Generalized Scaling and the Critical Eigenvector in Ideal Bose Condensation

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The condensation of an ideal Bose gas, in which the phase symmetry of the boson states has been removed by a linear coupling of the creation and annihilation operators $\psi^\dagger(\vec{r})$, $\psi(\vec{r})$ to a fictitious external field $C(\vec{r})$, is considered as a model of a second-order phase transition. In the vicinity of the transition, the spontaneous-order parameter (thermal expectation value of ψ) is related to the conjugate field by a series of terms of which the first represents a scaled equation of state that exhibits a power-law behavior with nonclassical exponents. The expectation value for the ordered product of boson operators $\langle\psi^\dagger(1)\,\psi^\dagger(2)\cdots\psi^\dagger(M)\,\psi(1')\cdots\psi(N')\rangle_\beta$ is determined and its behavior across the coexistence curve discussed. The existence of a conjectured critical eigenvector is demonstrated and a related asymptotic property of the ordered product determined.

I. INTRODUCTION

In this paper we reconsider the condensation of an ideal Bose-Einstein gas using an approach derived from a suggestion of Bogolubov.[1] With this point of view, the Bose transition acquires a very close resemblance to the liquid-vapor critical point as well as the magnetic transition about the Curie point. Our analysis is motivated by a desire to provide a definite model with which one can test certain of the recently conjectured descriptions of a phase transition. Since the Bose gas is one of the few nontrivial many-body systems which exhibits a change in phase whose properties can be calculated explicitly, much can be learned about the physical behavior in the vicinity of the transition.

Several other authors have also recently investigated the condensation of an ideal Bose gas as a model cooperative transition. Gunton and Buckingham,[2] using ideas like those contained here in Secs. I–III, have studied the dependence of the thermodynamic variables and pair-correlation function of both the dimensionality and the nature of the individual particle energy spectrum. Casher and Revzen[3] have shown that the Bose condensation is naturally representable in terms of the coherent states introduced by Glauber.[3]

In this paper we wish to present several points of general importance in the theory of critical phenomena not emphasized in previous work. First, the equation of state may be expressed as an ordered series in appropriately scaled thermodynamic variables. In the vicinity of the transition, the first term in the expansion exhibits the homogeneous form of a scaled equation of state,[4,5] which shows a power-law behavior with nonclassical exponents. Unlike the liquid-vapor transition, in this model the $\vec{k}$-dependent susceptibilities (variation in the complex-order parameter with respect to the conjugate field) are infinite for $\vec{k}=0$ along the coexistence curve.[6] Secondly, we have determined the thermal expectation value for the ordered product of boson-field operators in the neighborhood of the transition and have used them to demonstrate the existence of a "critical eigenvector" as has been postulated by one of us, MSG,[7] as a universal feature of critical points. The expressions for the expectation values are shown to have asymptotic properties related simply to the critical eigenvector. This relationship is a special case of a more general asymptotic formula for the distribution functions near the critical point that has been discussed elsewhere.[7]

The system of noninteracting Bose particles is described in terms of the complex boson-field operators $\psi^\dagger(\vec{r})$, $\psi(\vec{r})$ and the existence of a condensate (macroscopic-order parameter) characterized by a nonvanishing thermal expectation value for the single-field operator (denoted Ψ).[8] After removing the degeneracy associated with the continuous-phase symmetry of the boson states by introducing a fictitious external field linear in the boson operators,[9] a coherent-state representation is used to calculate the thermodynamic properties.[3] The mathematical description for the order parameter as an appropriate thermodynamic quantity is like that of the spherical model of a ferromagnet,[10] with the analysis of both systems reducing to computing Gaussian functional integrals with a physical external constraint. The critical indices for the Bose condensation are also those found in the spherical model.[10]

II. HAMILTONIAN

Consider a gas composed of N noninteracting, spinless Bose particles contained inside a d-dimensional volume $V(=L^d)$ in thermal equilibrium at a temperature $T(=\beta^{-1})$. The field operators associated with such a collection of bosons $\psi(\vec{r})$, $\psi^\dagger(\vec{r})$ satisfy the equal-time commutation relation

$$[\psi(\vec{r}),\psi^\dagger(\vec{r}\,')]=\delta(\vec{r}-\vec{r}\,'), \tag{1}$$

with the physical properties of the system determined by the Hamiltonian

$$H_0=\int d\vec{r}\;\psi^\dagger(\vec{r})\,[(-\hbar^2/2m)\vec{\nabla}^2]\,\psi(\vec{r}) \tag{2}$$

for a fixed particle number

$$N=\int d\vec{r}\;\psi^\dagger(\vec{r})\psi(\vec{r}). \tag{3}$$

Since H_0 and N commute, the states of the system are invariant to a gauge transformation generated by $\exp i\varphi N$, where φ is the continuous phase variable, $0 \leq \varphi \leq 2\pi$, and consequently the expectation value of a single-field operation ψ must vanish identically. To remove this degeneracy, we destroy the phase symmetry of the boson states by introducing a linear coupling of the boson operators to a fictitious local field $C(\vec{r})$ thereby introducing additional terms into the Hamiltonian

$$H = H_0 + \int d\vec{r}[C(\vec{r})\psi^\dagger(\vec{r}) + C^*(\vec{r})\psi(\vec{r})]. \qquad (4)$$

(The existence of a condensate is then exhibited by the presence of a spontaneous-order parameter as this external field vanishes.)

Expanding the boson-field operators in terms of the usual creation and annihilation operators of the Fock momentum-space representation $a_{\vec{k}}^\dagger$, $a_{\vec{k}}$ and introducing the Fourier components of the external field $c_{\vec{k}}$, the grand Hamiltonian operator $\mathcal{H} = H - \mu N$ is found to be

$$\mathcal{H} = \sum_{\vec{k}}(\epsilon_{\vec{k}} - \mu)a_{\vec{k}}^\dagger a_{\vec{k}} + c_{\vec{k}} a_{\vec{k}}^\dagger + c_{\vec{k}}^* a_{\vec{k}} \qquad (5)$$

where $N = \sum a_{\vec{k}}^\dagger a_{\vec{k}}$, and $\epsilon_{\vec{k}} = \frac{1}{2}k^2$ in units of $m = \hbar = 1$. The value of the chemical potential μ is restricted to $\mu \leq 0$ by the condition that $\mathcal{H}$ be positive definite.

