PHYSICAL REVIEW VOLUME 176, NUMBER 2 10 DECEMBER 1968

Effect of Orbital Quantization on the Critical Field of Type-II Superconductors

L. W. GRUENBERG*

Department of Electrical Engineering and Center for Materials Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

L. GUNTHER†

Department of Physics, Tufts University, Medford, Massachusetts 02155

(Received 1 July 1968)

We discuss the effect of the discreteness of the Landau levels on the upper critical field of a type-II superconductor. At $T=0°K$ an ideal superconductor will remain superconducting in an arbitrarily large magnetic field. However, the high-field state is destroyed by a minute amount of impurity scattering or by a small misalignment of up- and down-spin Landau levels. The residual effects are (a) a small shift in the low-temperature critical field $[\Delta H_{c2}/H_{c2}\approx(k_F\xi_0)^{-1}]$ and (b) oscillatory structure in the transition temperature $T_{c2}(H)$ for temperatures $T\lesssim\hbar\omega_c/2\pi^2k$.

INTRODUCTION

RECENTLY Helfand and Werthamer[1] have developed an elegant method for solving the linearized gap equation for a superconductor in a magnetic field. This enabled them to obtain the temperature and mean-free-path dependence of the upper critical field of a type-II superconductor. The only assumption made was to treat the effects of the field semiclassically, that is, to assume that the only effect of the field on a particle moving from x_1 to x_2 is to multiply its wave function by the phase factor

$$\exp[(ie/\hbar c)\int_1^2 \mathbf{A}\cdot d\mathbf{s}].$$

The purpose of this paper is to investigate the validity of this approximation, and find corrections to it.

Quantum effects in a magnetic field are only important when an electron can complete many orbits before scattering, i.e., when $\omega_c\tau$ is sufficiently greater than 1. However, Gor'kov[2] argued that even when this condition is satisfied, quantum effects on H_{c2} should be small. He pointed out that at H_{c2} the center-of-mass wave function of a Cooper pair (i.e., the order parameter) is spread over a coherence length ξ_0, which is approximately equal to the orbit radius $r_c=(\hbar c/eH)^{1/2}$ of the lowest Landau level. This wave function is constructed from pairs of electrons near the Fermi level, and these electrons have a much larger radius of curvature, i.e., the cyclotron radius $k_Fr_c^2\approx k_F\xi_0^2$. Thus one can neglect this curvature with an accuracy of order $(k_F\xi_0)^{-1}$. This leads directly to the semiclassical approximation.

On the other hand, the quantization of the orbits leads to singularities in the single-electron density of states at energy values $\epsilon_n=(n+\frac{1}{2})\hbar\omega_c$, where $\omega_c\simeq eH/mc$. The density of states at the Fermi level is thus a periodic function of $1/H$. Since the critical temperature is dependent on the interaction between electrons near the Fermi level [and hence on the density of states $N(\epsilon_F)$], it too should be an oscillatory function of $1/H$. It was with this idea in mind that we began this work.

In an earlier paper,[3] we made use of the method of Sondheimer and Wilson[4] and determined the quantum corrections to the kernel of the linearized gap equation. Such corrections were also derived by Rajagopal and Vasudevan,[5] but these authors then made an approximation that is valid only near $H=0$, $T=T_{c0}$, in which case quantum effects are negligible. Their method relies on an expansion of the single-particle Green's function in terms of harmonic-oscillator wave functions. Since this approach lends itself more directly to a physical picture of what is going on, we will employ it here. In Appendix A, it is shown that the two methods lead to equivalent results.

In Sec. 1, we review the Helfand-Werthamer method and generalize it to take into account quantum effects. The kernel of the gap equation is found in Sec. 2, and a qualitative discussion of the effects of quantization is given. An explicit expression for $T_{c2}(H)$ is derived in Sec. 3. We include the effects of electron spin and nonmagnetic impurity scattering, but assume that the field is small enough that the normal-state Pauli paramagnetism is not an important factor in limiting the upper critical field.

* Supported by the Office of Naval Research.

† Supported by National Science Foundation Grant No. GP-4322.

[1] E. Helfand and N. R. Werthamer, Phys. Rev. **147**, 288 (1966).

[2] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **37**, 833 (1959) [English transl.: Soviet Phys.—JETP **14**, 628 (1962)].

[3] L. Gunther and L. W. Gruenberg, Solid State Commun. **4**, 329 (1966).

[4] E. H. Sondheimer and A. H. Wilson, Proc. Roy. Soc. (London) **A210**, 173 (1951).

[5] A. K. Rajagopal and R. Vasudevan, Phys. Letters **20**, 585 (1966); **23**, 539 (1966).

1. QUANTUM MODIFICATIONS OF THE GAP EQUATION

The transition temperature $T_{c2}(H)$ of a pure type-II superconductor in a uniform applied magnetic field H is determined by the linearized gap equation

$$\Delta(r)=\int K(r,r')\Delta(r')d^3r', \tag{1.1}$$

where the kernel K is given by

$$K(r,r')=VkT_{c2}\sum_{\nu=-\infty}^{\infty}G_{\omega_\nu-}(\mathbf{r},\mathbf{r}')G_{-\omega_\nu+}(\mathbf{r},\mathbf{r}'), \tag{1.2}$$

$G_{\omega\pm}$ is the thermal Green's function for an electron in the normal state whose z component of spin is $\pm\frac{1}{2}$, and $\omega_\nu=\pi kT(2\nu+1)$.

In the presence of a magnetic field, G has the following form (using units in which $\hbar=c=1$):

$$G_{\pm|\omega|\sigma}(\mathbf{r},\mathbf{r}')=\exp\left(ie\int_{r'}^{r}\mathbf{A}\cdot d\mathbf{s}\right)\tilde{G}_{\pm|\omega|\sigma}(\mathbf{r}-\mathbf{r}'). \tag{1.3}$$

The function $\tilde{G}_{\pm\omega}$ depends only on the relative coordinate $(\mathbf{r}-\mathbf{r}')$.[3] This is a result of the invariance of the Schröedinger equation to a simultaneous translation and gauge transformation.

