s = 2(\ln B^2)^{-2}[-\ln B^2 - N^{-1}(\lambda_0 x_0^{-1} - x_0)/(\lambda_0 x_0^{-1} + x_0)],$$
$$= 2(\ln B^2)^{-2}\{-\ln B^2 + N^{-1}(a^2+b^2-\lambda_0) \times [(a^2+b^2-\lambda_0)^2 + 4\lambda_0 a^2]^{-1/2}\}, \quad (4.5a)$$

$$t = 2(\ln B^2)^{-2} N^{-1} B^{-2} \lambda_0 x_0^{-1} (\lambda_0 x_0^{-1} + x_0)^{-1}$$
$$= (\ln B^2)^{-2} N^{-1} B^{-2} \{1 - (a^2+b^2-\lambda_0) \times [(a^2+b^2-\lambda_0)^2 + 4\lambda_0 a^2]^{-1/2}\}, \quad (4.5b)$$

$$u = 2(\ln B^2)^{-2} N^{-1} B^2 x_0 (\lambda_0 x_0^{-1} + x_0)^{-1}$$
$$= (\ln B^2)^{-2} N^{-1} B^2 \{1 + (a^2+b^2-\lambda_0) \times [(a^2+b^2-\lambda_0)^2 + 4\lambda_0 a^2]^{-1/2}\}, \quad (4.5c)$$

and $C_N(B^2, x_0)$ is an appropriate normalization constant. Therefore, when $T\sim T_c$ and $\theta\sim 0$, we obtain

$$X(\eta) \sim C_N(B^2, x_0^2) \eta^{-s-1} \exp(-t\eta - u/\eta). \quad (4.6)$$

This approximation to $X(\eta)$ forms the basis of all further considerations of this section.

To give insight into the structure of $X(\eta)$ we first remark that as $N\to\infty$ the power-law distribution (3.11) approaches $\delta(\lambda-\lambda_0)$. Therefore when $T\neq T_c$ and $\theta\neq 0$, the exact $\nu(x)$ must approach $\delta(x-x_0)$ as $N\to\infty$ [and hence $X(\eta)$ must approach $\delta(\eta)$]. Our approximation to $X(\eta)$ was derived under the assumption that $T-T_c$ and θ were small and hence it is not obvious that (4.6) will approach $\delta(\eta)$ as $N\to\infty$. To see that this is in fact the case we note that the maximum of $\eta X(\eta)$ occurs at

$$\eta_m = (2t)^{-1}[-s + (s^2+4tu)^{1/2}] \quad (4.7a)$$

and its width (in the sense of a steepest descent integral) is

$$w_\eta = \{(\partial^2/\partial_\eta^2)[s\ln\eta + t\eta + u\eta^{-1}]\}^{1/2} \mid \eta=\eta_m,$$
$$= \eta_m^{3/2}(2u - s\eta_m)^{-1/2}. \quad (4.7b)$$

If $T\neq T_c$ and $\theta\neq 0$ are fixed and $N\to\infty$, $s=-2(\ln B^2)^{-1}+O(N^{-1})$, $t=O(N^{-1})$, and $u=O(N^{-1})$ so that $\eta_m=O(N^{-1})$ and $w_\eta=O(N^{-1})$. Therefore $X(\eta)\to\delta(\eta-\eta_m)$ in the sense that in the evaluation of the normalization integral of $X(\eta)$ we may replace $X(\eta)$ by $\delta(\eta-\eta_m)$.

We are interested in the contributions of order 1 to the specific heat when N is large. This is found by using our approximation for $X(\eta)$ in the integral (2.20) which in terms of η is

$$\lim_{\mathfrak{M}\to\infty} \mathfrak{M}^{-1} \ln C_{\mathfrak{M}}(\theta) = \int_0^{B^2} d\eta\, X(\eta) \int_0^1 d\lambda\, \mu(\lambda) \ln[a^2+b^2+a\lambda(1-\eta)/(\eta\lambda_0 x_0^{-1}+x_0)]. \quad (4.8)$$

Define $f(\eta)$ by

$$f(\eta) = \ln[a^2+b^2+a\lambda(1-\eta)/(\eta\lambda_0 x_0^{-1}+x_0)]$$
$$= \ln\{a^2+b^2+2a^2\lambda[(a^2+b^2-\lambda_0)+(1+\eta)(1-\eta)^{-1}((a^2+b^2-\lambda_0)^2+4\lambda a^2)^{1/2}]^{-1}\}. \quad (4.9)$$