The grand Hamiltonian is bilinear in the creation and annihilation operators of the individual $\vec{k}$ modes can be diagonalized exactly by simply completing the square, i.e., adding $\pm\sum c_{\vec{k}}^* c_{\vec{k}}(\epsilon_{\vec{k}} - \mu)^{-1}$. The unitary operator[3] $\mathcal{G}(\{c_{\vec{k}}\}) = \prod \exp(c_{\vec{k}} a_{\vec{k}}^\dagger - c_{\vec{k}}^* a_{\vec{k}})$ that generates this linear canonical transformation provides a new set of operators

$$\begin{aligned} A_{\vec{k}} &= \mathcal{G} a_{\vec{k}} \mathcal{G}^\dagger = a_{\vec{k}} + c_{\vec{k}}/(\epsilon_{\vec{k}} - \mu), \\ A_{\vec{k}}^\dagger &= \mathcal{G} a_{\vec{k}}^\dagger \mathcal{G}^\dagger = a_{\vec{k}}^\dagger + c_{\vec{k}}^*/(\epsilon_{\vec{k}} - \mu), \end{aligned} \qquad (6)$$

which also satisfy a boson commutation relation

$$[A_{\vec{k}}, A_{\vec{k}'}^\dagger] = \delta(\vec{k}, \vec{k}'). \qquad (7)$$

Thus, in the $\vec{A}$ representation, $\mathcal{H}$ takes the diagonal form

$$\mathcal{H} = \sum_{\vec{k}}(\epsilon_{\vec{k}} - \mu)A_{\vec{k}}^\dagger A_{\vec{k}} - \sum_{\vec{k}} \frac{c_{\vec{k}}^* c_{\vec{k}}}{\epsilon_{\vec{k}} - \mu}. \qquad (8)$$

This expression corresponds to the usual ideal boson system in which the Hamiltonian has been displaced by an amount $-\sum_{\vec{k}} |c_{\vec{k}}|^2/(\epsilon_{\vec{k}} - \mu)$. In the following sections, we shall make use of this diagonal $\vec{A}$ representation to calculate the thermodynamic properties of the system and also the expectation value for the ordered product of Bose-field operators. Although for a spatially uniform system only the $\vec{k} = 0$ component of the symmetry breaking field $C(\vec{r})$ exists ($c_{\vec{k}} = V^{-1/2} C \delta_{k0}$), in order to maintain generality, $C(\vec{r})$ will be taken to consist of all $\vec{k}$ components so as to allow for spatial variations.

III. THERMODYNAMIC PROPERTIES

All of the thermodynamic properties of the system can be obtained directly from the grand partition function $\Xi(\beta, \mu, V, C) \equiv \mathrm{Tr}(\exp - \beta\mathcal{H})$. Using the results of the previous section [diagonal form of $\mathcal{H}(A)$ with energy spectrum quadratic in $\vec{k}$] the calculation may be done explicitly, and in d dimensions, we find

$$\frac{1}{V}\ln\Xi(\beta, \mu, V, C) = \frac{1}{V}\ln\Xi_0 + \frac{1}{V}\sum_{\vec{k}} \frac{\beta |c_{\vec{k}}|^2}{\epsilon_{\vec{k}} - \mu}. \qquad (9)$$

Here $\Xi_0(\beta, \mu, V)$ is the grand partition function for an ideal Bose gas in zero field with

$$V^{-1}\ln\Xi_0 = \lambda^{-d} F_{\frac{1}{2}d+1}(z) - V^{-1}\ln(1 - z), \qquad (10)$$

where λ^{-1} is the thermal wavelength of the particles $(2\pi mkT)^{1/2}$ and with the fugacity $e^{\beta\mu}$ denoted by the variable z, $0 \leq z \leq 1$.

The Bose function $F_s(z)$ is defined in terms of its power-series expansion[11]

$$F_s(z) = \sum_{\sigma=1}^{\infty} \sigma^{-s} z^\sigma. \qquad (11)$$

It is a positive, monotonically increasing function of z which reduces to the Riemann zeta function at $z = 1$, $F_s(1) = \zeta(s)$.

The particle-number density n defined as

$$n = \lim_{\substack{N \to \infty \\ V \to \infty}} \frac{N}{V} = \lim_{V \to \infty} z\frac{\partial}{\partial z}\left(\frac{1}{V}\ln\Xi\right) \qquad (12)$$

is given by

$$n = (2\pi T)^{d/2} F_{d/2}(z) + \lim_{V \to \infty} \frac{1}{V}\frac{z}{1 - z} + \frac{1}{V}\sum_{\vec{k}} \frac{\beta^2 |c_{\vec{k}}|^2}{(\beta\epsilon_{\vec{k}} - \ln z)}. \qquad (13)$$

In more than two dimensions ($d > 2$), the quantity $F_{d/2}(z)$ is bounded above on the interval $0 \leq z \leq 1$ by $\zeta(d/2)$. Thus for any fixed-number density n less than $n_c = (2\pi T)^{d/2}\zeta(d/2)$, there exists a unique solution to Eq. (13) such that $z < 1$, with terms of $O(1/V)$ vanishing as $V \to \infty$. In the absence of any field, $C(\vec{r}) \to 0$, the number density is just

$$n(z) = (2\pi T)^{d/2} F_{d/2}(z), \qquad (14)$$

with $z < 1$, for $n < n_c$ along $C \equiv 0$. For values of the number density greater than the critical value n_c, the solution $n(z)$ requires a finite nonvanishing contribution of the form $n - n_c \equiv \Psi^2$, where

$$\Psi^2 = \frac{\beta^2 c_0^2}{(\ln z)^2} + \int_{\vec{k} > 0} d\vec{k}\, \beta^2 |c_{\vec{k}}|^2 (\beta\epsilon_{\vec{k}} - \ln z)^{-2} + \lim_{V \to \infty} \frac{1}{V}\frac{z}{1 - z}. \qquad (15)$$

