Simple Microscopic Theory of Surface Plasmons

PETER J. FEIBELMAN*
Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette, France

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We give a simple microscopic description of the surface modes of a finite high-density electron liquid. The random-phase approximation is reduced to a second-order differential equation with a surface solution at the expected frequency. We discuss the excitation of the surface plasmons, and also describe the analogous modes of systems with a short-range interaction.

I. INTRODUCTION

THE phenomenon of surface plasma oscillation has been discussed semiclassically by Ferrell and Stern,[1] and by Ritchie,[2] and more recently has been treated microscopically by Fedders.[3] It is our purpose here to give a microscopic description of the surface plasmons, which is based on the random-phase approximation (RPA) as is the work of Fedders, but which is more general in that approximations are introduced at a later point, and which is considerably simpler, in that lengthy algebraic manipulations are completely avoided. The RPA is ultimately reduced to a second-order differential equation plus small corrections, and the surface plasmon is shown to be a solution with the expected frequency.

We discuss the excitation of surface plasmons through inelastic electron scattering. And finally we study the analogous surface mode for a system with a short-range interaction.

II. RANDOM-PHASE APPROXIMATION

The dynamics of the electron gas at high density is described exactly by the random-phase approximation, or time-dependent self-consistent field equations. The electrons move against an isotropic background of very heavy positively charged particles. And thus the energy of the system is given by

$$E=\int d^3x\, d^3x'\{[-(\nabla^2/2m)\,\delta(x-x')]\rho(xx') + (e^2/2\,|\,x-x'\,|)[n(x)-n_B(x)][n(x')-n_B(x')]\},\quad (1)$$

where $\rho(xx')$ is the single-electron density matrix, $n(x)\equiv\rho(xx)$ is the average electron density, and $n_B(x)/Z$ is the density of ions, Z being their charge. Exchange terms have been neglected in writing Eq. (1). The charge neutrality of the system implies that

$$\int d^3x[n(x)-n_B(x)]=0.$$

The single-electron Hamiltonian is defined by $h(xx')=\delta E/\delta\rho(xx')$. Thus we have from (1),

$$h(xx')=-(\nabla^2/2m)+\int d^3x''\,(e^2/|\,x-x''\,|)\times[n(x'')-n_B(x'')]\delta(x-x').\quad (2)$$

The self-consistent field equation is then

$$i[\partial\rho(xx')/\partial t]=\int dx''\times[h(xx'')\rho(x''x')-\rho(xx'')h(x''x')],\quad (3)$$

or in matrix notation, $i\partial\rho/\partial t=[h,\,\rho]$. This equation describes both the ground state and the excitations of the electron gas. The ground state is static, and is therefore the solution to $[h,\rho]=0$. Thus in the ground state h and ρ may be simultaneously diagonalized. Let ψ_s satisfy

$$[-\nabla^2/2m+V_H(x)-\omega_s]\psi_s(x)=0,\quad (4)$$

where

$$V_H(x)=\int d^3x''\,(e^2/|\,x-x''\,|)[n^{(0)}(x'')-n_B(x'')],\quad (5)$$

and $n^{(0)}$ is the ground-state electron density. Clearly h is diagonal in the ψ_s representation if $n(x)=n^{(0)}(x)$. The ground-state density matrix is then simply

$$\rho^{(0)}(xx')=\sum_s\theta(\epsilon_F-\omega_s)\psi_s(x)\psi_s^*(x'),\quad (6)$$

which is diagonal in the s representation too. We have defined $\theta(x)=1$, if $x>0$, and $\theta(x)=0$, if $x\leq 0$. The Fermi energy ϵ_F is determined by the condition that the total number of electrons

$$N=\int d^3x\,\rho(xx).$$

Clearly $[h,\rho]=0$ is satisfied with the choice (6), for $\rho^{(0)}$, and E is minimized within the symmetry constraint that no more than one particle occupy each state.