When $T\neq T_c$ and $\theta\neq 0$ are fixed and $N\to\infty$ we see that if $\eta=O(N^{-1})$

$$f(\eta)=\ln[a^2+b^2+a\lambda x_0^{-1}]+O(N^{-1}).$$

Therefore our approximation $X(\eta)$ may be replaced by $\delta(\eta-\eta_m)$ for the purpose of obtaining the leading order term of (4.8). This is the behavior expected of the exact $X(\eta)$ in this limit so we conclude that approximation (4.6) may be used for *all* T and θ for the purpose of obtaining the leading order term in the specific heat when $N\to\infty$.

In fact, for our limited purpose $X(\eta)$ is well approximated by $\delta(\eta-\eta_m)$ not only when $T\neq T_c$ and $\theta\neq 0$ are fixed and $N\to\infty$ but also when a is of the order of $N^{-\epsilon}$ and $b(0)^2-\lambda_0$ is of the order of $N^{-\gamma}$, where $0<\epsilon<1$ and $0<\gamma<1$. There are three cases to be considered. (a) $\gamma>\epsilon$, (b) $\gamma<\epsilon$ and $b(0)^2-\lambda_0<0$, and (c) $\gamma<\epsilon$ and $b(0)^2-\lambda_0>0$. We find

case (a),

$$\eta_m=O(N^{-1+\epsilon}),\qquad w_\eta=O(N^{-1+\epsilon/2}),$$
$$f(\eta)=\ln b^2+O(N^{-\epsilon}),$$

case (b),

$$\eta_m=O(N^{-1-2\epsilon+3\gamma}),\qquad w_\eta=O(N^{-1-2\epsilon+5\gamma/2}),$$
$$f(\eta)=\ln b^2+O(N^{-\gamma}),\text{ and}$$

case (c),

$$\eta_m=O(N^{-1+\gamma}),\qquad w_\eta=O(N^{-1+\gamma/2}),$$
$$f(\eta)=\ln b^2+O(N^{-2\epsilon+\gamma}).$$

Since the variations in $f(\eta)$ caused by deviations in η on the order of w_η vanish as $N\to\infty$, we conclude that in these cases, for our purpose, $X(\eta)$ may be replaced by $\delta(\eta-\eta_m)$.

Because of the fact that even if $T\sim T_c$ the contribution of $\nu(x)-\delta(x-x_m)$ to the specific heat is very small for $|a|\gg N^{-1}$, it is useful to consider the contributions of $\delta(x-x_m)$ and $\nu(x)-\delta(x-x_m)$ separately. It is easily seen from (4.7a) and (3.12) that x_m obeys

$$x_m^2a[1+b^{-2}G]-x_m[a^2+b^2-\lambda_0+G(1+2a^2b^{-2})]-\lambda_0a[1-G(a^2+b^2)\lambda_0^{-1}b^{-2}]=0,\quad(4.10\text{a})$$

where

$$G=-N^{-1}(\ln B^2)^{-1}[(a^2+b^2-\lambda_0)^2+4a^2\lambda_0]^{1/2}.\quad(4.10\text{b})$$

If we use $\delta(x-x_m)$ for $\nu(x)$ in (2.20) we find, correct to order N^{-1},

$$-\beta\bar{F}_r=\int_0^\infty dE_2p(E_2)\ln(2\cosh\beta E_1\cosh\beta E_2)+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\int_{-\infty}^{\infty}dx\,\delta(x-x_m)\int_0^1 d\lambda\,\mu(\lambda)$$
$$\times\ln\{|1+z_1e^{i\theta}|^2[a^2+b^2+a\lambda x^{-1}]\}\sim\ln(2\cosh\beta E_1\cosh\beta E_2^0)+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta$$
$$\times\ln\{|1+z_1e^{i\theta}|^2\tfrac{1}{2}[1-G(a^2+b^2)\lambda_0^{-1}b^{-2}]^{-1}(a^2+b^2+\lambda_0-Gb^{-2}\lambda_0^{-1}[2(a^2+b^2)^2+\lambda_0(b^2+2a^2)]$$
$$+\{[a^2+b^2-\lambda_0+G(1+2a^2b^{-2})]^2+4a^2\lambda_0(1+b^{-2}G)[1-G(a^2+b^2)\lambda_0^{-1}b^{-2}]\}^{1/2})\}.\quad(4.11)$$