The semiclassical approximation results from replacing $\tilde{G}_{\pm|\omega|\sigma}(\mathbf{R})$ by $G_{\pm|\omega|}{}^0(R)\exp(\pm i\mu_0HR/V_F)$, where G^0 is the Green's function in the absence of a magnetic field:

$$G_{\pm|\omega|}{}^0(R)=(-m/2\pi R)\exp(\pm ik_FR-|\omega|R/V_F). \tag{1.4}$$

Helfand and Werthamer[1] make this approximation and then show that the appropriate eigenfunction of Eq. (1.1) is the ground-state wave function for a particle with charge $2e$ moving in a magnetic field. They proceed to show that (1.1) can be reduced to a simple algebraic equation:

$$1=VkT_{c2}\sum_{\nu=0}^{\infty}S_{\omega_\nu}{}^0, \tag{1.5}$$

where

$$S_\omega{}^0=2\,\mathrm{Re}\int d^3R\,G_\omega{}^0(R)G_{-\omega}{}^0(R)$$
$$\times\exp[(2i\mu_0HR/V_F)-\tfrac{1}{2}eH\rho^2],\qquad \rho^2=x^2+y^2. \tag{1.6}$$

In order to take into account impurity scattering, it is necessary to replace the kernel K of Eq. (1.2) by its impurity configuration average. When this is done, an equation of the form (1.5) is found,[1] with $S_\omega{}^0$ replaced by

$$S_\omega=S_{(\omega+\Gamma)}{}^0/[1-\Gamma S_{(\omega+\Gamma)}{}^0/2\pi N(\epsilon_F)], \tag{1.7}$$

where $(2\Gamma)^{-1}=\tau$ is the collision time and $N(\epsilon_F)$ is the density of states at the Fermi level.

In order to arrive at Eqs. (1.5) and (1.6) it is not necessary to know the exact form of G^0. It is only necessary to assume that except for the semiclassical phase factor

$$\exp\left(ie\int_{r'}^{r}\mathbf{A}\cdot d\mathbf{s}\right),$$

G depends only on the relative coordinate $(\mathbf{r}-\mathbf{r}')$. Since this is also a property of the exact Green's function, all the arguments leading to Eqs. (1.5) and (1.6) go through, except that we must replace S^0 by

$$\bar{S}_\omega{}^0=2\,\mathrm{Re}\int d^3R\,\tilde{G}_\omega(R)\tilde{G}_{-\omega}(R)\exp(-\tfrac{1}{2}eH\rho^2). \tag{1.8}$$

In order to take impurity scattering into account, it is necessary to find how $\tilde{G}_\omega$ is affected. We will assume that G_ω can be replaced by $\tilde{G}_{\omega+\Gamma}$. The validity of this approximztion has been discussed by Dworin.[6] We will neglect the vertex corrections that lead to the denominator of Eq. (1.7), since these are only important when $l/\xi_0\lesssim1$; the quantum effects we seek are only important when $\omega_c\tau\gtrsim1$ and this leads to the criterion [assuming $H_{c2}\approx hc/e\xi_0{}^2$]$(k_F\xi_0)^{-1}(l/\xi_0)\gtrsim1$, so that $l/\xi_0\gg1$. We thus find

$$1=VkT_{c2}\sum_{\nu=0}^{\infty}\bar{S}_{\omega_\nu},$$

$$\bar{S}_\omega=2\,\mathrm{Re}\int d^3R\,\tilde{G}_{\omega+\Gamma}(R)\tilde{G}_{-(\omega+\Gamma)}(R)\exp(-\tfrac{1}{2}eH\rho^2). \tag{1.9}$$

2. EVALUATION OF $\bar{S}_\omega$

We begin by neglecting scattering. A representation for the single-particle Green's function has been found by Rajagopal[7] and by Dworin.[6] The Green's function is expressed as a sum over harmonic-oscillator wave functions. After integrating over orbit centers, one obtains

$$\tilde{G}_{\omega\sigma}(R)=(eH/2\pi)\exp(-\tfrac{1}{2}t)\sum_{r=0}^{\infty}L_r(t)\int\frac{dk}{2\pi}$$
$$\times\exp(ikz)(i\omega-\bar{\epsilon}_{rk\sigma})^{-1}, \tag{2.1}$$

where $L_r(t)$ is the Laguerre polynomial of order r, $t=eH\rho^2/2$, and

$$\bar{\epsilon}_{rk\sigma}=(r+\tfrac{1}{2})\omega_c+k^2/2m+\mu_0\boldsymbol{\sigma}\cdot\mathbf{H}-\epsilon_F. \tag{2.2}$$

Substituting Eq. (2.1) into Eq. (1.9) and making use

[6] L. Dworin, Ann. Phys. (N.Y.) 38, 431 (1966).
[7] A. K. Rajagopal, Phys. Letters 5, 40 (1962).

of the relation[8]

$$\int_0^\infty e^{-2t}L_n(t)L_m(t)\,dt=\tfrac{1}{2}^{n+m+1}(m+n)!/(m!n!) \quad (2.3)$$

we obtain

$$\bar{S}_\omega{}^0=(eH/2\pi)\sum_{r,l=0}^{\infty}\tfrac{1}{2}^{r+l}\frac{(r+l)!}{r!l!}$$
$$\times\int\frac{dk}{2\pi}[(i\omega+\bar{\epsilon}_{rk+})(-i\omega+\bar{\epsilon}_{lk-})]^{-1}. \quad (2.4)$$

Finally, after making use of Eq. (1.9) and the identity

$$\tanh\tfrac{1}{2}(\beta\epsilon)=2kT\sum_{\nu-\infty}^{\infty}(-i\omega_\nu+\epsilon)^{-1}, \quad (2.5)$$

we find

$$1=(VeH/4\pi c)\sum_{r,l=0}^{\infty}\tfrac{1}{2}^{r+l}\frac{(r+l)!}{r!l!}$$
$$\times\int\frac{dk}{2\pi}[\tanh(\tfrac{1}{2}\beta\epsilon_{rk+})+\tanh(\tfrac{1}{2}\beta\epsilon_{lk-})]/(\epsilon_{rk+}+\epsilon_{lk-}). \quad (2.6)$$

This result was first obtained by Rajagopal *et al.*[5]

It will be shown later that the results of the semiclassical approximation can be obtained from Eq. (2.6) by replacing the sum over r and l by integrals and making a Gaussian approximation on the factorials. We can now discuss qualitatively the effects due to the discreteness of the Landau levels.