Here, the last term $z/(1 - z)V$ corresponds to the usual $\vec{k} = 0$ mode contribution that is traditionally required to support an indefinitely large number occupancy, but which is of a vanishingly small magnitude compared with those terms arising from the coupling of the phase of the boson states to the external field $C(\vec{r})$. Thus, in the thermodynamic limit of $V \to \infty$ first, as a uniform C tends to zero, the variable z approaches unity such that the following limit exists

$$\Psi^2 = \lim_{\substack{C\to 0\\ z\to 1_-}} \lim_{V\to\infty} \beta^2 c_0{}^2/(\ln z)^2. \quad (16)$$

Therefore for values greater than n_C the number density in zero external field is given by

$$n = (2\pi T)^{d/2} F_{d/2}(z=1) + \Psi^2. \quad (17)$$

The point of equality of the two different analytic branches for n [Eqs. (14) and (17)] occurs at $z=1$, $\Psi^2 = 0$ and defines a critical-value temperature

$$T_C = [2\pi/\zeta(d/2)n]^{2/d}. \quad (18)$$

The correspondence between Ψ^2 and the thermal expectation value of the single Bose-field operator $\langle\psi\rangle_\beta$ as a spontaneous order parameter comes from noting the identity

$$\langle\psi(\vec{r})\rangle_\beta = -\frac{\partial}{\partial\beta C^*(\vec{r})}\ln\Xi(\beta,\mu,V,C) = (\Psi^2)^{\frac{1}{2}}. \quad (19)$$

In two dimensions or less, $d \leq 2$, no such order parameter as Ψ exists, since $F_{d/2}(z)$ is unbounded as z tends to unity.

The pressure is related to the number density parametrically through the variable z by

$$\beta p = \lim_{C\to 0} \lim_{V\to\infty} V^{-1}\ln\Xi \quad (20)$$

and is given by

$$p = (2\pi)^{d/2} T^{d/2+1} F_{d/2}(z) \quad (21)$$

$$z<1, \quad T>T_C\,; \quad z=1, \quad T<T_C\,.$$

For the three-dimensional system $d=3$, the transition occurs along a critical isotherm $pn^{-5/3}$ constant – the condensed region degenerating to a simple coexistence line owing to the lack of repulsive interactions between the particles prevents a collapse to a state of zero volume.

The energy density, like that of any ideal gas is proportional to the pressure

$$E/V = -(\partial/\partial\beta)V^{-1}\ln\Xi = \tfrac{3}{2}k \quad (22)$$

and like the pressure is a continous function of the temperature. The specific heat at fixed number density (volume) is therefore continous across the transition, although its derivative

$$\Delta c_{v,\Psi}/\Delta T = [\partial^2(E/V)/\partial T^2]_\Psi$$

displays a finite jump at T_C along the coexistence line.

IV. SCALING PROPERTIES

The Legendre transformation which eliminates the variables conjugate to n and Ψ, μ and C, respectively, defines a new thermodynamic function

$$G(\beta,n,\Psi) \equiv \left(1 - \mu\frac{\partial}{\partial\mu} - C\frac{\partial}{\partial C}\right)\frac{1}{V}\ln\Xi(\beta,\mu,V,C) \quad (23)$$

$$= (2\pi T)^{d/2}[F_{\frac{1}{2}d+1}(z) - \ln z F_{d/2}(z)],$$

$$- 2T^{-1}\int_{\vec{k}>0} d\vec{k}\,\epsilon_{\vec{k}}\Psi_{\vec{k}}^*\Psi_{\vec{k}}, \quad (24)$$

where the fugacity z is to be expressed as a function of (β, n, Ψ). The equation of state relating the thermodynamic conjugate variable C to Ψ is obtained by differentiating G with respect to the order parameter Ψ at fixed T.

In the absence of a spontaneous condensate $\Psi=0$, at temperatures above T_C, $n<n_C$, inversion of $n(z)$ finds the leading terms of the G function correspond to an ideal-gas free-energy expansion $(d=3)$

$$G(\beta,n,\Psi=0) = \zeta(\tfrac{3}{2})n_C{}^{-1}T^{3/2}[n - n\ln n + O(n^2)]. \quad (25)$$

Below the critical temperature $T<T_C$ the relationship between the number density n and the fugacity z may be written as

$$F_{3/2}(z \lesssim 1) = T_C{}^{-3/2} n\left(\frac{1-\Psi^2/n}{(1+t)^{3/2}}\right), \quad (26)$$

where we have measured the temperature in terms of a reduced variable t defined by

$$T = T_C(1+t). \quad (27)$$

Thus along the coexistence curve (fixed $C=0$, $z\to 1$) the ratio of the reduced normal-state density $1-(\Psi^2/n)$ to the distance from the reduced critical temperature t is given by a homogeneous function of the chemical potential μ as had been generally conjectured by Widom.[5]

In the vicinity of the transition as z tends to unity, the series for $F_S(z)$ of Eq. (11) converges very slowly and it becomes necessary to use an alternate expression defined in terms of $\eta = -\ln z$,[11]

$$F_S(z\to 1) = \Gamma(1-s)\eta^{S-1} + \sum_{\sigma=0}^{\infty}\frac{(-1)^\sigma}{\sigma!}\zeta(s-\sigma)\eta^\sigma. \quad (28)$$