Weakly excited modes of the system are described by linearizing Eq. (3). We have thus

$$i\dot{\rho}^{(1)}=[h^{(0)},\rho^{(1)}]+[h^{(1)},\rho^{(0)}],\quad (7)$$

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[1] E. A. Stern and R. A. Ferrell, Phys. Rev. **120**, 130 (1960).
[2] R. H. Ritchie, Phys. Rev. **106**, 874 (1957).
[3] P. A. Fedders, Phys. Rev. **153**, 438 (1967).

where

$$h^{(1)}(xx')=\left(\int d^3x''\frac{e^2}{|x-x''|}n^{(1)}(x'')\right)\delta(x-x'). \quad (8)$$

Fourier transforming the time variable and taking the matrix element of Eq. (7) between states ψ_s and $\psi_{s'}$, we reduce it to

$$\rho_{ss'}^{(1)}=[(\theta_{s'}-\theta_s)/(\omega-\omega_s+\omega_{s'})]h_{ss'}^{(1)}, \quad (9)$$

where $\theta_s\equiv\theta(\epsilon_F-\omega_s)$. But $h^{(1)}$ depends only on $\rho^{(1)}(xx)$, not on the full density matrix $\rho^{(1)}(xx')$, so Eq. (9) is conveniently expressed in coordinate space as

$$\begin{aligned}n^{(1)}(x,\omega)&\equiv\sum_{ss'}\psi_s(x)\psi_{s'}^*(x)\rho_{ss'}^{(1)}\\&=\sum_{ss'}(\theta_{s'}-\theta_s)/(\omega-\omega_s+\omega_{s'})\psi_s(x)\psi_{s'}^*(x)\\&\times\int d^3x'd^3x''\psi_s^*(x')\psi_{s'}(x')(e^2/|x'-x''|)n^{(1)}(x'',\omega),\end{aligned} \quad (10)$$

which is an integral equation in $\mathbf{x}$, for $n^{(1)}(\mathbf{x},\omega)$.

III. SLAB GEOMETRY

For definiteness we choose a slab geometry for our system. Then $n_B(\mathbf{x})=n_B(z)$, which implies through Eqs. (4)–(6) that $n^{(0)}(\mathbf{x})$, is also a function only of z, the coordinate perpendicular to the slab surfaces.

We see that

$$V_H(z)=e^2\int d^2x'dz'\{[(z-z')^2+x'^2]^{-1/2}-|x'|^{-1}\}\times[n^{(0)}(z')-n_B(z')], \quad (11)$$

where the $|x'|^{-1}$ term gives no contribution because of charge neutrality and has been subtracted just to make convergence in x' explicit.

The wave functions $\psi_s(\mathbf{x})$ may now be written as

$$\psi_{kn}(x,z)=\psi_n(z)e^{ikx}/2\pi,$$

where

$$[-2m^{-1}(d^2/dz^2)+V_H(z)-\omega_n]\psi_n(z)=0. \quad (12)$$

The corresponding energies are thus $\omega_{kn}=\omega_n+k^2/2m$, so Eq. (10) may be Fourier analyzed in the coordinate $\mathbf{x}$ parallel to the slab. We obtain

$$n^{(1)}(qz\omega)=\int\frac{d^2k}{(2\pi)^2}\sum_{nn'}\frac{\theta_{kn'}-\theta_{k+qn}}{\omega-\omega_{k+qn}+\omega_{kn'}}\psi_n(z)\psi_{n'}^*(z)\times\int dz'dz''\psi_n^*(z')\varphi_n(z')(2\pi e^2/q)e^{-q|z'-z''|}n^{(1)}(qz''\omega). \quad (13)$$

Or, defining the fluctuation potential as

$$\delta\varphi(qz\omega)=\int dz''(2\pi e^2/q)e^{-q|z-z''|}n^{(1)}(qz''\omega),$$

we may rewrite Eq. (13) as

$$\delta\varphi(qz\omega)=\frac{2\pi e^2}{q}\int\frac{d^2k}{(2\pi)^2}\sum_{nn'}\frac{\theta_{kn'}-\theta_{k+qn}}{\omega-\omega_{k+qn}+\omega_{kn'}}\times\int dz'e^{-q|z-z'|}\psi_n(z')\psi_{n'}^*(z')\times\int dz''\psi_n^*(z'')\psi_{n'}(z'')\delta\varphi(qz''\omega). \quad (14)$$

IV. SURFACE PLASMONS

To see the appearance of surface plasmons, we examine Eq. (14) in a high-frequency limit. That is, we assume that terms in the kernel with $(\omega_{k+qn}-\omega_{kn'})$ comparable to ω play little role, an assumption we check at the end, and keep only the leading terms in $1/\omega^2$.