To begin with, we neglect spin and impurity scattering. There are two situations that we wish to consider:

(a) Suppose that the magnetic field is adjusted to make $(r+\frac{1}{2})\omega_c=\epsilon_F$ for some integer $r=n_0$. Then the term $r=l=n_0$ in the sum of Eq. (1.6) is proportional to

$$I_r=\int_0^{q_c}dq\,\tanh(q^2/4mkT)/q^2. \quad (2.7)$$

This integral diverges like $T^{-1/2}$ as $T\to 0$.

(b) Suppose now that the field does not satisfy the special criterion. Let us consider any diagonal term $r=l$ for which $\epsilon_r=(r+\frac{1}{2})\omega_c<\epsilon_F$. This term will be proportional to

$$I_r'=\int_0^{q_c}dq$$
$$\times\tanh\{[q^2-2m(\epsilon_F-\epsilon_r)]/4mkT\}/[q^2-2m(\epsilon_F-\epsilon_r)]. \quad (2.8)$$

This integral diverges like $\ln T$ as $T\to 0$.

These low-temperature divergences imply that at sufficiently low temperatures there is a stable superconducting state, no matter how strong the field. It is important to notice that these divergences occur as a result of the discreteness of the energy levels.

To proceed further, we will assume that $\omega_c\ll\epsilon_F$. The most important contributions to the sum over r and l will come from the region ϵ_r and $\epsilon_l\approx\epsilon_F$. Thus r and l are quite large and we can make use of the Gaussian approximation

$$\tfrac{1}{2}^{r+l}[(r+l)!/r!l!]\approx(r\pi)^{-1/2}\exp[-(r-l)^2/4r]. \quad (2.9)$$

Next we make use of the Poisson sum formula and Eqs. (2.4) and (2.9), which enable us to write

$$\bar{S}_\omega{}^0=\sum_{n,m=-\infty}^{\infty}S_{0\omega}{}^{nm}, \quad (2.10)$$

$$S_{0\omega}{}^{nm}\approx(-1)^{n+m}(eH/2\pi)\int_0^\infty dx\int_0^\infty dy$$
$$\exp[2\pi i(nx-my)-(x-y)^2/4x](x\pi)^{-1/2}$$
$$\times\int\frac{dk}{2\pi}[(i\omega+\bar{\epsilon}_{kx+})(-i\omega+\bar{\epsilon}_{ky-})]^{-1},$$

where

$$\epsilon_{kx\pm}=x\omega_c+k^2/2m\pm\mu_0H-\epsilon_F. \quad (2.11)$$

There are three types of terms in Eq. (2.10) that we will now consider separately.

(a) $n=m=0$: The energies $\omega_c x$ and $\omega_c y$ may be regarded as components of the kinetic energy. To make this explicit, we introduce new variables q and q' which are defined by the relations $\omega_c x=q^2/2m$ and $\omega_c y=q'^2/2m$. Since q and q' are both of order k_F, which is much larger than $(r_c)^{-1}$, we can approximate the exponent in Eq. (2.10) by

$$(x-y)^2/4x\approx(q-q')^2c/2eH. \quad (2.12)$$

It is convenient to regard q and q' as vectors. We introduce new variables θ and θ' and write

$$\int d(q^2)\,d(q'^2)\,f(q,q')=\pi^{-2}\int d^2q\,d^2q'\,f(q,q'). \quad (2.13)$$

We also take note of the following approximate relation:

$$\exp[-(q-q')^2/2eH]=(qq'/2eH)^{1/2}$$
$$\times\int_0^{2\pi}d\theta_{qq'}\exp[-(\mathbf{q}-\mathbf{q}')^2/2eH]. \quad (2.14)$$

Finally, we regard $(\mathbf{q},k)$ as a three vector and change variables from $\mathbf{q}'$ to $\mathbf{Q}=\mathbf{q}-\mathbf{q}'$. We then find

$$S_{0\omega}{}^{\infty}\approx(4\pi/eH)\int d^3k\,d^2Q\,(2\pi)^{-2}$$
$$\exp(-Q^2/2eH)[(i\omega+\bar{\epsilon}_{k+})(-i\omega+\bar{\epsilon}_{k-}-\mathbf{k}\cdot\mathbf{Q}/m)], \quad (2.15)$$

[8] J. S. Gradsteyn and J. M. Ryshik, *Table of Integrals, Series and Products* (Academic Press Inc., New York, 1965), p. 844.

where $\bar{\epsilon}_{k\pm}=k^2/2m\pm\mu_0 H-\epsilon_F$. In order to obtain the last factor on the right, we have dropped a term $Q^2/2m$ that is small compared to $\mathbf{k}\cdot\mathbf{Q}/m$ everywhere except in a very small angular region since $Q\approx r_c^{-1}$ and $k\approx k_F$.

The $\mathbf{k}$ integral can now be changed to an integral over energy and angle. After doing the integral over ϵ_k, we find

$$S_{0\omega}^{\infty}=(N_0/eH)\int d^2Q\,dz\;\exp(-Q^2/2eH)/[2\omega+i(2\mu_0H+QV_Fz)],\quad(2.16)$$

where N_0 is the density of states at the Fermi level. This term is the same as the semiclassical solution as is shown in Appendix A.