Using this series for $F_{3/2}(z)$, the inversion of Eq. (26) finds a natural expansion parameter

$$\frac{\xi}{\xi_0} = 1 - \frac{(1-\Psi^2/n)}{(1+t)^{3/2}}, \quad \xi_0 = -\frac{\zeta(\frac{3}{2})}{\Gamma(-\frac{1}{2})} \quad (29)$$

in which to expand the thermodynamic function $G(\beta,n,\Psi)$,

$$G(\beta,n,\Psi) = (2\pi T_C)^{3/2}(1+t)^{3/2}$$

$$\times[\zeta(\tfrac{5}{2}) + \tfrac{4}{5}\pi^{1/2}\xi^3 - \tfrac{17}{10}\zeta(\tfrac{1}{2})\xi^4 + \cdots]$$

$$- 2T_C{}^{-1}(1+t)^{-1}\int d\vec{k}\,\epsilon_{\vec{k}}\Psi_{\vec{k}}^*\Psi_{\vec{k}} \quad (30)$$

and the equation of state (C uniform, $\Psi_{\vec{k}\neq 0}=0$),

$$C = \left(\frac{\partial G}{\partial\Psi}\right)_T = 2\Psi(\tfrac{12}{5}\pi^{1/2}\xi^2)$$

$$\times[1 - \tfrac{17}{6}\zeta(\tfrac{1}{2})\pi^{-1/2}\xi + O(\xi^2)]. \quad (31)$$

As $t \to 0$, this parameter ξ may be expanded in a binomial series and written in terms of Ψ^2/n and the variable $x = t(n/\Psi^2)$

$$\xi/\xi_0 = \Psi^2(1+\tfrac{3}{2}x) + O(x^2\Psi^4). \tag{32}$$

Substituting for ξ and rearranging terms we find that in the vicinity of the transition, the equation of state is given by

$$C/C_0 = \Psi^5 H_2(x) + \Psi^7 H_3(x) + \ldots + \Psi^{2s+1} H_s(x) \tag{32}$$

with $C_0 = 6\zeta^2(\tfrac{3}{2})/5\pi^{1/2}$. Here $H_s(x)$ denotes a polynomial of order s in the variable x of which the first few are explicitly

$$H_2(x) = (1+\tfrac{3}{2}x)^2, \tag{33}$$

$$H_3(x) = \gamma_3(1+\tfrac{3}{2}x)^3 - 3(1+\tfrac{3}{2}x)(1+\tfrac{25}{8}x)x,$$

$$H_4(x) = \gamma_4(1+\tfrac{3}{2}x)^4 - \tfrac{9}{8}\gamma_3(1+\tfrac{3}{2}x)^2(1+\tfrac{25}{8}x)$$

$$+\tfrac{15}{4}(1+\tfrac{3}{2}x)(1+\tfrac{7}{6}x)x^2 + \tfrac{9}{4}x^2(1+\tfrac{25}{8}x)^2$$

with the constants $\gamma_3 = -\tfrac{17}{12}\zeta(\tfrac{3}{2})\zeta(\tfrac{1}{2})/\pi \sim 1.72$, and $\gamma_4 = 7\zeta^2(\tfrac{3}{2})/24\pi^{3/2} \sim 0.35$.

A relation similar to Eq. (32) has been suggested for the Ising model equation of state by Domb and Hunter[4] from studies of the power-series developments of the partition function. The first term of the rhs represents the homogeneous functional form for the equation of state proposed by Widom.[5] It is of the form later asserted by Griffiths[12] to relate the applied magnetic field and the net magnetization of a simple ferromagnet and rewritten as the scaling-law equation of state

$$C(\Psi, t) = \Psi|\Psi|^{\delta-1} h(x = t\Psi^{-1/\beta}), \tag{34}$$

In our model, the critical indices have the values $\beta = \tfrac{1}{2}$ and $\delta = 5$, while $h(x)$ is given by the polynomial $H_2(x) = (1+\tfrac{3}{2}x)^2$ with $x = t\Psi^{-2}$. This function vanishes identically along the two-phase boundary (coexistence curve $C = 0$, $\Psi \neq 0$, $t < 0$) of $x_0 = -\tfrac{2}{3}$. Schematic representations of $G(\beta, n, \Psi)$ together with the equation of state $C(\beta, \Psi)$ are shown in Figs. 1 and 2.

The variation of the thermodynamic function $G(\beta, n, \Psi)$ with temperature defines a fixed density (volume) specific heat $c_{v,\Psi}$ whose behavior about the transition as Ψ vanishes is characterized by the critical exponent α,

$$c_{v\Psi} = (\partial G/\partial T)_{\Psi \to 0} \sim \text{const} + t^{-\alpha}. \tag{35}$$

In this model, we thus find a negative value of $\alpha = -1$ which describes a specific heat both continuous and finite up to the transition. A finite discontinuity in slope at T_c occurs like that in the spherical model and classical van der Waals fluid.

The isothermal susceptibility χ_T, defined as the change in the order parameter Ψ with the field C at constant temperature, is described along the coexistence curve by the critical index γ,

$$\chi_T = (\partial\Psi/\partial C)_{\Psi \to 0} \sim t^{-\gamma}. \tag{36}$$

Using the previous expression for $G(\beta, n, \Psi)$ with $\Psi^2 = \sum_{\vec{k}} \Psi_{\vec{k}}^* \Psi_{\vec{k}}$ and noting the $\Psi_{\vec{k}}$ are complex quantities so that any change in them must be separated into independent variations[6]