By comparing $\bar{F}_r$ with the Onsager free energy F_0 of (3.4), we conclude that to order N^{-1} $\bar{F}_r$ is the same as F_0 except that the critical temperature has been shifted from $b(0)^2-\lambda_0=0$ to

$$b(0)^2-\lambda_0=-G(0)=-[b(0)^2-\lambda_0]N^{-1}[\ln b(0)^2\lambda_0^{-1}]^{-1},\quad(4.12)$$

which is exactly the same equation for T_c as (4.1). Therefore over the range of T where $\nu(x)$ is well approximated by $\delta(x-x_m)$, F_r is dominated by a term which has an apparent logarithmic divergence at T_c.

To study the behavior very near T_c we write

$$F_r-\bar{F}_r\sim-(\beta 4\pi)^{-1}\int_{-\pi}^{\pi}\theta\int_{-\infty}^{\infty}dx\,[\nu(x)-\delta(x-x_m)]\ln[a^2+b^2+a\lambda_0x^{-1}],\quad(4.13)$$

where the λ integral has been evaluated to $O(N^{-1})$. We consider only T close enough to T_c so that $b^2-\lambda_0<0$. The $\sim$ sign is to mean that both sides lead to the same specific heat to order 1 as $N\to\infty$. Furthermore we have seen that unless $a^2+b^2-\lambda_0=O(N^{-1})$ we cannot have a contribution of order 1 to the specific heat obtained from (4.13). Therefore, $ax_0^{-1}=O(N^{-1})$ and, since $|x_0|<|x|$, we may expand the logarithm of (4.13) to obtain

$$F_r-\bar{F}_r\sim-(\beta 4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\int_{-\infty}^{\infty}dx[\nu(x)-\delta(x-x_m)]a\lambda_0(a^2+b^2)^{-1}x^{-1}$$
$$=(-\beta\pi)^{-1}\int_{-\pi}^{0}d\theta\,a^2\lambda_0(a^2+b^2)^{-1}\int_0^{B^2}d\eta\,[X(\eta)-\delta(\eta-\eta_m)]$$
$$\times\{a^2+b^2-\lambda_0+(1+\eta)(1-\eta)^{-1}[(a^2+b^2-\lambda_0)^2+4a^2\lambda_0]^{1/2}\}^{-1}.\quad(4.14)$$

To evaluate this approximately to order 1, we need to find the correct range of θ, η, and $T-T_c$. We have previously considered the case $a \gg N^{-1}$ and $|b^2-\lambda_0| \gg N^{-1}$ and seen via a steepest-descent integration that the contribution is negligible. In a similar fashion we may show that the cases (1) a is of order N^{-1}, (2) $a=o(N^{-1})$ and $s=O(N)$ give only contributions of an order N^{-1} to the specific heat. The only region that gives a contribution of order one to the specific heat as $N\to\infty$ is $a=O(N^{-2})$ and $T-T_c=O(N^{-2})$.

When $T-T_c=O(N^{-2})$ and $a=O(N^{-2})$, $X(\eta)$ and $\delta(\eta-\eta_m)$ do not cancel at all so that F_r and $\bar{F}_r$ must be evaluated separately. Because $\bar{C}_v^r=-T\bar{F}_r''(T)$ is to leading order in N the same as Onsager's specific heat, we may use Onsager's result[1] to find

$$\bar{C}_v^r \sim 4k\beta_c^2\pi^{-1}\{z_{1c}z_{2c}^0(1-z_{1c})^{-2}(1-z_{2c}^0)^{-2}[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]^2 \times[\ln T_c/|T-T_c|-\ln\tfrac{1}{8}\beta_c(E_1(z_{1c}+z_{1c}^{-1})+E_2^0(z_{2c}^0+z_{2c}^{0-1}))] -[E_1^2 4z_{1c}^2 z_{2c}^{02}(1-z_{2c}^0)^{-4}gd2\beta_cE_1+2E_1E_2^0+E_2^{02}4z_{1c}^2z_{2c}^{02}(1-z_{1c})^{-4}gd2\beta_cE_2^0]\}, \quad (4.15)$$

where gd stands for the Gudermannian [gd$x=\tan^{-1}\sinh x$], the subscript c means $T=T_c$, and $z_{2c}^0=\tanh\beta_cE_2^0$.