(b) $n=m$, $n\neq0$: Once again we go through the steps leading to Eq. (2.16). This time we obtain

$$S_{0\omega}^{nn}=(N_0/eH)\int d^2Q\,dz\times\exp[-(Q^2/2eH)+2\pi niQV_Fz/\omega_c]\times[2\omega+i(2\mu_0H+QV_Fz)]^{-1}.\quad(2.17)$$

We do the z integration by closing the contour as shown in Fig. 1. The integrals along the vertical lines are proportional to $\exp(\pm2\pi inQV_F/\omega_c)$ and, when the Q integral is done, the result is proportional to $\exp[-(k_Fr_c)^2]$ and can be neglected. The integral from A to B vanishes when this path is moved up to infinity. Thus, we need only find the residue at $z=(-2\mu_0H+2i\omega)/QV_F$. Then the Q integral can be done, and we find

$$S_{0\omega}^{nn}=2N_0(\pi^5/\omega_c\epsilon_F)^{1/2}\mathrm{Re}\times\exp[-4\pi|n|(\omega+i\mu_0H)/\omega_c],\qquad n>0$$
$$S_{0\omega}^{nn}=0,\qquad n<0.\quad(2.18)$$

(c) $n\neq m$: We first evaluate the integral in Eq. (2.10) for nonzero values of n and m. The main contribution to $S_{0\omega}^{nm}$ comes from the region of integration around $\omega_cx\approx\omega_cy\approx\epsilon_F$. We can thus approximate $(x\omega_c)^{-1/2}\exp[-(x-y)^2/4x]$ by $\epsilon_F^{-1/2}$. The remaining

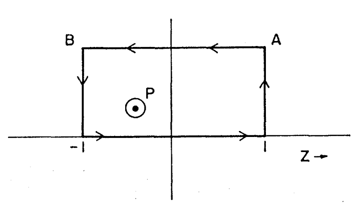

FIG. 1. Contour used in evaluating the z integral of Eq. (2.18). There is a pole at P where $z=(-2\mu_0H+2i\omega)/QV_F$.

integration is quite straightforward and we obtain for n and m greater than zero

$$S_{0\omega}^{nm}+S_{0\omega}^{mn}=\mathrm{Re}(-1)^{n+m}N_0(2\pi)^{3/2}\epsilon_F^{-1}(|n-m|)^{-1/2}\times\exp[-2\pi(n+m)(\omega+i\mu_0H)/\omega_c]\times\cos[2\pi(n-m)(\epsilon_F/\omega_c)-\tfrac{1}{4}\pi];\quad(2.19)$$

while S vanishes for n or m less than zero.

Finally, we can show that $S_{0\omega}^{n0}$ and $S_{0\omega}^{0n}$ are negligibly small. The important region of integration in Eq. (2.10) for $S_{0\omega}^{n0}$ is around $x\omega_c\approx\epsilon_F$, but there is no restriction on y. After doing the x integration, we are left with an integral of the form

$$I=\int\frac{dE}{2\omega+iE}\exp[2\pi in(E/\omega_c)-E^2/4\epsilon_F\omega_c],\quad(2.20)$$

which is proportional to $\exp[-(k_Fr_c)^2]$ on account of the oscillatory factor.

We note that we can include the effects of impurity scattering by replacing ω by $\omega+\Gamma$ on the right-hand side of Eqs. (2.16), (2.18), and (2.19).

3. DETERMINATION OF $T_{c2}(H)$

We now proceed to discuss the corrections to the semiclassical results. These will only be important at sufficiently low temperatures, i.e., when the damping factor $\exp(-2\pi^2kT/\omega_c)$ is not too small. We can thus evaluate the semiclassical contribution to the gap equation in the low-temperature limit. In Appendix A it is shown that when the paramagnetic effect on H_{c20} is small, the semiclassical contribution is

$$\ln(2\omega_0/\Delta_0)-(kT/N_0)\sum_{\nu=0}^{\infty}S_{0\omega_\nu}^{00}=\ln(H/H_{c20})+a(T_{c2}/T_{c0})^2,\quad(3.1)$$

where H_{c20} is the $T=0$ critical field, T_{c0} is the critical temperature in the absence of field and a is a constant of order unity that is given by Eq. (A5).

Making use of Eqs. (2.18), (2.19), and (3.1), we can write the equation for the transition temperature as follows:

$$\ln(H/H_{c20})=-a(T/T_{c0})^2+2\pi^{3/2}(\omega_c/\epsilon_F)^{1/2}S_1+2(2\pi)^{1/2}(\omega_c/\epsilon_F)S_2,\quad(3.2)$$

where

$$S_1=2\pi(kT/\omega_c)\sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\exp(-4\pi n\tilde{\omega}_\nu/\omega_c),$$

$$S_2=2\pi(kT/\omega_c)\sum_{n=1}^{\infty}\sum_{m>n}^{\infty}\sum_{\nu=0}^{\infty}(-1)^{n+m}\times\exp[-2\pi(n+m)\tilde{\omega}_\nu/\omega_c]\times\cos[2\pi|n-m|(\epsilon_F/\omega_c)-\tfrac{1}{4}\pi](|n-m|)^{-1/2},\quad(3.3)$$

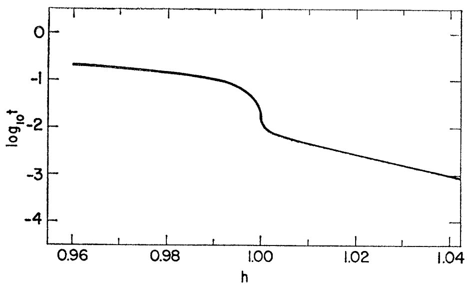

FIG. 2. $t=T_{c2}/T_{c0}$ as a function of $h=H/H_{c20}$. $\omega_c\tau$ is assumed to be infinite and the spin-splitting parameter η is equal to zero. The oscillatory fine structure is not shown here. The parameter a of Eq. (3.2) is taken to be 1.0 and $k_F\xi_0=53$.

with

$$\tilde{\omega}_\nu=(2\nu+1)\pi kT+\Gamma+i\mu_0H. \qquad (3.4)$$

S_1 and S_2 represent the quantum corrections to the critical-temperature–versus–field relation. Their properties are discussed in Appendix C.