With no approximation, Eq. (14) may be rewritten as

$$\delta\varphi(qz\omega)=\frac{2\pi e^2}{q\omega^2}\int\frac{d^2k}{(2\pi)^2}\sum_{nn'}(\theta_{kn'}-\theta_{k+qn})\times\{(\omega_{k+qn}-\omega_{kn'})+[(\omega_{kn'}-\omega_{k+qn})^3/(\omega_{kn'}-\omega_{k+qn})^2-\omega^2]\}\times\int dz'e^{-q|z-z'|}\psi_n(z')\psi_{n'}^*(z')\times\int dz''\psi_n^*(z'')\psi_{n'}(z'')\delta\varphi(qz''\omega). \quad (15)$$

In lowest order we drop the term $(\Delta\omega)^3/[(\Delta\omega)^2-\omega^2]$. But in the term we retain, the sums can be done explicitly using the completeness of the ψ_n. That is, noting that $\omega_{k+qn}-\omega_{kn'}=\omega_n-\omega_{n'}+(2kq+q^2)/2m$, we have

$$(q\omega^2/2\pi e^2)\delta\varphi(qz\omega)=\frac{q^2}{m}\int dz'e^{-q|z-z'|}n^{(0)}(z')\delta\varphi(qz'\omega)+(2m)^{-1}\int\frac{d^2k}{(2\pi)^2}\sum_{nn'}(\theta_{kn'}-\theta_{kn})\int dz'e^{-q|z-z'|}\psi_{n'}^*(z')\psi_n(z')\times\int dz''\left[\psi_n^*(z'')\frac{a^2\psi_{n'}}{dz''^2}-\frac{d^2\psi_n^*}{dz''^2}\psi_{n''}(z'')\right]\delta\varphi(qz''\omega), \quad (16)$$

where in the first term we have used the fact that

$$n^{(0)}(z)=\int\frac{d^2k}{(2\pi)^2}\sum_n\psi_n^*(z)\psi_n(z)\theta_{kn}, \quad (17)$$

and in the second term have made explicit use of Eq. (12) for the $\psi_n(z)$. This latter term of Eq. (16) is further reduced using completeness, integration by parts where necessary, and using Eq. (17) once again. We thus obtain simply

$$(m\omega^2 q/2\pi e^2)\,\delta\varphi(qz\omega) = \int dz' e^{-q|z-z'|}\left[n_0(z')\left(q^2-\frac{d^2}{dz'^2}\right)\delta\varphi(qz'\omega) - \frac{dn_0}{dz'}\frac{d}{dz'}\,\delta\varphi(qz'\omega)\right]. \quad (18)$$

Equation (18) may, however, still be reduced further. We note that

$$(d^2/dz^2)\,e^{-q|z-z'|} = q^2 e^{-q|z-z'|} - 2q\delta(z-z'). \quad (19)$$

Thus Eq. (18) is equivalent to the differential equation

$$[(m\omega^2/2\pi e^2) - 2n_0(z)](d^2/dz^2 - q^2)\,\delta\varphi(qz\omega) - 2(dn_0/dz)(d/dz)\,\delta\varphi(qz\omega) = 0, \quad (20)$$

together with boundary conditions, which may be satisfied implicitly by substituting solutions of Eq. (20) into (18) and requiring satisfaction of the latter equation.

We see immediately that if $n^{(0)}$ = constant, we get ordinary plasmons and $\omega^2 = \omega_p^2 = 4\pi e^2 n^{(0)}/m$. To find the surface plasmon solutions requires more work, however. Integrating by parts, it is easy to see that Eq. (18) reduces to

$$(m\omega^2 q/2\pi e^2)\,\delta\varphi(z) = \int dz' e^{-q|z-z'|} n_0(z') \times [q^2\delta\varphi(z') + q\,\mathrm{sgn}(z-z')\,(d/dz')\,\delta\varphi(z')]. \quad (21)$$