We evaluate C_v^r by splitting the θ integration into two regions, one for $0<|\theta|\le 0(N^{-2})$ and the other $|\theta|\gg N^{-2}$. In the second region the integrand of (4.14) is small. Therefore, to leading order the parts of C_v^r and $\bar{C}_v^r$ coming from angular integration over this second region are equal. It is easily seen that the large angles give a contribution to C_v^r that is a constant. The point of separation between the two regions is not well defined but, as will shortly be verified, the precise choice of this cutoff does not affect the temperature-dependent part of C_v^r but only the constant. Therefore, we may evaluate the leading contribution to C_v^r as $N\to\infty$ when $T-T_c=O(N^{-2})$ up to a constant by integrating θ only up to $O(N^{-2})$. We then may determine the constant by the requirement that, when $|T-T_c|\gg N^{-2}$, $C_v^r\sim\bar{C}_v^r$ where $\bar{C}_v^r$ is given by (4.15).

To explicitly carry out this evaluation, we define

$$\phi=-8\lambda_0^{-1/2}z_1(1+z_1)^{-2}N^2\theta, \quad (4.16)$$

so that

$$a=\tfrac{1}{4}N^{-2}\lambda_0^{1/2}\phi+O(N^{-4}). \quad (4.17)$$

We further define δ by

$$\lambda_0^{-1}(1-z_1)^2(1+z_1)^{-2}-e^{-N^{-1}}=\tfrac{1}{2}N^{-2}\delta. \quad (4.18)$$

As $N\to\infty$ δ is to be of order 1. Explicitly,

$$\delta=(T-T_c)N^2 4k\beta_c^2(1+z_{2c}^0)z_{2c}^{0-1} \times\{[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]+O(N^{-1})\}. \quad (4.19)$$

We then have

$$C_v^r=-T(\partial^2F_r/\partial T^2)\sim-k\beta_c^3 16(1+z_{2c}^0)^2z_{2c}^{0-2} \times\{E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)\}^2N^4(\partial^2F_r/\partial\delta^2). \quad (4.20)$$

Furthermore,

$$a^2+b^2-\lambda_0=-\lambda_0N^{-1}[1-(2N)^{-1}(\delta+1)]+O(N^{-3}) \quad (4.21)$$

and

$$\ln B^2=-N^{-1}[1-(2N)^{-1}\delta]+O(N^{-3}). \quad (4.22)$$

We then find

$$s=-\delta+O(N^{-1}), \quad (4.23a)$$

$$t=2N+O(1), \quad (4.23b)$$

$$u=(8N)^{-1}\phi^2+O(N^{-2}). \quad (4.23c)$$

Therefore,

$$C_v^r\sim-k\beta_c^2(8\pi)^{-1}(1+z_{1c})^2(1+z_{2c}^0)^2z_{2c}^{0-1}z_{1c}^{-1}\{E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)\}^2N^{-1} \times\frac{\partial^2}{\partial\delta^2}\int_0^{N^2}d\phi\,\phi^2\int_0^1 d\eta\,\eta^{\delta-1}\exp(-2N\eta-\phi^2/8N\eta)C_N(\phi,\delta)\left(1-\frac{1+\eta}{1-\eta}\right)^{-1}+\bar{K}, \quad (4.24)$$

where the coefficient of N^2 is arbitrary and $\bar{K}$ is determined by the requirement that as $\delta\to\infty$, $C_v^r\to\bar{C}_v^r$.

The normalization constant $C_N(\phi,\delta)$ is determined from the requirement

$$1=C_N\int_0^1 d\eta\,\eta^{\delta-1}\exp(-2N\eta-\phi^2/8N\eta). \quad (4.25)$$

Letting $\xi=4N\phi^{-1}\eta$, we have

$$C_N^{-1}=(4N)^{-1}\phi\int_0^{4N\phi^{-1}}d\xi\,(\xi\phi/4N)^{\delta-1} \times\exp[-\tfrac{1}{2}\phi(\xi+\xi^{-1})]. \quad (4.26)$$

The upper limit may be extended to ∞ and we find

$$C_N^{-1}\sim 2(\phi/4N)^\delta K_\delta(\phi), \quad (4.27)$$

where $K_\delta(\phi)$ is the modified Bessel function of the third kind[8] of order δ. Similarly, we find

$$\int_0^1 d\eta\,\eta^{\delta-1}\exp(-2N\eta-\phi^2/8N\eta)[1-(1+\eta)(1-\eta)^{-1}]^{-1} \sim-(\phi/4N)^{\delta-1}K_{\delta-1}(\phi). \quad (4.28)$$