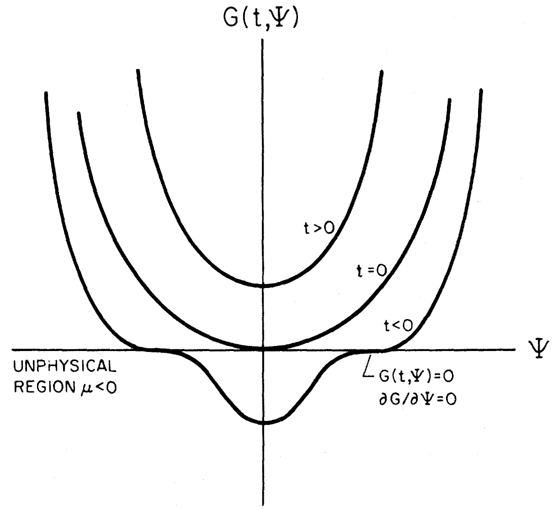

FIG. 1. Representation of $G(t, \Psi) = \Psi^6(1+\tfrac{3}{2}x)^3$ lines of constant temperature in the vicinity of the critical point. Above T_c, the function is bounded asymptotically by the value along the critical isotherm $G(0, \Psi) = \Psi^6$. The locus of the intercept $\partial_\Psi G(t<0, \Psi>0)$ determines the coexistence curve and the amount of ordering present. The region of $G<0$ is unstable and excluded by requiring $\mu \leq 0$ always (see Sec. II of text).

$$\delta\Psi_{\vec{k}}/\Psi_0 = \delta\alpha_{\vec{k}} + i\delta\beta_{\vec{k}},$$

it can be shown that

$$\frac{1}{\Psi_0^2}\frac{\partial^2 G}{\partial\alpha_{\vec{k}}\partial\alpha_{\vec{k}'}} = \frac{1}{\Psi_0^2}\frac{\partial^2 G}{\partial\beta_{\vec{k}}\partial\beta_{\vec{k}'}}$$

$$= 2\delta_{\vec{k}\vec{k}'}[3(\Psi^2)^2(1+\tfrac{3}{2}x)^2 - 2\beta\epsilon_{\vec{k}}] \tag{37}$$

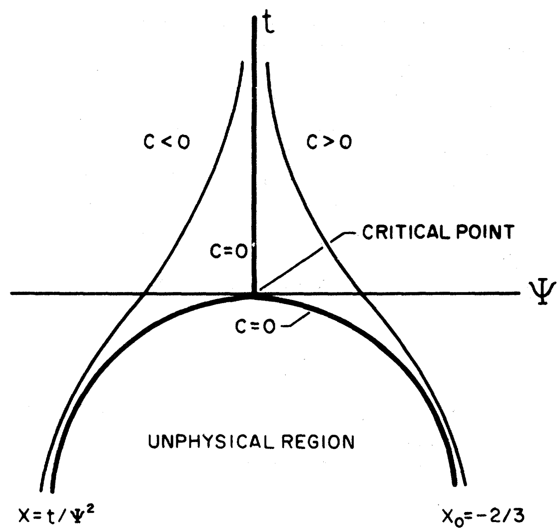

FIG. 2. Schematic representation of order parameter Ψ versus reduced temperature t for various values of the applied field C/C_0 together with lines at constant $x = t\Psi^{-2}$. The coexistence curve is given by the parabola of $x_0 = -\tfrac{2}{3}$.

with $\partial^2 G/\partial\alpha_{\vec{k}}\partial\beta_{\vec{k}'}=0$. The inverse of these second derivatives defines two orthogonal susceptibilities χ_α, χ_β everywhere equal and displaying a nonclassical form of the $\vec{k}=0$ susceptibility diverging as t^{-2} ($\gamma=2$) at the transition.

The complete set of critical indices $(-1, \frac{1}{2}, 2, 5)$ satisfy the thermodynamic inequalities as exact equalities

$$2-\alpha=\beta(\delta+1)=2\beta+\gamma.$$

These values correspond to those found for the Spherical Model of Ferromagnetism.[10] A discussion of the similarities between the Bose condensation and the Spherical Model is contained in the recent work of Gunton and Buckingham[2] and Langer.[3]

The complete expression for the equation of state in the vicinity of the transition is given by Eq. (32). Deviations from the scaled homogeneous form are given by the general term $\Psi^{2s+1}H_s(x)$. Thus the corrections along the critical isotherm ($t=0$, $x=0$) form a power series in Ψ^2 while those along the critical "isochore" ($\Psi=0$, $t<0$) a series in t. It is not possible to provide any further prescription as to corrections for general x since the roots of the polynomials $H_s(x)$ are all different. Along the coexistence curve, $x_0=-\frac{2}{3}$, where both $H_2(x_0)=H_3(x_0)=0$, the first nonvanishing contribution enters as $H_4(x)$ and is of order $(\frac{13}{12})^2\Psi^9$. The correction to the coexistence curve x_0 will vanish like $\frac{13}{36}\Psi^2$ as the critical point is approached.

V. EXPECTATION VALUES

In this section we consider the thermal expectation value for the ordered product of the field operators

$$\langle\psi^\dagger(\vec{r}_1)\psi^\dagger(2)\cdots\psi^\dagger(M)\psi(1')\cdots\psi(N')\rangle_\beta .$$

These quantities contain information about the spatial correlations of the Bose field and may be related to the occurence of the spontaneous-order parameter Ψ below the transition. In view of a recent conjecture by MSG,[7] it is of particular interest to determine the behavior for these expectation values in the vicinity of the critical point.

It is most convenient to work in the Fourier transform space of the creation and annihilation operators by first defining a matrix $\mathfrak{A}$ whose elements are given by the sequence of direct products

$$\mathfrak{A}(M|N)\equiv\langle\prod_{\mu=1}^{M}a_{\vec{k}_\mu}^{\dagger}\prod_{\nu=1}^{N}a_{\vec{k}_\nu}\rangle_\beta. \tag{38}$$

Products of this form are most easily found within the diagonal $\vec{A}$ representation, where the usual techniques of operator algebra find

$$\langle A_{\vec{k}_\mu}^{\dagger}\rangle=\langle A_{\vec{k}_\mu}\rangle=0,$$

$$\langle\prod_{\mu=1}^{M}A_{\vec{k}_\mu}^{\dagger}\prod_{\nu=1}^{N}A_{\vec{k}_\nu}\rangle=\sum_{\substack{0\subseteq\mu\subseteq M\\ 0\subseteq\nu\subseteq N}}\prod_{\substack{\tilde\mu\subseteq\mu\\ \tilde\nu\subseteq\nu}}\delta(\tilde\mu|\tilde\nu)b(\tilde\mu). \tag{39}$$

