criterion for its applicability. As mentioned in Sec. III C, there are practical difficulties in testing the dependence on the angle between the oscillating currents and the steady field, owing to the normal regions at the surface where the flux emerges.

As we have already mentioned in the Introduction, it is possible to reinterpret some of the results of the skin effect measurements of Alais and Simon[22] in the light of the "step" in Sec. III B and the inclination and amplitude dependences discussed in Sec. III C. The screening or shielding by the surface sheath may not be appreciable if the surface is rough, or if the field is not aligned parallel to it and the flux pierces the sheath in the form of vortices. Provided the amplitude of the oscillating field is sufficiently large to overcome pinning, the oscillating flux can easily move the vortices along the sheath—thus enabling flux to pass from above the sheath to below, in the manner found by Boato *et al.*[44] If, however, the pinning is strong or the surface smooth, and the field accurately aligned parallel to it, small amplitude oscillating fields do not cause any flux flow, and large shielding effects can be observed. It seems that if vortices exist above H_{c2}, they must be confined to the sheath. However, if they move, they can give rise to the appearance of vortex motion similar to that observed below H_{c2}.

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[44] G. Boato, G. Gallinaro and C. Rizzuto, Solid State Commun. **3**, 173 (1965).

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Anomalous Thermal Conductivity of Superconductors due to Impurity Spin Ordering

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It is shown that spontaneous and also a magnetic-field-induced ordering of the paramagnetic impurity spins in superconductors gives rise to an anomalous temperature dependence of the electronic thermal conductivity.

ONE of the most important aspects of a study of superconductors containing paramagnetic impurities is the problem of the coexistence of superconductivity and magnetic ordering. There exists now some experimental and theoretical evidence for the coexistence of superconductivity and impurity spin ordering.[1–3] It is the purpose of this paper to show that the electronic thermal conductivity should strongly reflect the coexistence of superconductivity and magnetic ordering, possibly as strongly as the specific heat, and to stimulate experimental studies bearing upon the theoretical predictions presented here. The anomalous dependence of the superconducting transition temperature on the concentration of paramagnetic impurities,[1,2] which has been explained as resulting from spontaneous ordering of the impurity spins,[4] directly reflects the coexistence of superconductivity and impurity spin ordering only at temperatures close to the superconducting transition temperature. However, a study of the electronic thermal conductivity and the specific heat,[5,6] for example, should clearly reflect the coexistence of superconductivity and magnetic ordering at all temperatures below the superconducting transition temperature.

If the paramagnetic impurity spins order, then the theory by Gor'kov and Rusinov,[3] using a static *s-d* exchange coupling between the conduction electrons and the paramagnetic impurities, needs to be extended by using a dynamic *s-d* exchange coupling. The dynamics of the *s-d* exchange coupling become very important if the superconducting transition temperature

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[1] R. A. Hein, R. L. Falge, Jr., B. T. Matthias, and E. Corenzwit, Phys. Rev. Letters **2**, 500 (1959).
[2] J. E. Crow and R. D. Parks, Phys. Letters **21**, 378 (1966).
[3] L. P. Gor'kov and A. I. Rusinov, Zh. Eksperim. i Teor. Fiz. **46**, 1363 (1964) [English transl.: Soviet Phys.—JETP **19**, 922 (1964)].
[4] K. H. Bennemann, Phys. Rev. Letters **17**, 438 (1966).
[5] N. E. Phillips and B. T. Matthias, Phys. Rev. **121**, 105 (1961).
[6] D. K. Finnemore, D. C. Hopkins, and P. E. Palmer, Phys. Rev. Letters **15**, 891 (1965); K. H. Bennemann and J. W. Garland, Phys. Rev. **159**, 369 (1967).

is of the same order as the magnetic ordering temperature, since then the electron spin-flip relaxation time is comparable to the spin relaxation time of the paramagnetic impurities. If spontaneous ordering of the impurity spins occurs, or if the impurity spins are polarized by an external magnetic field, then the electron spin-flip scattering by the paramagnetic impurities becomes suppressed. Since the electron spin-flip scattering is largely responsible for the drastic breaking up of Cooper pairs, we expect that the reduction of the electronic spin-flip scattering arising from the impurity spin ordering favors superconductivity. However, if the impurity spins order, there is also a Zeeman splitting of the Fermi surface which is destructive of superconductivity. It is interesting to notice that spin-orbit scattering reduces the destructive effect of the Zeeman splitting of the Fermi surface, since it causes frequent electron transitions between the spin-up and the spin-down Fermi surface, leading to a hybridization of the two electron spin states at the Fermi surface. Therefore, the spin-orbit coupling can be used for regulating the two opposite effects resulting from the reduction of the spin-disorder scattering and the Zeeman splitting of the Fermi surface. Notice that in the case of antiferromagnetic-like ordering of the impurity spins, with small areas of uniform magnetization, only the reduction of the spin-disorder scattering is present.