If we also note that $(1-z_{1c}^2)(1-z_{2c}^{02})=4|z_{1c}||z_{2c}^0|+$

$O(N^{-1})$, we have

$$C_v{}^r \sim 4k\beta_c{}^2\pi^{-1}(1-z_{1c})^{-2}(1-z_{2c}{}^0)^{-2}z_{1c}z_{2c}{}^0 \times\{E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)\}^2 \times\frac{\partial^2}{\partial\delta^2}\int_0^{N^2} d\phi\,\frac{\phi K_{\delta-1}(\phi)}{K_\delta(\phi)}+\bar{K}. \tag{4.29}$$

As $\phi\to\infty$,

$$K_{\delta-1}(\phi)/K_\delta(\phi)\to 1-\phi^{-1}(\delta-\tfrac{1}{2})+\tfrac{1}{2}\phi^{-2}(\delta^2-\tfrac{1}{4})+O(\phi^{-3}), \tag{4.30}$$

so there is a term in (4.29) that behaves as $\ln N^2$ (where we remember that the coefficient of N^2 is arbitrary). We explicitly extract this term to write

$$\frac{\partial^2}{\partial\delta^2}\int_0^{N^2} d\phi\,\frac{\phi K_{\delta-1}(\phi)}{K_\delta(\phi)} = \int_0^\infty d\phi\left\{\phi\,\frac{\partial^2}{\partial\delta^2}\frac{K_{\delta-1}(\phi)}{K_\delta(\phi)}-(\phi+1)^{-1}\right\}+\ln N^2+O(N^{-1}), \tag{4.31}$$

which, using the recurrence relation

$$K_{\delta-1}(\phi)=-\delta\phi^{-1}K_\delta(\phi)-(d/d\phi)K_\delta(\phi), \tag{4.32}$$

may be reexpressed as

$$\int_0^\infty d\phi\left[\frac{\partial^2}{\partial\delta^2}\ln K_\delta(\phi)-(\phi+1)^{-1}\right]+\ln N^2-1+O(N^{-1})=R(\delta)+\ln N^2-1+O(N^{-1}), \tag{4.33}$$

which defines the function $R(\delta)$.

For convenience we absorb the -1 term of (4.33) into $\bar{K}$ of (4.29) by defining

$$K=\bar{K}-4k\beta_c{}^2\pi^{-1}(1-Z_{1c})^{-2}(1-Z_{2c}{}^0)^{-2}Z_{1c}Z_{2c}{}^0\{E_1(1-Z_{1c})+E_2{}^0(1-Z_{2c}{}^0)\}^2. \tag{4.34}$$

To determine K we must expand (4.33) as $\delta\to\infty$. We use the expansion[9]

$$K_\delta\sim(\tfrac{1}{2}\pi)^{1/2}(\delta^2+\phi^2)^{-1/4}\exp[-(\delta^2+\phi^2)^{1/2}+\delta\sinh^{-1}(\delta/\phi)]\{1+[-\tfrac{1}{8}+(5/24)(1+\phi^2/\delta^2)^{-1}](\delta^2+\phi^2)^{-1/2}\}, \tag{4.35}$$

to find that as $\delta\to\infty$

$$\int_0^\infty d\phi\left[\frac{\partial^2}{\partial\delta^2}\ln K_\delta(\phi)-(\phi+1)^{-1}\right]+\ln N^2=\ln 2+\ln N^2\delta^{-1}-\tfrac{1}{8}\delta^{-2}+O(\delta^{-3}). \tag{4.36}$$

Therefore we find as $\delta\to\infty$

$$C_v{}^r(\delta)\to -4k\beta_c{}^2\pi^{-1}(1-z_{1c})^{-2}(1-z_{2c}{}^0)^{-2}z_{1c}z_{2c}{}^0\{E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)\}^2 \times\ln\{2k\beta_c{}^2\,|T-T_c|\,(z_{2c}{}^{0-1}+1)[E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)]\}+K. \tag{4.37}$$