The $\delta(m|n)$ denotes a generalized delta function in $\vec{k}$ space over the product of all possible pairwise permutations containing elements from the sets $[m]$, $[n]$, and $b(m)$ is the usual boson distribution factor

$$(z^{-1}\exp\beta\epsilon(\vec{k}_m)-1)^{-1}.$$

Using the linear algebraic relations of Eq. (6) to transform back into the $\vec{A}$ representation, together with the restriction to uniform external $C(\vec{r})$, the general matrix element becomes

$$\mathfrak{A}(M|N)=\sum_{\substack{0\subseteq\mu\subseteq M\\ 0\subseteq\nu\subseteq N}}\prod_{\substack{\tilde\mu\subseteq\mu\\ \tilde\nu\subseteq\nu}}\delta(\tilde\mu|\tilde\nu)b(\tilde\mu)$$
$$\times(-\beta c_0/\ln z)^{M-\mu+N-\nu}\delta_{\vec{k}_M 0}\delta_{\vec{k}_N 0}. \tag{40}$$

Here we note that the expectation value for the single boson-field operator $\psi^\dagger(\vec{r})$ or $\psi(\vec{r})$ obtained by the Fourier inversion of a single creation/annihilation operator is given by

$$\langle\psi^\dagger(\vec{r})\rangle=V^{-1/2}\sum_{\vec{k}}e^{-i\vec{k}\vec{r}}\langle a_{\vec{k}}^{\dagger}\rangle=-\beta c_0^{*}/\ln z. \tag{41}$$

The quantity $\beta c_0/\ln z$ is, however, just the square root of the condensate number density Ψ^2 discussed in the preceding sections. Thus as the field $C(\vec{r})$ tends to zero in the single-phase region $z<1$, the expectation value of the boson field $\langle\psi\rangle$ vanishes identically. However, within the condensed phase ($\mu\to0$, $z=1$ line) along the coexistence curve, $\langle\psi\rangle$ remains finite as the order parameter exhibits a nonvanishing macroscopic value up to the critical temperature T_c even in the absence of any external field, $C\to0$.

Preforming the $(M+N)$-fold Fourier inversion of $\mathfrak{A}(M|N)$, one finds the expectation value for the product of boson-field operators in the coordinate space

$$B(M|N)\equiv\langle\psi^\dagger(\vec{r}_1)\psi^\dagger(2)\cdots\psi^\dagger(M)\psi(1')\cdots\psi(N')\rangle_\beta$$
$$=\sum_{\substack{0\subseteq\mu\subseteq M\\ 0\subseteq\nu\subseteq N}}\prod_{\substack{\tilde\mu\subseteq\mu\\ \tilde\nu\subseteq\nu}}\delta(\tilde\mu|\tilde\nu)b(\vec{r}_{\tilde\mu}-\vec{r}_{\tilde\nu})\Psi^{M-\mu+N-\nu}. \tag{42}$$

Each $b(\vec{r}_m-\vec{r}_n)$ is the Fourier transform into the coordinate space of the boson distribution function

$$b(\vec{R})=\langle\psi^\dagger(\vec{r}+\vec{R})\psi(\vec{r})\rangle$$
$$=\int d\vec{k}\,e^{-i\vec{k}\vec{R}}[z^{-1}\exp\tfrac{1}{2}\beta k^2-1]^{-1}. \tag{43}$$

In d dimensions, evaluation of this integral can be shown to involve the Hankel transform[13]

$$\int_0^\infty dk\,k^{d/2}J_{\frac{1}{2}d-1}(kR)[z^{-1}\exp\tfrac{1}{2}\beta k^2-1]^{-1}, \tag{44}$$

where $J_s(x)$ is the s-order Bessel function of the first type. As $R\to0$, this integral reduces to the local number density of particles

$$b(R\to0)=\langle\psi^\dagger(\vec{r})\psi(\vec{r})\rangle=(2\pi T)^{d/2}F_{d/2}(z). \tag{45}$$

For very large spatial separations, $\vec{R} \gg 0$, an asymptotic evaluation of the integral expression finds[13]

$$b(R \gg 0) \sim (2zT)R^{1-\frac{1}{2}d}[2T(1-z)R^2]^{\frac{1}{2}(\frac{1}{2}d-1)} \times K_{\frac{1}{2}d-1}\{[2T(1-z)R^2]^{\frac{1}{2}}\} \quad (46)$$

with $K_s(x)$ the modified Bessel function of order s.

Thus in three dimensions ($d=3$) the boson-field correlation function $b(R)$ may be written as $b(R \gg 0) \sim R^{-1/2}\mathcal{G}_3(y)$, where the argument $y=(1-z)R^2$ depends on the thermodynamic state through the fugacity z. Above the transition ($z<1$), the function $\mathcal{G}_3(y)$ decays rapidly as

$$\mathcal{G}_3(y \gg 0) \sim y^{-1/4} \exp -(2Ty)^{1/2} \quad (47)$$

so that the correlation exhibits an Ornstein-Zernike type of behavior

$$b(R) \sim R^{-1} \exp[-(1-z)^{1/2}R].$$

For temperatures less than T_c, where $z=1$, function $\mathcal{G}_3(y=0)$ is constant and therefore independent of the spatial separation.[13]

In the absence of any external field, above the critical temperature ($T>T_c$), there is no spontaneous order parameter as Ψ is identically zero. Thus the only nonvanishing contributions in Eq. (42) arise from the empty sets $M-\mu=N-\nu=0$ and the $B(M|N)$ decompose into a product over-all permutations of the boson factors

$$B_+(M|N) = \sum_{\substack{\mu\subseteq M\\ \nu\subseteq N}} \prod_{\substack{\tilde{\mu}\subseteq\mu\\ \tilde{\nu}\subseteq\nu}} \delta(\tilde{\mu}|\tilde{\nu})b(\vec{r}_{\tilde{\mu}}-\vec{r}_{\tilde{\nu}}). \quad (48)$$

The expression for $B(M|N)$ below T_c along the coexistence curve, where Ψ is finite, is given by Eq. (42) which we denote by $B_-(M|N)$.