For simplicity suppose that

$$n_0(z) = n_\infty(-z), \quad (22)$$

i.e., that the electron density is uniform below $z=0$ and drops sharply to zero at the surface. This is a reasonable approximation if the behavior of $\delta\varphi$ right at the surface is not too important. We shall return to this point. Using Eq. (22) in (21) we see that for $z<0$,

$$(m\omega^2 q/2\pi e^2 n_\infty)\,\delta\varphi(z) = q^2 \int_{-\infty}^{0} dz' e^{-q|z-z'|}\delta\varphi(z') + q\int_{-\infty}^{0} e^{-q|z-z'|}\,\mathrm{sgn}(z-z')\,(d/dz')\,\delta\varphi(z'). \quad (23)$$

Integrating the last term by parts we obtain

$$(m\omega^2 q/2\pi e^2 n_\infty)\,\delta\varphi(z) = 2q\delta\varphi(z) - q\delta\varphi(0)\,e^{qz}, \quad (24)$$

so clearly $\delta\varphi(z) = \delta\varphi(0)\,e^{qz}$ and $\omega^2 = 2\pi n_\infty e^2/m = \frac{1}{2}\omega_p^2$, just as in the approximations of previous authors. For $z>0$ we see that $\delta\varphi(z) = \delta\varphi(0)\,e^{-qz}$, as expected.

For a more realistic density distribution $n^{(0)}(z)$, a closed-form solution to Eq. (18) has not been obtained. However, it would not be difficult to obtain a numerical solution using the equivalent differential equation (20). We note in passing that if $n^{(0)}(z)$ had been chosen to represent a slab of finite thickness rather than a semi-infinite one, we would have obtained from Eq. (18) the expected modification of the surface plasmon energy due to "interference of the surfaces." This is a straightforward exercise.

We also remark that the approximation of electron bound in a well with infinitely high walls at the edges, used by Fedders,[3] has been completely avoided in the present work. This approximation, which apparently leads to the correct surface plasmon energy, nevertheless suffers from an ambiguity, namely, whether or not one is to include the self-energy corrections in the particle propagators. In the present method this ambiguity is automatically resolved, and the "tedious labor" of the older method is avoided, besides.

V. LANDAU DAMPING OF SURFACE PLASMONS

In order to estimate the correction to $\delta\varphi(z)$ due to the term in Eq. (15) that was neglected, we must formulate a perturbation method. It is convenient to start from Eq. (13), which is completely equivalent to (15). Equation (13) may be written formally as

$$\mathcal{L}_{q\omega} f_q n^{(1)}(q\omega) = n^{(1)}(q\omega), \quad (25)$$

where

$$f_q(zz') = (2\pi e^2/q)\,e^{-q|z-z'|},$$

and $\mathcal{L}$ is a Hermitian kernel. In order to define a perturbation theory we write the auxiliary equation

$$\mathcal{L}_{q\omega} f_q u_l = \lambda_l(q\omega)\,u_l, \quad (26)$$

which reduces to Eq. (13) if $\lambda_l=1$. For fixed q and ω a complete set of u_l's can be defined; the condition $\lambda_l(q\omega)=1$ is then the dispersion relation for the lth mode.

The advantage of Eq. (26) over Eq. (25) is that in the former we readily see how to carry out a perturbation theory for λ, and can thus readily obtain successive approximations to the dispersion relations of the various modes. In particular, if $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}$, and $\mathcal{L}^{(0)} f u_l^{(0)} = \lambda_l^{(0)} u_l^{(0)}$, we have immediately that $\lambda_l = \lambda_l^{(0)} + \lambda_l^{(1)} + \cdots$, and

$$\lambda_l^{(1)} = \frac{\langle u_l^{(0)} \,|\, f\mathcal{L}^{(1)} f \,|\, u_l^{(0)}\rangle}{\langle u_l^{(0)} \,|\, f \,|\, u_l^{(0)}\rangle}. \quad (27)$$

This result depends on the hypothesis that the eigenfunctions of $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ are in one-to-one correspondence, which should certainly be true if $|\lambda_l^{(1)}| \ll |\lambda_l^{(0)}|$. In our case, of course, $\mathcal{L}^{(0)}$ corresponds to the piece of the kernel of Eq. (13) that was retained, and $\mathcal{L}^{(1)}$ corresponds to the rest.