Comparing this with (4.15) gives

$$\begin{aligned}K=&-4k\beta_c{}^2\pi^{-1}[(1-z_{1c})^{-2}(1-z_{2c}{}^0)^{-2}z_{1c}z_{2c}{}^0[E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)]^2\\&\times\{-\ln 2(z_{2c}{}^{0-1}+1)[E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)]+\ln\tfrac{1}{8}[E_1(z_{1c}+z_{1c}{}^{-1})+E_2{}^0(z_{2c}{}^0+z_{2c}{}^{0-1})]\}\\&+E_1{}^2 4z_{1c}{}^2z_{2c}{}^{02}(1-z_{2c}{}^0)^{-4}\,\mathrm{gd}\,2\beta_cE_1+2E_1E_2{}^0+E_2{}^{02}4z_{1c}{}^2z_{2c}{}^{02}(1-z_{1c})^{-4}\,\mathrm{gd}\,2\beta_cE_2{}^0]\\=&-4k\beta_c{}^2\pi^{-1}\{\tfrac{1}{2}[E_1{}^2\sinh 2\beta_cE_2{}^0+2E_1E_2{}^0+E_2{}^{02}\sinh 2\beta_cE_1]\\&\times[\ln\tfrac{1}{4}(E_1\coth 2\beta_cE_1+E_2{}^0\coth 2\beta_cE_2{}^0)-\ln 2\{(1+\coth\beta_cE_2{}^0)[E_1(1-\tanh\beta_cE_1)+E_2{}^0(1-\tanh\beta_cE_2{}^0)]\}]\\&+E_1{}^2\sinh^2 2\beta_cE_2{}^0\,\mathrm{gd}\,2\beta_cE_1+2E_1E_2{}^0+E_2{}^{02}\sinh^2 2\beta_cE_1\,\mathrm{gd}\,2\beta_cE_2{}^0\}.\end{aligned} \tag{4.38}$$

We therefore have as our final result that when $T-T_c=O(N^{-2})$

$$C_v{}^r\sim 2\pi^{-1}k\beta_c{}^2\{E_1{}^2\sinh 2\beta_cE_2{}^0+2E_1E_2{}^0+E_2{}^{02}\sinh 2\beta_cE_1\}\left\{\int_0^\infty d\phi\left[\frac{\partial^2}{\partial\delta^2}\ln K_\delta(\phi)-(\phi+1)^{-1}\right]+\ln N^2\right\}+K, \tag{4.39}$$

where δ is defined by (4.19) and K by (4.38). We note in particular that $C_v{}^r$ is an even function of δ.

The specific heat (4.39) has been calculated for the random lattice characterized by $\mu(\lambda)$ given by (3.11).

[8] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 2, p. 82.
[9] Reference 8, Vol. 2, p. 26.

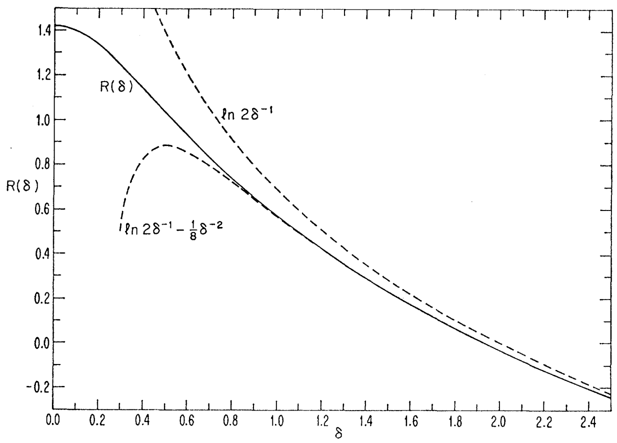

FIG. 2. The integral $R(\delta)$ as a function of δ.

The corresponding probability density $P(E_2)$ is

$$P(E_2)=2N(\tanh E_2{}^0\beta)^{-2N}\beta(\mathrm{sech}^2\beta E_2)(\tanh\beta E_2)^{2N-1} \quad \text{for} \quad 0\leq E_2\leq E_2{}^0. \quad (4.40)$$

This probability clearly depends on temperature. The probabilities envisioned in the original statement of the model were to be temperature-independent. Fortunately, because the only significant deviations of $C_v{}^r$ from $\bar{C}_v{}^r$ occurs when $T-T_c=O(N^{-2})$, we see that, in the temperature range where (4.39) is valid,

$$P(E_2)=\beta_c(\mathrm{sech}^2\beta_c E_2)\,2N(\tanh E_2{}^0\beta_c)^{-2N} \times(\tanh\beta_c E_2)^{2N-1}[1+O(N^{-1})]. \quad (4.41)$$

Therefore $P(E_2)$ as given by (4.41) differs, when $T-T_c=O(N^{-2})$, from a temperature-independent probability by a term of $O(N^{-1})$, which is negligible.