According to the mechanisms above, which become operative if the impurity spins are not free to rotate but are fixed by molecular fields, we expect that the electronic thermal conductivity $K(T)$ of superconductors containing paramagnetic impurities will exhibit an anomalous increase for small Δ and $\tau_{tr}\approx\tau_{tr}^{ex}$. For $\tau_{tr}\ll\tau_{tr}^{ex}$, it will exhibit an anomalous decrease with decreasing temperature as a result of the drastic increase of the electronic collision time τ_{tr}^{ex} arising from the suppression of the spin-disorder scattering caused by spontaneous impurity spin ordering (or by an external magnetic field in the case of a type-II superconductor and thin films). Notice that $K(T)\alpha\tau_{tr}(T)$. This anomalous temperature dependence of the electronic thermal conductivity should become very pronounced if the spin-orbit scattering is such that the increase in the superconducting order parameter Δ due to the reduction of the electronic spin-flip scattering is nearly the same as or smaller than the decrease in the order parameter Δ arising from the Zeeman splitting of the Fermi surface. Regarding the thermal resistance

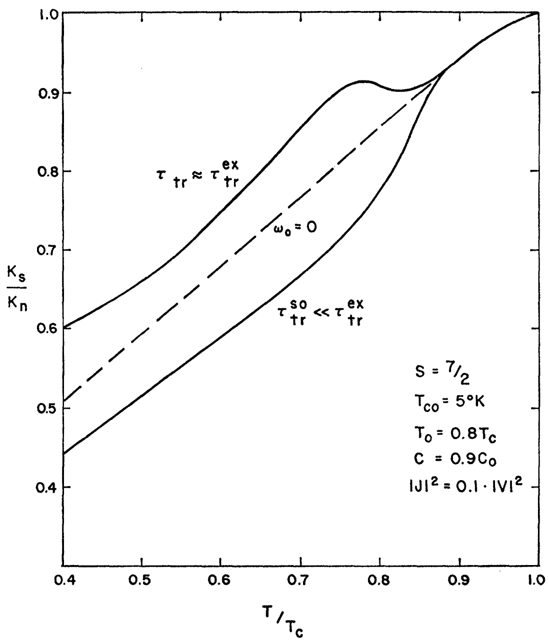

FIG. 1. The ratio of the electronic thermal conductivity in the superconducting and in the paramagnetic normal state. The molecular field ω_0 acting on the impurity spins is given by $\omega_0 = T_0[2-(T/T_0)^2]$ for $T_0<T<1.4T_0$, and $\omega_0=T_0$ for $T<T_0$. c_0 denotes the concentration of paramagnetic impurities which destroys superconductivity for $\omega_0=0$. K_s/K_n is calculated using Eqs. (1) and (2).

$\rho_{th}=1/K$ as a functional of the entropy, one finds, by expanding ρ_{tr} in terms of the electronic entropy fluctuations associated with the spin ordering, that the temperature coefficient of the thermal resistance should behave, for small Δ and at temperatures near the magnetic ordering temperature, like the electronic magnetic specific heat. The coexistence of superconductivity and magnetic ordering is reflected by the anomalous decrease of K for $\tau_{tr}\ll\tau_{tr}^{ex}$, and for $\tau_{tr}\approx\tau_{tr}^{ex}$ by the fact that the predicted anomalous enhancement of the low-temperature electronic thermal conductivity is smaller for the superconducting state than it would be for the normal state.