First we will neglect effects due to electron spin and impurity scattering. S_1 then depends only on the ratio $x=2\pi^2kT/\omega_c$. It has the following asymptotic behavior:

$$S_1=(x/2\pi)e^{-2x}, \qquad x\gg 1$$

$$S_1=(\gamma-\ln x)/4\pi, \qquad x\ll 1 \qquad (3.5)$$

where γ is Euler's constant. For large x, S_1 is smaller than the T^2 term in Eq. (3.2). However, for $x\lesssim 1$, S_1 dominates. Also as a first approximation we can neglect the oscillatory term S_2 since there is an extra factor $(\omega_c/\epsilon_F)^{1/2}$ multiplying it in Eq. (3.2). Thus, for $x\lesssim 1$, we find

$$\ln(H/H_{c20})=2\pi^{3/2}(\omega_c/\epsilon_F)^{1/2}S_1. \qquad (3.6)$$

For $x\ll 1$ we can solve for T_c with the aid of Eq. (3.5). The result is

$$kT_{c2}=0.09\omega_c(H/H_{c20})^{-1.1(\epsilon_F/\omega_c)^{1/2}}. \qquad (3.7)$$

Thus superconductivity can exist at arbitrarily large fields, but the transition temperature is exponentially small when $H>H_{c20}$. This behavior is a consequence of the discreteness of the energy levels. Our result is similar to that found by Shapoval[9] for the critical temperature of a very thin film in a parallel magnetic field.

The deviation of T_{c2} from its semiclassical behavior is shown in Fig. 2. An interesting feature of (3.7) is that T_{c2} goes through a minimum when $H\approx 7.3H_{c20}$ and then *increases* with increasing field. This can be understood as follows: From Eqs. (2.6) and (2.9) it can be seen that an electron in state $(n, k_z, \uparrow)$ where $n\approx\epsilon_F/\omega_c$ is paired with electrons in states $(m, -k_z, \downarrow)$ providing $|n-m|\lesssim n^{1/2}=(\epsilon_F/\omega_c)^{1/2}\approx k_Fr_c$. Thus the number Δn of Landau levels involved in the pairing decreases with increasing H. The term that leads to the low-temperature divergence (and hence to superconducting pairing in very high fields) is the one for which $n=m$. It becomes relatively more important as Δn decreases. According to Eq. (3.7), this effect dominates when $H>7.3H_{c20}$.

The transition temperature remains vanishingly small until the field is increased to make $\omega_c\approx\epsilon_F$. This situation may be achieved in a semiconducting superconductor. However, the validity of Eq. (3.7) depends on the Stirling approximation so our result is not applicable. Moreover, it is unreasonable to use the phenomenological Bardeen-Cooper-Schrieffer (BCS) electron-electron interaction when the field is so large. We will, therefore, not pursue this question any further.

The above results are drastically modified by impurity scattering. The Landau levels are broadened and this destroys the superconducting correlations for large H. To see this, we note that S_1 does not diverge at $T=0$ unless $\omega_c\tau=\infty$. In Appendix A it is shown that

$$S_1(T=0)=(4\pi)^{-1}\ln(\omega_c\tau/2\pi), \qquad (3.8)$$

providing $\omega_c\tau\gg 1$. Using Eqs. (3.2) and (3.8) we find the maximum field H_m for which superconductivity can exist:

$$(H_m-H_{c20})/H_{c20}=(\pi\omega_c/4\epsilon_F)^{1/2}\ln(\omega_c\tau/2\pi). \qquad (3.9)$$

Thus, even for very pure materials ($\omega_c\tau=10^3$), the maximum critical field does not differ considerably from H_{c20}.

Electron spin can also have a drastic effect. First, we note that if $2\mu_0H/\omega_c$ is an integer, S_1 is not affected by spin. When this condition is satisfied, there is spin degeneracy that facilitates pairing. We set

$$\mu_0H/\omega_c=\eta+\tfrac{1}{2}q, \qquad q=0, 1, 2\cdots; \qquad |\eta|<\tfrac{1}{2}. \qquad (3.10)$$

S_1 diverges at $T=0$ only if $\eta=0$. It is shown in Appendix

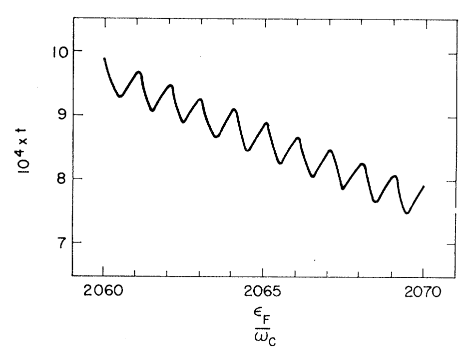

FIG. 3. Sketch of t versus ϵ_F/ω_c for $(2\pi^2kT/\omega_c)\approx 0.3$ and $(H/H_{c20})\approx 1.04$. The amplitude of the oscillations has been estimated with the aid of Eq. (3.2). Impurity scattering and spin splitting are neglected. We have taken $k_F\xi_0=53$.

[9] E. A. Shapoval, Zh. Eksperim. i Teor. Fiz. **51**, 669 (1966) [English transl.: Soviet Phys.—JETP **24**, 43 (1967)]

C that

$$S_1(T=0) = -(4\pi)^{-1}\ln(4\pi\eta), \qquad |\eta|\ll\tfrac{1}{2}. \quad (3.11)$$

By following the arguments that led to Eq. (3.8) we find

$$(H_m - H_{c20})/H_{c20} = (\pi\omega_c/4\epsilon_F)^{1/2}\ln(4\pi\eta)^{-1}. \quad (3.12)$$

Thus, even a small amount of spin misalignment will suppress superconductivity for H much greater than H_{c20}.

Next, let us consider the oscillatory term S_2. Its effect is negligible unless $x<1$. Unfortunately, in the region $x\approx 1$, where the oscillations are first observable, it is very difficult to evaluate S_2. A sketch of the behavior of $T_{c2}(H)$ in the vicinity of $x=0.3$ is shown in Fig. 3.

The behavior of S_2 when $x\ll 1$ is discussed in Appendix C. We let $2\pi\epsilon_F/\omega_c = n+\frac{1}{2}+\delta$, with $|\delta|\leq\frac{1}{2}$. S_2 is very sharply peaked about $\delta=0$, where it is given by

$$S_2 = (\pi x)^{-1/2}(1-2^{-3/2})\zeta(\tfrac{3}{2}). \quad (3.13)$$

Furthermore, when $x\lesssim 36\omega_c/\epsilon_F$ the $S_2(\delta=0)$ term in Eq. (3.2) dominates and T_{c2} is then a very sharply oscillatory function of H^{-1},

$$kT_{c2} = 0.14\omega_c[\omega_c/\epsilon_F \ln(H/H_{c20})]^2, \qquad \delta\approx 0. \quad (3.14)$$

For $\delta\neq 0$, T_{c2} is given by Eq. (3.7). This behavior is illustrated in Fig. 4.