VI. CRITICAL EIGENVECTOR

The thermal expectation value for the ordered product of the boson-field operators

$$\langle\psi^\dagger(1)\cdots\psi^\dagger(M)\psi(1')\cdots\psi(N')\rangle_\beta$$

is different on either side of the coexistence curve (above and below T_c) with the difference being given by

$$\Delta B(M|N) = B_-(M|N) - B_+(M|N),$$

where the subscripts refer to the two phases. At the critical point, the properties of the coexisting phases become identical and thus the quantity $\Delta B(M|N)$ must itself vanish. The difference of the expectation value for the single Bose-field operator $\Delta\langle\psi\rangle$ is just the spontaneous-order parameter Ψ which also goes to zero as the transition point is approached from below with T tending to T_c. In fact, from Eqs. (42) and (48), it may be seen that the difference ΔB behaves like the order parameter Ψ,

$$\Delta B(M|N) = \sum_{\substack{0\subseteq\mu\subseteq M\\ \nu=N}} \prod_{\substack{\tilde{\mu}\subseteq\mu\\ \tilde{\nu}\subseteq N}} \delta(\tilde{\mu}|\tilde{\nu})b(\vec{r}_{\tilde{\mu}}-\vec{r}_{\tilde{\nu}})\Psi^{M-\mu}$$

$$+ \sum_{\substack{0\subseteq\nu\subseteq N\\ \mu=M}} \prod_{\substack{\tilde{\mu}\subseteq M\\ \tilde{\nu}\subseteq\nu}} \delta(\tilde{\mu}|\tilde{\nu})b(\vec{r}_{\tilde{\mu}}-\vec{r}_{\tilde{\nu}})\Psi^{N-\nu}$$

$$+ \sum_{\substack{0\subseteq\mu\subset M\\ 0\subseteq\nu\subset N}} \prod_{\substack{\tilde{\mu}\subseteq M-1\\ \tilde{\nu}\subseteq N-1}} \delta(\tilde{\mu}|\tilde{\nu})b(\vec{r}_{\tilde{\mu}}-\vec{r}_{\tilde{\nu}}) \times \Psi^{M-\mu+N-\nu}. \quad (49)$$

Thus their limiting ratio

$$\mathcal{V}(M|N) \equiv \lim_{\Psi\to 0} \frac{\Delta B(M|N)}{\Psi}$$

$$= \delta(M-1|N) \prod_{\substack{\tilde{\mu}\subseteq\mu=M-1\\ \tilde{\nu}\subseteq\nu=N}} b(\vec{r}_{\tilde{\mu}}-\vec{r}_{\tilde{\nu}})$$

$$+ \delta(M|N-1) \prod_{\substack{\tilde{\mu}\subseteq\mu=M\\ \tilde{\nu}\subseteq\nu=N-1}} b(\vec{r}_{\tilde{\mu}}-\vec{r}_{\tilde{\nu}})$$

$$+ \delta(M\pm 2|N)O(\Psi) \quad (50)$$

exists and is finite (nonzero) for an odd number of elements $M=N\pm 1$, while vanishing of $O(\Psi)$ for an even $M=N$.

It has been conjectured by Green[7] that along the coexistence curve, the direct correlation matrix has an eigenvector whose components behave like the difference of the multiparticle distribution functions across the phase boundary. Thus, at the critical point because of the identity of the two phases this eigenvector has zero value. (See the defining remarks of Eq. (53) below.) The results of the previous paragraph substantiate these more general ideas about the behavior of the appropriate averages in the neighborhood of the transition for this model system. And, the very existence of the limiting ratio $\mathcal{V}(M|N)$ supports the characterization of the critical-point properties in terms of the critical eigenvector.

From the structure of the expression for the correlation function $B(M|N)$, [see Eq. (42)] which contains the sum over all possible permutations of the boson factors or bonds, $b(\vec{r}_{mn})$ it follows that the dominant spatial behavior is determined by that $\vec{r}_{mn}$ which represents the minimal separation between members of the groups. (This is equivalent to noting that it is just the direct two-body correlation function between closest members of separated groups which determines the asymptotic correlation between groups of particles.) If the groups $[M]$ and $[N]$ are divided into distinct, widely separated subsets $[M'+M'']$, $[N'+N'']$, respectively, where $[M']$, $[N']$, and $[M''][N'']$ are close, then the general $B(M|N)$ may be written as a sum of terms involving the products of bonds between $[M'][N']$, $[M''][N'']$, and $[M'][N'']$, $[M''][N']$,

$$B(M'\oplus M''|N'\oplus N'')$$

$$= \Bigg(\sum_{\substack{0\subseteq\mu'\subseteq M'\\ 0\subseteq\nu'\subseteq N'}} \prod_{\substack{\tilde{\mu}'\subseteq\mu'\\ \tilde{\nu}'\subseteq\nu'}} \delta(\tilde{\mu}'|\tilde{\nu}')b(\vec{r}_{\tilde{\mu}'}-\vec{r}_{\tilde{\nu}'}) \times \Psi^{M'-\mu'+N'-\nu'}$$

$$\times \sum_{\substack{0\subseteq\mu''\subseteq M''\\ 0\subseteq\nu''\subseteq N''}} \prod_{\substack{\tilde\mu''\subseteq\mu''\\ \tilde\nu''\subseteq\nu''}} \delta(\tilde\mu''|\tilde\nu'')b(\vec{r}_{\tilde\mu''}-\vec{r}_{\tilde\nu''}) \times \Psi^{M''-\mu''+N''-\nu''}\Bigg)$$