By employing the Kubo formula and treating the electron heat-current response function within the ladder approximation,[7] one obtains for the ratio of the electronic thermal conductivities in the superconducting state and in the paramagnetic normal state the approximate expression[8]

$$K_s/K_n=(3/4\pi^2)(k_BT)^{-3}\int_0^\infty d\omega\,\omega^2\,\mathrm{sech}^2(\omega/2k_BT)\sum_{\alpha=+,-}\left(1+\frac{|u_\alpha(\omega)|^2-1}{|u_\alpha^2(\omega)-1|}\right)\frac{(\tau_{tr})_\alpha}{(\tau_{tr})_0},\tag{1}$$

where $(\tau_{tr})_\alpha$ denotes the electron transport collision time which for the paramagnetic normal state is given by

[7] V. Ambegaokar and L. Tewordt, Phys. Rev. 134, A805 (1964).
[8] V. Ambegaokar and A. Griffin, Phys. Rev. 137, A1151 (1965).

$(\tau_{\rm tr})_0$ and where $u_\alpha(\omega)$ is given by

$$\frac{\omega}{\Delta}\pm\left(I+\frac{V}{J}\frac{1}{\tau_{\rm ex}}\langle S_z\rangle\cos\vartheta_{I,H}\right)=u_\pm\left(1-\frac{\xi}{(1-u_\pm^2)^{1/2}}\right)+(2\tau_{\rm so}\Delta)^{-1}\frac{u_\pm-u_\mp}{(1-u_\mp^2)^{1/2}}$$
$$-\Delta^{-1}\{u_\pm(\Sigma_{t\pm}^{(2)}(\omega)+\Sigma_{l\pm}^{(2)}(\omega))+(\Sigma_{t\pm}^{(1)}(\omega)+\Sigma_{l\pm}^{(1)}(\omega))\},\qquad I\equiv cJ\langle S_z\rangle+\mu_BH\cos\varphi_{I,H}.\quad(2)$$

Here, c is the concentration of paramagnetic impurities, J is the average exchange integral, and S_z is the component of the impurity spins in the direction of the average exchange field I. V is the nonexchange part of the paramagnetic impurity scattering potential. The superconducting order parameter is determined by

$$\ln(T/T_{c0})=\int_0^\infty d\omega\,\tanh(\omega/2k_BT)$$
$$\left[(2\Delta)^{-1}\sum_{\alpha=+,-}\mathrm{Im}\left(\frac{1}{[1-u_\alpha^2(\omega)]^{1/2}}-\frac{1}{\omega}\right)\right],\qquad(3)$$

where T_{c0} denotes the superconducting transition temperature in the absence of paramagnetic impurities and for $H=0$. The spin-orbit collision time $\tau_{\rm so}$ is given by

$$\tau_{\rm so}^{-1}=\tfrac{1}{3}c_{\rm so}N(0)\int d\Omega\,\sin^2\varphi\,|\,V_{\rm so}(\Omega)\,|^2,$$

where $c_{\rm so}$ denotes the concentration of spin-orbit scatters, which potential is denoted by $V_{\rm so}$, and $N(0)$ is the electron normal-state density of states at the Fermi surface. The parameter ξ for a bulk type-I superconductor is equal to zero, for a thin superconducting film equal to[9]

$$\frac{\pi T_{c0}}{2\Delta}\left(\frac{H}{H_c(T=0,c=0)}\right)^2,$$

and for a type-II superconductor equal to

$$\frac{\pi T_{c0}}{2\Delta}\frac{H}{H_{c2}(T=0,c=0)}.$$

H_{c2} is the upper critical magnetic field. The electron self-energies resulting from the exchange scattering are given by

$$\Sigma_{t+}^{(1)}(\omega)=\frac{N_-(0)c}{2p_{F-}^2}\int_{-\infty}^{\infty}\frac{dz}{2\pi}\int_{-\infty}^{\infty}d\omega'\int_{q_1}^{q_2}dq\,q\,|\,J(q)\,|^2B_t(q,z-\omega')\,\mathrm{Im}\frac{u_-(\omega')}{[1-u_-^2(\omega')]^{1/2}}\frac{(1-f_{\omega'}+N_{z-\omega'})}{z-\omega},\qquad(4)$$

$$\Sigma_{t-}^{(1)}(\omega)=\frac{N_+(0)c}{2p_{F+}^2}\int_{-\infty}^{\infty}\frac{dz}{2\pi}\int_{-\infty}^{\infty}d\omega'\int_{q_1}^{q_2}dq\,q\,|\,J(q)\,|^2B_t(q,\omega'-z)\,\mathrm{Im}\frac{u_+(\omega')}{[1-u_+^2(\omega')]^{1/2}}\frac{f_{\omega'}+N_{\omega'-z}}{z-\omega},\qquad(5)$$