Impurity scattering will have a drastic effect on these oscillations. The $T=0$ divergence of S_2 is removed. When $\delta=0$, S_2 is of order $(\omega_c\tau)^{1/2}$. In order to observe the behavior shown in Fig. 4, it is necessary that the S_2 and S_1 terms in Eq. (3.2) be comparable. This will happen only if $\omega_c\tau > (\epsilon_F/\omega_c)[\ln(\omega_c\tau)]^2$. As long as $\omega_c\tau>1$, the behavior shown in Fig. 3 should be observable.

Spin misalignment can also suppress the strong oscillations. For $\delta=0$, $S_2(T=0)$ is proportional to $\eta^{-1/2}$. The strong oscillations will not appear unless $(\ln\eta)^2 < \omega_c/\epsilon_F$.

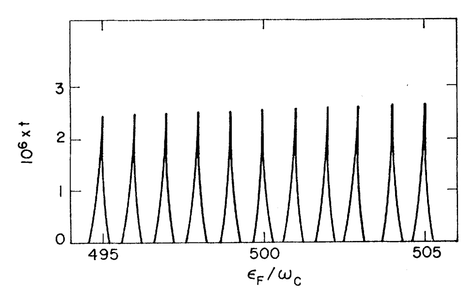

FIG. 4. Sketch of t versus ϵ_F/ω_c for $(H/H_{c20})\approx 40$. Impurity scattering and spin splitting are neglected. We have taken $k_F\xi_0=53$.

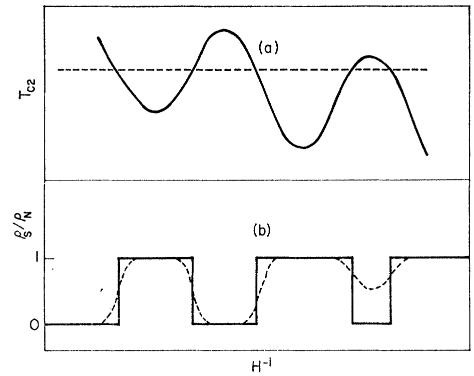

FIG. 5. (a) Sketch of T_{c2} versus H^{-1}. (b) Sketch of the resistivity ρ_s of the superconductor divided by the normal-state resistivity ρ_N as a function of H^{-1}. The temperature is held fixed as indicated by the dashed line in (a). The solid curve results when the transition is perfectly sharp; the dotted curve is obtained when the transition is broadened.

4. CONCLUSIONS

The most striking feature of our results is the prediction of superconductivity for fields $H\gg H_{c20}$. Due to the discreteness of the Landau levels, an "ideal" superconductor (i.e., one in which the collision time $\tau=\infty$ and the up- and down-spin Landau levels are degenerate) will be superconducting in arbitrarily high fields at zero temperature. The transition temperature is very small $[kT_c\leq\omega_c/(k_Fr_c)^4]$ and is a sharply peaked oscillating function of H^{-1}. However, this state is of academic interest only, since it is destroyed by a minute amount of impurity scattering or spin misalignment.

The residual effects are a small shift in the zero temperature critical field $[\delta H_{c2}/H_{c2}$ of order $(k_F\xi_0)^{-1}]$ and a small oscillatory correction to the critical temperature $[\delta T_c/T_c$ of order $(k_F\xi_0)^{-1}]$. These oscillations appear only when $2\pi^2kT/\omega_c<1$ and $\omega_c\tau>1$. The most promising method of observing them is to look for oscillations in the resistance in the transition region as in the Little-Parks experiment.[10] A measurement of the resistance as a function of field at a fixed temperature would exhibit the behavior shown in Fig. 5. The number of anomalous peaks can be shown to be of order $(\omega_c/2\pi^2kT)^{1/2}$.

APPENDIX A

The semiclassical contribution to the gap equation can be written as

$$S_\omega^{00} = (m^2/2\pi^2)\ \mathrm{Re}\int d^3R\,[\exp(-2\tilde{\omega}R/V_F - \rho^2/2r_c^2)]/R^2, \quad (\mathrm{A1})$$

where $\tilde{\omega}=\omega+\Gamma+i\mu_0H$. We have neglected the vertex

[10] W. A. Little and R. D. Parks, Phys. Rev. Letters **9**, 9 (1962).

corrections because we are interested in the kernel only when $l \gg \xi_0$.

We follow Helfand and Werthamer and make a Fourier transform of $[\exp(-2\tilde{\omega}R/V_F)]/R^2$. After integrating over d^3R we obtain

$$S_{0\omega}{}^{00} = \frac{2\pi N_0}{eHV_F} \operatorname{Im} \int dQ \exp(-\tfrac{1}{2}(Qr_c)^2) \ln \frac{2\tilde{\omega}+iQV_F}{2\tilde{\omega}-iQV_F}. \tag{A2}$$

We need to evaluate

$$VkT \sum_{\nu=0}^{\infty} S_{\omega_\nu}{}^{00}.$$

At low temperatures, we may use the relation

$$2\pi kT \sum_\nu F(\omega_\nu) = \int_0^\infty d\omega\, F(\omega) + \tfrac{1}{6}\pi^2 (kT)^2 \left(\frac{\partial F}{\partial \omega}\right)_{\omega=0}. \tag{A3}$$