To justify calling the temperature given by the solution of (4.1) the critical temperature we must show that $C_v{}^r$ as given by (4.39) is not analytic at $\delta=0$. For this purpose we integrate the integral in (4.39) by parts once and consider

$$\int_0^\infty d\phi\left[\frac{\partial^2}{\partial\delta^2}\frac{\phi K_\delta'(\phi)}{K_\delta(\phi)}+(\phi+1)^{-1}\right]. \quad (4.42)$$

When $\delta\to 0$ the singular behavior of this function comes from the region near $\phi=0$. Thus we may obtain the most singular behavior of (4.42) if we integrate only from 0 to some small positive upper limit ϵ and expand for ϕ and δ near zero

$$K_\delta(\phi)\sim(2\delta)^{-1}[(\phi/2)^{-\delta}-(\phi/2)^{\delta}]. \quad (4.43)$$

Therefore we study

$$\int_0^\epsilon d\phi\,\delta(\phi^{-\delta}+\phi^\delta)/(\phi^{-\delta}-\phi^\delta)=\int_0^\epsilon d\phi\,2\delta(1-\phi^{2\delta})^{-1}-\delta\epsilon. \quad (4.44)$$

Now if $\epsilon<1$,

$$\int_0^\epsilon d\phi/\ln\phi$$

exists, so the analytic properties of (4.43) are the same as

$$I(\delta)=\int_0^1 d\phi\,[\delta(1-\varphi^\delta)^{-1}+(\ln\phi)^{-1}]. \quad (4.45)$$

We have been able to replace ϵ by 1 in (4.45) because the integrand is regular at $\phi=1$. It is easily seen that $I(\delta)-I(-\delta)=\delta$ so we need consider only $\delta>0$. Then if we let

$$\phi=e^{-\xi/\delta}, \quad (4.46)$$

we find[10]

$$I(\delta)=\int_0^\infty d\xi\,e^{-\xi/\delta}[(1-e^{-\xi})^{-1}-\xi^{-1}] -\ln\delta-\psi(\delta^{-1}), \quad (4.47)$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. For small δ we may use the asymptotic expansion[11] for $\psi(\delta^{-1})$ to obtain the formal power series valid for both positive and negative δ,

$$I(\delta)=\tfrac{1}{2}\delta+\sum_{n=1}^{\infty}B_{2n}\delta^{2n}(2n)^{-1}. \quad (4.48)$$

Here B_{2n} are the Bernoulli numbers. Therefore, the most singular part of $C_v{}^r$ is an *infinitely differentiable* function of T even at $T=T_c$. However, because (4.48) diverges for all $\delta\neq 0$, $C_v{}^r$ is *not* an analytic function of T at $T=T_c$. It is therefore correct to call T_c as determined from (4.1) the critical temperature.

Finally, it is useful to supplement these analytic considerations with a numerical evaluation of $C_v{}^r$ when $T-T_c=O(N^{-2})$. We have done this on an IBM 7044 computer and present the results in Fig. 2 and 3.

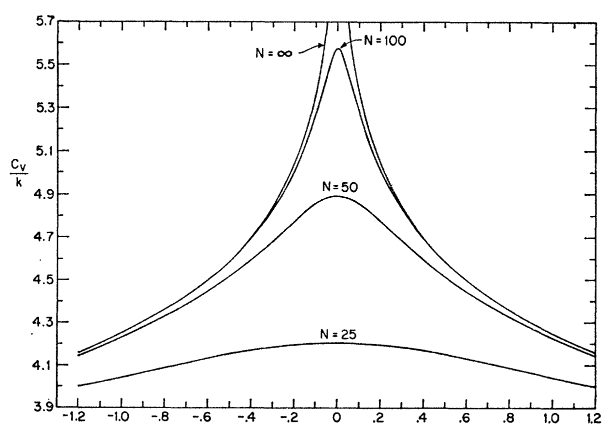

FIG. 3. Comparison of C_v of the pure Onsager lattice and $C_v{}^r$ for several values of N for the case $E_1=E_2{}^0$.

[10] Reference 8, Vol. 1, p. 18.
[11] Reference 8, Vol. 1, p. 47.