and

$$\Sigma_{l\pm}^{(1)}(\omega)=\frac{N_\pm(0)c}{2p_{F\pm}^2}\int_{-\infty}^{\infty}\frac{dz}{2\pi}\int_{-\infty}^{\infty}d\omega'\int_0^{q_2}dq\,q\,|\,J(q)\,|^2B_l(q,z-\omega')\,\mathrm{Im}\frac{u_\pm(\omega')}{[1-u_\pm^2(\omega')]^{1/2}}\frac{1-f_{\omega'}+N_{z-\omega'}}{z-\omega}.\qquad(6)$$

Here, the Fermi momentum $p_{F\pm}$ and the normal-state electron density of states, $N_\pm(0)$, refer to electrons with spins $+$ and $-$, respectively; $q_1\equiv p_{F+}-p_{F-}\cong 2I/v_F$; and $q_2\equiv p_{F-}+p_{F+}$. The Fermi and Bose distribution functions are denoted by f_ω and N_ω, respectively. The spectral density functions of the transverse and longitudinal impurity spin excitations are denoted by $B_t(q,z)$ and $B_l(q,z)$, respectively. The self-energies $\Sigma_{t\pm}^{(2)}$ and $\Sigma_{l\pm}^{(2)}$ are obtained from Eqs. (4)–(6) by replacing

$$\mathrm{Im}\frac{u_\pm}{(1-u_\pm^2)^{1/2}}$$

with

$$\mathrm{Im}\frac{1}{(1-u_\pm^2)^{1/2}}.$$

It follows from Eqs. (4) and (5) that $\Sigma_{t\pm}^{(1)}$ and $\Sigma_{t\pm}^{(2)}$ become zero with increasing Zeeman splitting of the Fermi surface and decreasing temperature, in accordance with the intuitive physical idea that magnetic ordering freezes out the electron spin-flip scattering. The ratio of the electron transport collision times $(\tau_{\rm tr})_\alpha/(\tau_{\rm tr})_0$ is approximately given by

$$(\tau_{\rm tr})_\alpha/(\tau_{\rm tr})_0\cong 1+(c\,|\,J\,|^2S(S+1)/\sum_i c_i\,|\,V_i\,|^2)\Gamma_\alpha,$$

with $\Gamma_\alpha\equiv 1-(\tau_{\rm tr}^{\rm ex})_0/(\tau_{\rm tr}^{\rm ex})_\alpha$. Here, c_i and V_i denote the concentration and potential of the scatterers of kind i, and $(\tau_{\rm tr}^{\rm ex})_\alpha^{-1}$ and $(\tau_{\rm tr}^{\rm ex})_0^{-1}$ denote the contributions to $(\tau_{\rm tr})_\alpha^{-1}$ due to the exchange scattering in the presence and in the absence of magnetic ordering, respectively. Assuming Δ to be small, and consequently neglecting the terms involving

$$\mathrm{Im}\frac{1}{(1-u_\alpha^2)^{1/2}},$$

[9] K. Maki, Physics 1, 21 (1964); 1, 201 (1964).

one obtains approximately

$$\frac{1}{(\tau_{\mathrm{tr}}^{\mathrm{ex}})_-}=\int_{-\infty}^{\infty}d\omega'(f_{\omega'}+N_{\omega'-\omega})\left\{\mathrm{Im}\frac{u_+(\omega')}{[1-u_+^2(\omega')]^{1/2}}\frac{N_+(0)c}{8p_{F+}^4}\int_{q1}^{q2}dq\,q^3\,|J(q)|^2B_t(q,\omega'-\omega)\right.$$
$$\left.+\mathrm{Im}\frac{u_-(\omega')}{[1-u_-^2(\omega')]^{1/2}}\frac{N_-(0)c}{8p_{F-}^4}\int_0^{q2}dq\,q^3\,|J(q)|^2B_l(q,\omega-\omega')\right\},\quad(7)$$

and

$$\frac{1}{(\tau_{\mathrm{tr}}^{\mathrm{ex}})_+}=\int_{-\infty}^{\infty}d\omega'(1-f_{\omega'}+N_{\omega-\omega'})\left\{\mathrm{Im}\frac{u_-(\omega')}{[1-u_-^2(\omega')]^{1/2}}\frac{N_-(0)c}{8p_{F-}^4}\int_{q1}^{q2}dq\,q^3\,|J(q)|^2B_t(q,\omega-\omega')\right.$$
$$\left.+\mathrm{Im}\frac{u_+(\omega')}{[1-u_+^2(\omega')]^{1/2}}\frac{N_+(0)c}{8p_{F+}^4}\int_0^{q2}dq\,q^3\,|J(q)|^2B_l(q,\omega-\omega')\right\}.\quad(8)$$