We will assume that $l \gg \xi_0$ and $\mu_0 H \ll$ the BCS gap Δ. We can then neglect $\mu_0 H$ and Γ in the first term on the right-hand side of Eq. (A3). However, the last term diverges when $\mu_0 H = \Gamma = 0$. For'kov[2] has shown that this term has the form $T \ln T$ in this case. We therefore keep $\Gamma + i\mu_0 H$ in the T^2 term. We then obtain

$$S \equiv kT/N_0 \sum_{\nu=0}^{\infty} S_{0\omega_\nu}{}^{00} = 1 + \tfrac{1}{2}\gamma + \ln(2\omega_c r_c/\sqrt{2}V_F) + \tfrac{1}{6}(\pi kTr_c/V_F)^2 \operatorname{Re}[\exp(z)E_1(z)], \tag{A4}$$

where γ is Euler's constant, $E_1(z)$ is the exponential integral, and $z = -\frac{1}{2}(\mu_0 H - i\Gamma)^2 (r_c/V_F)^2$. Using the asymptotic expansion of $E_1(z)$ for small z, we obtain

$$\ln(2\omega_0/\Delta) - S = \tfrac{1}{2}[\ln(H/H_{c20}) + a(T/T_{c0})^2], \tag{A5}$$

where

$$H_{c20} = 6.7\Delta^2 c/\hbar V_F{}^2 e,$$
$$a = 0.34(1 + \tfrac{1}{2}\ln[2/\{(H/H_P)^2 + \tfrac{1}{2}\pi^2(\xi_0/l)^2\}]),$$
$$H_P = \Delta/\sqrt{2}\mu_0. \tag{A6}$$

APPENDIX B

In this Appendix we will compare the expression for the coefficients $S_{0\omega}{}^{nm}$ calculated in Sec. 2 with those obtained earlier by us[3] by a different approach.

We recall that the Green's function $G_{\pm|\omega|\sigma}(r, r')$ is related to the single-particle density matrix

$$\psi_\sigma(\mathbf{r}, \mathbf{r}'; \beta) = \sum_{n,\sigma} \phi_{n\sigma}(\mathbf{r}) \phi_{n\sigma}{}^*(\mathbf{r}') \exp(-\beta\epsilon_{n\sigma})$$

by a Laplace transformation:

$$G_{\pm|\omega|\sigma}(\mathbf{r}, \mathbf{r}') = \mp i \int dt \exp[(\pm i\epsilon_F - |\omega|)t] \psi_\sigma(r, r'; \pm it). \tag{B1}$$

An exact expression for ψ_σ has been found by Sondheimer and Wilson[4]:

$$\psi_\sigma(\mathbf{r}, \mathbf{r}'; \beta) = \exp(ie \int_{r'}^{r} \mathbf{A}(\mathbf{s}) \cdot d\mathbf{s}) \tilde{\psi}_\sigma(\mathbf{r} - \mathbf{r}'; \beta),$$
$$\tilde{\psi}_\sigma(\mathbf{R}; \beta) = (m/2\pi\hbar^2\beta)^{3/2} (\beta\omega_c/\sinh\beta\omega_c) \times \exp[-\tfrac{1}{4}m\omega_c\rho^2 \coth(\tfrac{1}{2}\beta\omega_c) - (mz^2/2\beta) - \beta\mu_0(\boldsymbol{\sigma}\cdot\mathbf{H})R]. \tag{B2}$$

The integral in Eq. (B1) can be evaluated by means of the method of stationary phase. Providing $k_F{}^{-1} < \rho < k_F r_c{}^2$ and $z < 2n\pi k_F r_c{}^2$, this leads to the following expression for $\tilde{G}$, the homogeneous part of G:

$$\tilde{G}_{\pm|\omega|\sigma}(R) = \sum_{n=0}^{\infty} \tilde{G}_{\pm|\omega|\sigma}{}^n(R),$$
$$\tilde{G}_{\pm|\omega|\sigma}{}^0(R) = -(m/2\pi R) \exp[\pm i(k_F - \mu_0\boldsymbol{\sigma}\cdot\mathbf{H}/V_F)R],$$
$$\tilde{G}_{\pm|\omega|\sigma}{}^n(R) = \pm(-1)^{n+1}(im/r_c)(2\pi^3 k_F \rho n)^{-1/2} \times \cos(k_F\rho - \tfrac{1}{4}\pi) \exp\{\pm i[z^2/4n\pi r_c{}^2 + (2\pi n\epsilon_F/\omega_c) - \tfrac{1}{4}\pi] - (2\pi n/\omega_c)(|\omega| \pm i\mu_0\boldsymbol{\sigma}\cdot\mathbf{H})\}. \tag{B3}$$

Making use of Eq. (1.8) we can now evaluate $\bar{S}^0$:

$$\bar{S}_\omega{}^0 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} S_{0\omega}{}^{nm},$$
$$S_{0\omega}{}^{nm} = 2 \operatorname{Re} \int d^3R\, \tilde{G}_{\omega+}{}^n(R) \tilde{G}_{-\omega-}{}^m(R) \exp(-\rho^2/2r_c{}^2). \tag{B4}$$

The expression for $S_{0\omega}{}^{00}$ is the same as that given by Eq. (1.6) for $S_\omega{}^0$, while Eq. (2.19) is obtained when $n \neq m$ by integrating in Eq. (B4). Also, $S_{0\omega}{}^{0n}$ and $S_{0\omega}{}^{n0}$ are vanishingly small, as shown by us previously.[3] However the z integration in Eq. (B4) for $S_{0\omega}{}^{nn}$ appears to diverge at large z. The reason for this is that the expression we have found for $\tilde{G} = \sum \tilde{G}^n$ is valid only for $|z| < 2n\pi k_F r_c{}^2$. When $|z|$ is larger, the corresponding stationary point in Eq. (B1), moves to the imaginary t axis, and $\tilde{G}^n$ no longer contributes to $\tilde{G}$. We therefore cut off the z integration in Eq. (B4) and obtain

$$S_{0\omega}{}^{nn} \approx 8\pi^{3/2} N_0 (\epsilon_F \omega_c)^{-1/2} \exp[-4\pi n(\omega + i\mu_0 H)/\omega_c]. \tag{B5}$$

This approximate result differs from Eq. (2.18) by a factor of $4/\pi$.

APPENDIX C

In this Appendix, the low-temperature behavior of S_1 and S_2 is discussed. We first note that if $2\mu_0 H/\omega_c$ is an integer, the electron spin has no effect on S_1. We write

$$\mu_0 H/\omega_c = \tfrac{1}{2}q + \eta, \qquad q = 0, 1, 2, \cdots, \qquad |\eta| \leq \tfrac{1}{2}. \quad \text{(C1)}$$

We will assume that $|\eta| \ll 1$ and $\Gamma \ll \omega_c$. After summing over n, S_1 becomes

$$S_1(T, \eta, \Gamma) = \mathrm{Re}(2\pi kT/\omega_c) \sum_{\nu=0}^{\infty} [\exp(4\pi\tilde{\omega}_\nu/\omega_c) - 1]^{-1}, \quad \text{(C2)}$$

where $\tilde{\omega} = \omega + \Gamma + i\eta$.