5. DISCUSSION

The principal result of this study is to exhibit explicitly a model with random impurities where, to order 1, the specific heat is an infinitely differentiable function of the temperature even at T_c. This behavior is qualitatively in beautiful agreement with the experimental findings previously quoted. We therefore feel justified in proposing[12] that random impurities can be the origin of the smooth specific heats observed experimentally.

On the other hand, this particular model of random impurities is obviously artificial. It is not realistic to assume that all vertical bonds $E_2(j)$ in a given row will be equal if we are allowing $E_2(j)$ to vary from row to row. A realistic model of random impurities should not have such a large amount of correlation between the random bonds. For this reason, we are extremely fortunate that the specific heat found in Sec. 4 is infinitely differentiable. If random impurities had increased the observed order of the phase transition to a higher but finite order, the question could very reasonably be asked if relaxation of this very stringent correlation requirement would further increase the observed order. However, even the limited amount of randomness allowed in this model has made the dominant contribution to all derivatives of the specific heat continuous, so no further increase of the observed order is possible. This very smooth specific heat is in distinct contrast to the result of Syozi's model.[13]

We are unable to ascertain which of our results are qualitatively dependent on the very special sort of randomness allowed in our model. For example, we find that if the impurities have a narrow width of order N^{-1}, there is an effect of order 1 on the specific heat only when $|T-T_c|=O(N^{-2})$. We do not know if this order of magnitude persists in general.

It must be emphasized that in this computation of the observed (order 1) specific heat we have in no way settled the question of the order of the phase transition. To answer this question, which admittedly is not very important physically, one must study the analytic behavior of all contributions to C_v^r. There are several reasons why this is not trivial. We mention two.

(1) The relationship of the differential equation (4.3) to the integral equation (3.28) must be studied in detail. The naive procedure of using more terms in the Taylor-series expansion of $U(\tau)$ leaves one with a differential equation of order larger than 1 and the problem of determining the boundary conditions must be resolved.

[12] B. M. McCoy and T. T. Wu, Phys. Rev. Letters, **21**, 549 (1968).

[13] I. Syozi, Progr. Theoret. Phys. (Kyoto) **34**, 189 (1965); I. Syozi and S. Miyazima, *ibid.* **36**, 1083 (1966); J. W. Essom and H. Garelick, Proc. Phys. Soc. (London) **92**, 136 (1967).

(2) Formally, it is possible to write down an exact iterative solution to the integral equation (3.28). This iterative solution is analytic only in the segments $B^{2n}<\eta<B^{2(n-1)}$. The approximation employed in Sec. 4 approximates the analytic function which this segmented solution approaches when $n\rightarrow\infty$. This is the important region for the order-1 contributions to C_v^r, but in general the effects (if any) of the segmented nature of the exact solution are not understood.

There are several ways in which the considerations of this paper may be generalized and extended.

(A) The present discussion has been concerned with the case in which E_2 is the random variable. However, the general formulation given in Sec. 2 applies equally well when E_1 (and even both E_1 and E_2) is random. A new feature which can occur when E_1 is random is that, as may be seen from (2.29), T_c may be zero even though $|E_1|$ is never small if $P(E_1)=P(-E_1)$.

(B) The specialization to the power-law distribution (3.11) has been made for clarity of presentation only and many of the above considerations may be generalized to any narrow distribution.

(C) It is also possible to consider distributions which do not have small widths. In particular, the power-law distribution (3.11) when N is not large may be analyzed in terms of a difference-differential equation.

(D) A more challenging problem is the study of spin-correlation functions. The average value of the nearest-neighbor correlation functions may be readily obtained if we combine the results of IV which express correlation functions in terms of C_n and D_n with the considerations of the present paper. However, the spin correlation functions themselves possess a probability distribution function. To study the moments of this distribution and also to study spin correlations other than nearest neighbor, it is necessary to consider joint stationary probability distributions. In particular for the second moment of the nearest-neighbor correlation-functions' probability function the relevant joint probability function satisfies a two-dimensional integral equation. This equation may be approximated by a partial differential equation in a manner analogous to that done in this paper.

(E) The most outstanding feature of this model that remains to be investigated is the behavior of the spontaneous magnetization near T_c. Without a doubt the eighth-root singularity found by Yang will be weakened by the presence of random impurities. The difficult question is, how much?

ACKNOWLEDGMENTS

The authors are deeply indebted to Professor C. N. Yang for the most helpful discussions. One of us (T.T.W) also wishes to thank Professor G. C. Rota and Professor H. McKean for fruitful conversations.