Assuming that Δ is small and that the energy distribution of the transverse impurity spin excitation is peaked at ω_0, and employing the dissipation-fluctuation theorem, we can rewrite Eqs. (4)–(6) approximately as

$$\Sigma_{t+}^{(1)}(\omega)=\frac{N_-(0)c\pi}{p_{F-}^2}\int_{q1}^{q2}dq\,q\,|J(q)|^2\left\{i[\langle S_z\rangle(1-f_{\omega-\omega_0})+\tfrac{1}{2}\langle S_-(q)S_+(-q)\rangle]\,\mathrm{Im}\frac{u_-(\omega-\omega_0)}{[1+u_-^2(\omega-\omega_0)]^{1/2}}\right.$$
$$\left.-\tfrac{1}{2}\langle S_-(q)S_+(-q)\rangle\,\mathrm{Re}\frac{u_-(\omega-\omega_0)}{[1-u_-^2(\omega-\omega_0)]^{1/2}}-\int_{-\infty}^{\infty}\frac{dz}{4\pi^2}\mathrm{Im}\frac{u_-(\omega-z)}{[1-u_-^2(\omega-z)]^{1/2}}B_t(q,z)\ln\left|\frac{z-\omega}{z}\right|\right\},\quad(9)$$

$$\Sigma_{t-}^{(1)}(\omega)=\frac{N_+(0)c\pi}{p_{F+}^2}\int_{q1}^{q2}dq\,q\,|J(q)|^2\left\{i[\langle S_z\rangle f_{\omega+\omega_0}+\tfrac{1}{2}\langle S_-(q)S_+(-q)\rangle]\,\mathrm{Im}\frac{u_+(\omega+\omega_0)}{[1-u_+^2(\omega+\omega_0)]^{1/2}}\right.$$
$$\left.-\tfrac{1}{2}\langle S_-(q)S_+(-q)\rangle\,\mathrm{Re}\frac{u_+(\omega+\omega_0)}{[1-u_+^2(\omega+\omega_0)]^{1/2}}-\int_{-\infty}^{\infty}\frac{dz}{4\pi^2}\mathrm{Im}\frac{u_+(\omega+z)}{[1-u_+^2(\omega+z)]^{1/2}}B_t(q,z)\ln\left|\frac{z+\omega}{z}\right|\right\},\quad(10)$$

and

$$\Sigma_{l\pm}^{(1)}(\omega)=-\frac{N_\pm(0)c\pi}{2p_{F\pm}^2}\frac{u_\pm(\omega)}{[1-u_\pm^2(\omega)]^{1/2}}\int_0^{q2}dq\,q\,|J(q)|^2\langle S_z(q)S_z(-q)\rangle.\quad(11)$$

For simplicity in the following we apply the molecular-field approximation and use $B_t(q,z)=4\pi\langle S_z\rangle\delta(z-\omega_0)$, with $\omega_0=(3nc/2\epsilon_{\mp})J^2\langle S_z\rangle+2\mu_BH$, where n is the number of electrons per atom and ϵ_F the Fermi energy. One finds then

$$\Sigma_{t\pm}^{(1)}(\omega)=-\frac{\langle S_z\rangle[\coth(\omega_0/k_BT)-1]}{2[\tau^{\mathrm{ex}}(q_1)]_{\mp}}\mathrm{Re}\frac{u_{\mp}(\omega\mp\omega_0)}{[1-u_{\mp}^2(\omega\mp\omega_0)]^{1/2}}$$
$$+\frac{\mathrm{Im}\{u_{\mp}(\omega\mp\omega_0)/[1-u_{\mp}^2(\omega\mp\omega_0)]^{1/2}\}}{[\tau^{\mathrm{ex}}(q_1)]_{\mp}}\left(i[\langle S_z\rangle[1-f_\pm(\omega\mp\omega_0)]+\tfrac{1}{2}\langle S_-(q)S_+(-q)\rangle]-\pi^{-1}\langle S_z\rangle\ln\left|\frac{\omega\mp\omega_0}{\omega_0}\right|\right),\quad(12)$$