It was pointed out in Sec. 3 that S_1 diverges logarithmically as $T \to 0$ when $\Gamma = \eta = 0$. To obtain the leading terms when $\Gamma \ll \omega_c$ and $|\eta| \ll 1$, we subtract and add $\sum_\nu \omega_c/[4\pi\tilde{\omega}_\nu(1+2\pi\tilde{\omega}_\nu/\omega_c)]$. Since this and its derivative both have the same behavior when $\omega_\nu \to 0$ we can write

$$S_1 \approx \mathrm{Re}(4\pi)^{-1} \int_0^\infty dx\{[\exp\tilde{x} - 1]^{-1} - [\tilde{x}(1+\tfrac{1}{2}\tilde{x})]^{-1}\} + (\omega_c/16\pi^3 kT) \sum_{\nu=0}^{\infty} A_\nu,$$

$$A_\nu = \{[\nu + \tfrac{1}{2} + (\Gamma + i\eta\omega_c)/2\pi kT] \times [\nu + \tfrac{1}{2} + (\Gamma + i\eta\omega_c + \omega_c/2\pi)/2\pi kT]\}^{-1}, \quad \text{(C3)}$$

where

$$\tilde{x} = x + 4\pi(\Gamma + i\eta\omega_c)/\omega_c. \quad \text{(C4)}$$

This approximation includes all terms that do not vanish when $|\pi kT + \Gamma + i\eta\omega_c| \Rightarrow 0$. The integrals in (C3) are elementary while the sum can be expressed in terms of the digamma function. We find

$$S_1 = \mathrm{Re}\{\psi[\tfrac{1}{2} + (a + \omega_c/2\pi)/2\pi kT] - \psi(\tfrac{1}{2} + a/2\pi kT) - \ln 2\}/4\pi, \quad \text{(C5)}$$

where

$$a = \Gamma + i\eta\omega_c. \quad \text{(C6)}$$

When $a = 0$, this reduces to

$$S_1(T, 0, 0) \approx (4\pi)^{-1}[\gamma + \ln(\omega_c/2\pi^2 kT)], \quad \text{(C7)}$$

where γ is Euler's constant and $(\omega_c/2\pi^2 kT) \gg 1$. When $T = 0$, (C6) leads to

$$S_1 = (4\pi)^{-1} \ln(\omega_c/4\pi|a|). \quad \text{(C8)}$$

Next, let us consider S_2, which is given by

$$S_2 = \mathrm{Re}(\pi kT/\omega_c) \sum_{\nu=0}^{\infty} \sum_{\substack{n,m=1 \\ n\neq m}}^{\infty} (-1)^{n+m} \times \exp[-2\pi(n+m)\tilde{\omega}_\nu/\omega_c] \times \cos\{[2\pi(n-m)\epsilon_F/\omega_c] - \tfrac{1}{4}\pi\}\, |n-m|^{-1/2}. \quad \text{(C9)}$$

The summation variables may be changed to n and $r = n - m$. After performing the n sum, we obtain

$$S_2 = \mathrm{Re}(2\pi kT/\omega_c) \sum_{r=1}^{\infty} \sum_{\nu=0}^{\infty} (-1)^r r^{-1/2} \times \cos[(2\pi r\epsilon_F/\omega_c) - \tfrac{1}{4}\pi] \times \exp[-2\pi r\tilde{\omega}_\nu/\omega_c]\{\exp[(4\pi\tilde{\omega}_\nu/\omega_c) - 1]\}^{-1}. \quad \text{(C10)}$$

The r sum depends crucially on the ratio ϵ_F/ω_c and it is necessary to consider two situations separately:

(a) $\quad \epsilon_F/\omega_c = p + \tfrac{1}{2}$ for some integer p.

After approximating the last factor in Eq. (C10) by $\omega_c/4\pi\tilde{\omega}_\nu$ and the r sum by an integral we find

$$S_2 = \mathrm{Re}(\omega_c/16\pi^3 kT)^{1/2}\zeta(\tfrac{3}{2}, \tfrac{1}{2} + a/2\pi kT), \quad \text{(C11)}$$

where

$$\zeta(n; z) = \sum_{m=0}^{\infty} (m+z)^{-n}.$$

For $a = 0$, this reduces to

$$S_2(T, 0, 0) = (\omega_c/2\pi^3 kT)^{1/2}(1 - 2^{-3/2})\zeta(\tfrac{3}{2}). \quad \text{(C12)}$$

For $a \neq 0$, $T = 0$ we find

$$S_2(0, \eta, \Gamma) = \mathrm{Re}(2\pi a^{1/2})^{-1}. \quad \text{(C13)}$$

(b) $\quad \epsilon_F/\omega_c = p + \tfrac{1}{2} + \delta, \qquad 0 < |\delta| \leq \tfrac{1}{2}$.

We can approximate $\exp[-2\pi r\tilde{\omega}_\nu/\omega_c]$ by 1. This leads to

$$S_2 = S_1 F(\delta), \quad \text{(C14)}$$

where

$$F(\delta) = \sum_r r^{-1/2} \cos(2\pi r\delta - \tfrac{1}{4}\pi). \quad \text{(C15)}$$

This sum may be evaluated in two limiting cases:

$$F(\delta) = (2|\delta|)^{-1/2}, \qquad 0 < |\delta| \ll \tfrac{1}{2}$$

$$F(\delta) = 2^{-1/2}(1 - 2^{-1/2})\zeta(\tfrac{1}{2}), \qquad |\delta| = \tfrac{1}{2}. \quad \text{(C16)}$$

By comparing Eqs. (C8), (C12), and (C14) we see that S_2 is sharply peaked about $\delta = 0$ when $2\pi^2 kT/\omega_c \ll 1$. It is large when $|\delta| \lesssim x$.