In our formal notation let $v_l = f u_l$. The v_l satisfy $f\mathfrak{L} v_l = \lambda_l v_l$, which is the formal version of Eq. (14). In terms of the $v_l^{(0)}$ we have

$$\lambda_l^{(1)} = \frac{\langle v_l^{(0)} | \mathfrak{L}^{(1)} | v_l^{(0)} \rangle}{\langle v_l^{(0)} | f^{-1} | v_l^{(0)} \rangle}. \tag{28}$$

Note that

$$f_q^{-1}(zz') = (4\pi e^2)^{-1}[q^2 - (d^2/dz^2)]\delta(z-z'). \tag{29}$$

This ansatz satisfies the requirement that

$$\int dz' f_q^{-1}(zz') f_q(z'z'') = \delta(z-z''), \tag{30}$$

as can be seen using Eq. (19).

It is now straightforward to compute $\lambda^{(0)}$ for the surface plasmon, and to study the modification in its propagation due to $\mathfrak{L}^{(1)}$. Assuming the square density profile (22), we obtain $v_l = e^{-q|z|}$, and

$$\lambda^{(0)} = 2\pi n_\infty e^2/m\omega^2 \tag{31}$$

for the surface wave. (Note that the normalization constant

$$\langle v_l | f^{-1} | v_l \rangle = \int dz\, e^{-q|z|} (4\pi e^2)^{-1} [q^2 - (d^2/dz^2)] e^{-q|z|}$$
$$= q/2\pi e^2.)$$

We obtain also

$$\lambda^{(1)} = \frac{2\pi e^2}{q\omega^2} \int \frac{d^2k}{(2\pi)^2} \sum_{nn'} \frac{(\theta_{kn'} - \theta_{k+qn})(\omega_{kn'} - \omega_{k+qn})^3}{(\omega_{kn'} - \omega_{k+qn})^2 - \omega^2} \times \left| \int dz\, \psi_n(z) \psi_{n'}^*(z) e^{-q|z|} \right|^2. \tag{32}$$

This correction term is expected to be small for small q and large ω, on the ground that when $(\omega_n - \omega_{n'})^2$ is large enough to be comparable to ω^2, the matrix element $\int \psi_n \psi_{n'}^* e^{-q|z|}$ has fallen off considerably, and conversely when $(\omega_n - \omega_{n'})^2$ is small enough for the matrix element to be large, the denominator is also large because of the large ω^2, and the numerator, $(\omega_{kn'} - \omega_{k+qn})^3$, is small besides.

Because of our use of the sharp density profile, we expect to overestimate $\lambda^{(1)}$ somewhat. If $e^{-q|z|}$ were replaced by a smooth function of z, the matrix element would fall off considerably more rapidly as $(\omega_n - \omega_{n'})^2$ increases.

It is particularly interesting to evaluate $\mathrm{Im}\lambda^{(1)}$, which enables us to compute the Landau damping of the surface plasmon. This damping, small in the present case, completely dominates the dispersion relation for the low-frequency surface waves of semi-infinite liquid ^{3}He; and as a result their propagation is not observable.[4]

[4] P. J. Feibelman, Phys. Letters **26B**, 601 (1968); Ann. Phys. (N.Y.) **48**, 369 (1968).

We have

$$\mathrm{Im}\lambda^{(1)} = +(\pi^2 e^2/q) \int [d^2k/(2\pi)^2] \sum_{nn'} (\theta_{kn'} - \theta_{k+qn}) [\delta(\omega_{kn'} - \omega_{k+qn} - \omega) - \delta(\omega_{kn'} - \omega_{k+qn} + \omega)] \times \left| \int dz\, \psi_n(z) \psi_{n'}^*(z) e^{-q|z|} \right|^2. \tag{33}$$