and

$$\Sigma_{l\pm}^{(1)}(\omega)=\frac{S(S+1)-\langle S_z\rangle\coth(\omega_0/2k_BT)}{2[\tau^{\mathrm{ex}}(0)]_\pm}\frac{u_\pm(\omega)}{[1-u_\pm^2(\omega)]^{1/2}},\quad(13)$$

with

$$\langle S_z\rangle=SB_S(S\omega_0/k_BT),\qquad\frac{1}{[\tau^{\mathrm{ex}}(q_1)]_\pm}\equiv\frac{N_\pm(0)c\pi}{p_{F\pm}^2}\int_{q1}^{q2}dq\,q\,|J(q)|^2.\quad(14)$$

Here, $B_s(x)$ is the Brillouin function. The expressions for $\Sigma_{t\pm}^{(2)}$ and $\Sigma_{l\pm}^{(2)}$ are again obtained from Eqs. (12) and (13) by replacing $u_\pm/(1-u_\pm^2)^{1/2}$ with $1/(1-u_\pm^2)^{1/2}$. It is obvious that $\Sigma_{t\pm}^{(1)}$ and $\Sigma_{t\pm}^{(2)}$ become zero for $(\omega_0/k_BT)\rightarrow\infty$. For $\omega_0=0$, one easily recaptures the expressions derived by Abrikosov and Gor'kov for the case that the impurity spins are free to rotate.[10] Using the same approximations as in deriving Eqs. (12) and (13), one

[10] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].

obtains for Eqs. (7) and (8)

$$\frac{1}{(\tau_{\mathrm{tr}}{}^{\mathrm{ex}})_{\pm}} = \mathrm{Im}\frac{u_{\mp}(\omega\mp\omega_0)}{[1-u_{\mp}{}^2(\omega\mp\omega_0)]^{1/2}}[\langle S_-S_+\rangle+2\langle S_z\rangle(1-f_{\pm(\omega\mp\omega_0)})]\frac{N_{\mp}(0)c}{4p_{F\mp}{}^4}\int_q^{q_0} dq\, q^3\,|\,J(q)\,|^2 + \mathrm{Im}\frac{u_{\pm}(\omega)}{[1-u_{\pm}{}^2(\omega)]^{1/2}}\langle S_zS_z\rangle\frac{N_{\pm}(0)c\pi}{4p_{F\pm}{}^4}\int_0^{q_2} dq\, q^3\,|\,J(q)\,|^2. \quad (15)$$

In the case where the dominant contribution to $(\tau_{\mathrm{tr}})_\alpha$ results from the paramagnetic impurities one finds

$$\frac{(\tau_{\mathrm{tr}})_{\pm}}{(\tau_{\mathrm{tr}})_3}\cong 1+|\,J/V\,|^2\left\{S(S+1)-[\langle S_-S_+\rangle+2\langle S_z\rangle(1-f_{\pm\,\omega\pm\omega_0})]\eta_{\mp}(q_1)\,\mathrm{Im}\frac{u_{\mp}(\omega\mp\omega_0)}{[1-u^{\pm 2}(\omega\mp\omega_0)]^{1/2}} -\langle S_zS_z\rangle\eta_{\pm}(0)\,\mathrm{Im}\frac{u_{\pm}(\omega)}{[1-u_{\pm}{}^2(\omega)]^{1/2}}\right\}, \quad (16)$$

with

$$\eta_{\pm}(q_1)\equiv\frac{N_{\pm}(0)\,p_F{}^4}{N(0)\,p_{F\pm}{}^4}\int_{q_1}^{q_2} dq\, q^3\Big/\int_0^{q_2} dq\, q^3. \quad (17)$$

Here, V denotes the nonexchange part of the scattering potential due to the paramagnetic impurities.