The θ functions in Eq. (33) require either ψ_n or $\psi_{n'}$ or both to be bound-state wave functions. Suppose ψ_n is a bound state. In order that one of the δ functions contribute, for ω large, it is clear that $\psi_{n'}$ must be a state high in the continuum. Thus $\omega_{n'} \approx k_{n'}^2/2m$ and $\psi_{n'} \approx \exp(ik_{n'}z)/(2\pi)^{1/2}$, and the sum on n' is actually an integral. The matrix element must be treated with some care. Suppose $n \neq n'$; then

$$M_{nn'}{}^q = \int dz\, \psi_n^*(z) \psi_{n'}(z) e^{-q|z|}$$
$$= \lim_{\epsilon\to 0} \int dz\, \psi_n^*(z) \psi_{n'}(z) (e^{-q|z|} - e^{-\epsilon q|z|}),$$

which has an explicit limit, zero, as $q \to 0$. Now if n is a bound state, it cuts off the integral at about $z=0$. But if n' is high in the continuum the spatial dependence of ψ_n is much slower than that of $\psi_{n'}$. And thus we have

$$M_{nn'}{}^q \approx \lim_{\epsilon\to 0} \int_\infty^0 dz\, \exp(ik_{n'}z) (e^{qz} - e^{\epsilon qz})$$
$$= q/(k_{n'}^2 - iqk_{n'}) \approx q/k_{n'}^2. \tag{34}$$

We now substitute into Eq. (33). The sum on bound states is converted to an integral for a "semi-infinite" system. We obtain

$$\mathrm{Im}\lambda^{(1)} \approx + \frac{\pi^2 e^2}{q} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_{n'}}{2\pi} \times \left[-2\theta\left(\epsilon_F - \frac{k^2}{2m}\right) \delta\left(\frac{k_{n'}^2}{2m} - \omega\right) \right] \frac{q^2}{k_{n'}^4}$$
$$= (-\pi e^2 n_\infty q / 2^{5/2} m^{3/2} \omega^{5/2}). \tag{35}$$

Neglecting the real part of $\lambda^{(1)}$ we have

$$\lambda = (2\pi e^2 n_\infty / m\omega^2) \{1 - [i\pi q / 8(2m\omega)^{1/2}] + \cdots\}, \tag{36}$$

which yields the dispersion relation

$$\omega^2 = (\omega_p^2/2) \{1 - [i\pi q / 2^{13/4} (m\omega_p)^{1/2}] + \cdots\}. \tag{37}$$

Thus for long wavelengths the damping is rather small, but more important than the next real correction to ω^2 which varies as q^2.

VI. COUPLING TO SURFACE PLASMONS

The excitation of surface plasmons is particularly easy to treat within the formalism we developed to carry out perturbation theory.

In the notation of Eq. (25), the modification of the equation of motion due to the introduction of a density coupled external field is simply

$$\mathcal{L}_{q\omega}[f_q n^{(1)}(q\omega)+V_{\text{ext}}(q\omega)]=n^{(1)}(q\omega). \quad (38)$$

This equation is solved by resolving it into solutions of the auxiliary equation, (26). Thus,

$$n^{(1)}(q\omega)=\sum_l \frac{\mathfrak{N}_l^{-1}\lambda_l(q\omega)\,\langle u_l\mid V_{\text{ext}}(q\omega)\rangle}{1-\lambda_l(q\omega)}\mid u_l\rangle, \quad (39)$$

where $\mathfrak{N}_l$ normalizes $\mid u_l\rangle$. And the density-density response function is therefore

$$S(q\omega)=\sum_l \frac{\lambda_l(q\omega)\mathfrak{N}_l^{-1}}{1-\lambda_l(q\omega)}\mid u_l\rangle\langle u_l\mid. \quad (40)$$

In order to describe the inelastic scattering of high-energy electrons it is sufficient to use the lowest Born approximation, in which case the rate of scatter from incident momentum $\mathbf{k}$ into momentum $\mathbf{k}'+d\mathbf{k}'$ is given by

$$\mathcal{R}d\Omega_{k'}=(2\pi/\hbar)\,d\Omega_{k'}\int\frac{dk'^2}{(2\pi)^5}k'^2$$
$$\times\int d^3x_1d^3x_2d^3x_3d^3x_4\,\psi_{k'}{}^*(x_1)\psi_k(x_1)v(x_1-x_2)$$
$$\times R(x_2,x_3;\epsilon_k-\epsilon_{k'})v(x_3-x_4)\psi_k{}^*(x_4)\psi_{k'}(x_4), \quad (41)$$

where v is the Coulomb potential and

$$R(x,x';\epsilon)=-(1/\pi)\,\text{Im}S(xx';\epsilon+i\delta) \quad (42)$$

is the spectral function which is, as we see, given in term of the density response function. For close to normal incidence and not too large angular deflection ψ_k the electron-incoming and -outgoing states may be taken as undistorted waves. (For reflection experiments at close to grazing incidence, waves distorted by V_H must clearly be used, with careful attention to the fact that the S matrix is given by $\langle\psi_{k'}{}^-\mid\psi_k{}^+\rangle$.) And Eq. (41) may be reduced to

$$\mathcal{R}d\Omega=(2\pi A/\hbar)\,d\Omega\,\frac{2\pi e^2}{q}\int\frac{k'^2dk'}{(2\pi)^2}\left[\frac{2q}{(k_z-k_z')^2+q^2}\right]^2$$
$$\times\left(\frac{-1}{\pi}\right)\text{Im}\frac{1}{1-\lambda_{\text{surf}}(q\omega+i\delta)}, \quad (43)$$

where A is the area of the slab, and where only the surface plasmon contribution has been included in R. In the lowest approximation $\lambda_{\text{surf}}=2\pi e^2n_\infty/m(\omega+i\delta)^2$. Substituting this into Eq. (42) we obtain the differential rate of scattering into angle (θ,φ) as

$$u=(e^2/\pi\hbar v)[\theta\theta_E/(\theta^2+\theta_E{}^2)^2], \quad (44)$$

where $\theta_E=\hbar\omega_{\text{surf}}/2E$, E being the incident electron's energy, and v its velocity. This is just the result of Ferrell and Stern, obtained without the use of a semiclassical approximation.

VII. SYSTEMS WITH SHORT-RANGE INTERACTIONS

The low-energy modes of a system with an attractive short-range interaction at its surface have been treated elsewhere.[4] Here we discuss the high-energy modes of a system with a repulsive interaction at the surface, such as for example the nuclear system, in the isospin$=$1 channel.

We simply replace the Coulomb interaction in our equations by the Yukawa force of strength g^2 and range a. Thus now

$$f_q(zz')=2\pi g^2\exp[-(q^2+a^2)\mid z-z'\mid]/(q^2+a^2)^{1/2}. \quad (45)$$

For long wavelengths $q\ll a$ and may be neglected. Thus

$$f_q(zz')\xrightarrow[q/a\to 0]{}(2\pi g^2/a)\,e^{-a|z-z'|}. \quad (46)$$

The above manipulations may be repeated, and at high frequencies, Eq. (18) is simply replaced by

$$(m\omega^2a/2\pi g^2)\,\delta\varphi(qz\omega)=\int dz'e^{-a|z-z'|}$$
$$\times\left[n_0(z')\left(q^2-\frac{d^2}{dz'^2}\right)\delta\varphi(qz'\omega)-\frac{dn_0}{dz'}\frac{d}{dz'}\delta\varphi(qz'\omega)\right], \quad (47)$$

which is again equivalent to a differential equation. The assumption of the sharp density profile (22) (which may not be such a good approximation here because a^{-1} the force range is of the same order as the surface diffuseness) leads to a surface mode whose spatial form $q/a\ll 1$ is given by

$$\delta\varphi(qz\omega)=e^{-az}, \qquad z>0$$
$$=e^{+2az}, \qquad z\leq 0. \quad (48)$$

The frequency of the mode is $\omega^2=4\pi g^2n_\infty/3m$. For the nuclear system the T(isospin)$=1$ force has $g^2\approx 50$ MeV F, the inverse mass is $m^{-1}=41$ MeV F^2, and the density is $n_\infty\approx 0.19$ (particles) F^{-3}. Thus the energy of the surface isospin wave should lie at around

$$\omega=[(4\pi/3)\times 50\times 41\times 0.19]^{1/2}\text{ MeV}\approx 40\text{ MeV} \quad (49)$$

for a semi-infinite nuclear system and at very long wavelength. For a spherical nucleus, for a finite surface diffuseness and a finite wavelength of course we would expect the above number to be modified. The dipole resonance is experimentally observed in nuclei at about 25 MeV.

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