It follows from Eqs. (1) and (16) that for $(\omega_0/k_BT)\to\infty$ the ratio $(\tau_{\mathrm{tr}})_\alpha/(\tau_{\mathrm{tr}})_0$ approaches the value $(1+S\,|\,J/V\,|^2)$ Similarly, for $(\omega_0/k_BT)\to\infty$ the Eq. (2) can be rewritten approximately as

$$\frac{\omega}{\Delta}\pm\left(I+\frac{V}{J}\frac{1}{\tau_{\mathrm{ex}}}\langle S_z\rangle\right)\left(1-\frac{\xi+\beta}{(1-u_{\pm}{}^2)^{1/2}}\right)+(2\tau_{\mathrm{so}}\Delta)^{-1}\frac{u_{\pm}-u_{\mp}}{(1-u_{\mp}{}^2)^{1/2}}, \quad (18)$$

with

$$\beta\equiv\frac{\langle S_zS_z\rangle}{S(S+1)}(2\tau^{\mathrm{ex}}\Delta)^{-1}\xrightarrow[k_BT\gg\omega_0]{}\frac{S}{S+1}\frac{\Delta(\omega_0=0)}{\Delta}\beta_0,$$

demonstrating that superconductivity is favored by the reduction of the spin-flip scattering. Here β_0 denotes the value of β for $\omega_0=0$. Hence it follows that K_s/K_n, which can also be put into the form

$$\frac{K_s}{K_n}=\frac{3}{4\pi^2}(k_BT)^{-3}\int_0^\infty d\omega\,\omega^2\,\mathrm{sech}^2(\omega/2k_BT)\sum_{\alpha=+,-}\times\left\{\left(\mathrm{Im}\frac{u_\alpha(\omega)}{[1-u_\alpha{}^2(\omega)]^{1/2}}\right)^2-\left(\mathrm{Im}\frac{1}{[1-u_\alpha{}^2(\omega)]^{1/2}}\right)^2\right\}\frac{(\tau_{\mathrm{tr}})_\alpha}{(\tau_{\mathrm{tr}})_0}, \quad (19)$$

increases with decreasing temperature in the limit of $\tau_{\mathrm{tr}}{}^{\mathrm{ex}}\approx\tau_{\mathrm{tr}}$ and $\Delta\to 0$, $k_BT\ll\omega_0$, as a result of impurity spin ordering. One obtains for $\tau_{\mathrm{so}}\to\infty$, $T\to 0$, and $\Delta\to 0$ $(k_BT\ll\omega_0)$ the expressions

$$\frac{K_s}{K_n}=\frac{4}{\pi^2}\frac{(\xi+\beta)^{-2/3}}{[1-(\xi+\beta)^{2/3}]^{1/2}}[2-(\xi+\beta)^{2/3}]\sum_{\alpha=+,-}\frac{2\omega_{00}{}^\alpha}{\Delta}\times\exp(-\omega_{00}{}^\alpha/k_BT)(1+S\,|\,J/V\,|^2),\quad \text{if } (\xi+\beta)<1 \quad (20)$$

and

$$\frac{K_s}{K_n}=\left\{1-(\xi+\beta)^{-2}\left[1+\left(\frac{I/\Delta}{\xi+\beta}\right)^2\right]^{-2}\right\}(1+S\,|\,J/V\,|^2),\quad \text{if } (\xi+\beta)>1. \quad (21)$$

Here, $\omega_{00}{}^\alpha$ denotes the energy gap in the electron excitation spectrum:

$$\omega_{00}{}^{\pm}\equiv[1-(\xi+\beta)^{2/3}]^{3/2}\Delta\pm\left(I+\frac{V}{J}\frac{1}{\tau_{\mathrm{ex}}}\langle S_z\rangle\right).$$

It follows from Eqs. (20) and (21) that K_s/K_n increases with decreasing temperature if

$$(\beta^2-1)S\,|\,J/V\,|^2>\left[1-\left(\frac{S}{S+1}\frac{\Delta(\omega_0=0)}{\Delta}\right)^2\right],$$

which should be possible to arrange experimentally without much difficulty.

In summary, it follows from the analysis presented that the electronic thermal conductivity of superconductors containing paramagnetic impurities should exhibit an anomalous decrease in dK/dT as a result of impurity spin ordering if the contribution to the transport collision time due to nonmagnetic impurities is much smaller than the contribution resulting from exchange scattering by the paramagnetic impurities. This effect corresponds to the enhancement of the superconducting transition temperature resulting from impurity spin ordering. However, if the electron transport collision time is essentially determined by the exchange scattering, then the electron thermal conductivity should exhibit an anomalous increase with decreasing temperature if impurity spin ordering occurs.

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