$$+\Bigg(\sum_{\substack{0\subseteq\mu'\subseteq M'\\ 0\subseteq\nu''\subseteq N''}} \prod_{\substack{\tilde\mu'\subseteq\mu'\\ \tilde\nu''\subseteq\nu''}} \delta(\tilde\mu'|\tilde\nu'')b(\vec{r}_{\tilde\mu'}-\vec{r}_{\tilde\nu''}) \times \Psi^{M'-\mu'+N''-\nu''}$$

$$\times \sum_{\substack{0\subseteq\mu''\subseteq M''\\ 0\subseteq\nu'\subseteq N'}} \prod_{\substack{\tilde\mu''\subseteq\mu''\\ \tilde\nu'\subseteq\nu'}} \delta(\tilde\mu''|\tilde\nu')b(\vec{r}_{\tilde\mu''}-\vec{r}_{\tilde\nu'}) \times \Psi^{M''-\mu''+N'-\nu'}\Bigg). \tag{51}$$

The first bracket on the rhs contains only those terms which arise from bonds between the groups $[M']$, $[N']$, and $[M'']$ $[N'']$ and thus represents the direct product of $B(M'|N')\cdot B(M''|N'')$. The second set of terms contains factors between the asymptotically far elements $[M']$, $[N'']$, and $[M'']$ $[N']$ and since each bond is a decreasing function of the relative separation, it follows that the dominant behavior is determined by that single bond which represents the minimum distance between $[M']$ $[N'']$ or $[M'']$ $[N']$. The remaining elements of the corresponding subsets must therefore have $(M'-1=N')$, $(M''=N''-1)$ or $(M'=N'-1)$, $(M''-1=N'')$ members, as is easily seen from the pictorial representation for Eq. (51) given in Fig. 3. These terms, however, represent the critical eigenvectors for the separate subsets, $\mathcal{V}(M|N')$ and $\mathcal{V}(M''|N'')$. Thus we find that $B(M|N)$ has the simple asymptotic property

$$B(M'\oplus M''|N'\oplus N'')\sim B(M'|N')B(M''|N'') + b(\min|\vec{r}'-\vec{r}''|)\mathcal{V}(M'|N')\mathcal{V}(M''|N''). \tag{52}$$

An analogous result has been proposed by Green[15] for the molecular distribution functions of a fluid near its critical point. Here the critical eigenvector is defined as the limiting ratio of the difference in the n-particle distribution function to that of the single-particle (density) distribution between the two coexisting phases

$$\mathcal{V}(n)\equiv \lim_{\Delta f(1)\to 0}\frac{f_L(\{n\})-f_V(\{n\})}{f_L(\{1\})-f_V(\{1\})}. \tag{53}$$

For a system containing $[M+N]$ particles, where the group $[M]$ is far removed from $[N]$, the difference between $f(\{M\oplus N\})$ and the product of the individual group distribution functions $f(\{M\})\cdot f(\{N\})$ has the asymptotic property

$$f(\{M\oplus N\})-f(\{M\})\cdot f(\{N\}) \sim \mathcal{V}(M)\mathcal{G}(\min|r_M-r_N|)\mathcal{V}(N). \tag{54}$$

$\mathcal{G}(\vec{r}_{MN})$ is the two-particle correlation function between least-distant members of the two sets, while $\mathcal{V}(M)$ and $\mathcal{V}(N)$ denote the critical eigenvectors for the separate groups of particles.

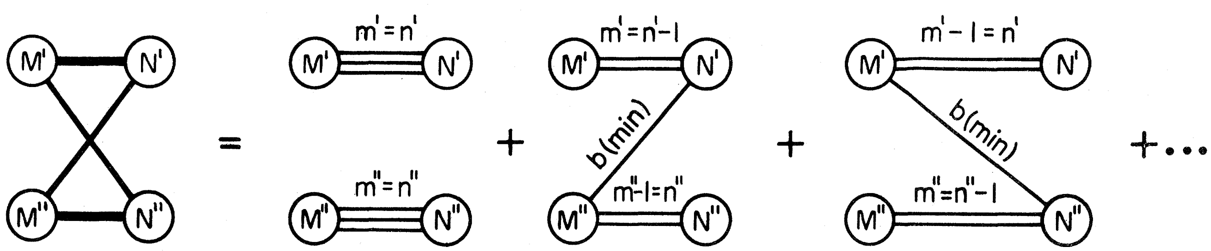

FIG. 3. Diagrammatic representation for the distribution function $B(M|N)$ between disjointed subsets of M and N.

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Conductivity of a Plasma in a Steady Magnetic Field. II

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With the aid of techniques used in our previous paper I, we prove, without recourse to any approximation, that $k_\mu k_\nu \sigma_{\mu\nu}(\vec{k},\omega)/k^2$ is independent of the magnetic field strength.

I. INTRODUCTION

The magnetic field dependence of the complex-conductivity tensor of a magnetoplasma was studied by us in a recent paper[1] (herein referred to as I) by employing the Kubo formalism. We considered the "frequency-dependent" conductivity tensor $\sigma_{\mu\nu}(\omega)$ in that paper. The problem of the magnetic field dependence of the "wave number and frequency-dependent" complex-conductivity tensor $\sigma_{\mu\nu}(\vec{k},\omega)$ is of much greater importance, particularly in the study of the propagation of electromagnetic waves in a magnetoplasma. The problem here is far more difficult than was the $k=0$ case dealt with in I. The extraction of the explicit magnetic field dependence of $\sigma_{\mu\nu}(\vec{k},\omega)$ has not so far proved to be a success. However, answers have been obtained to simpler questions such as how can the invariant (rotationally) of $\sigma_{\mu\nu}(\vec{k},\omega)$ that is independent of magnetic field strength be found. From a study of the problem in the Vlassov approximation, it is known that $k_\mu k_\nu \sigma_{\mu\nu}(\vec{k},\omega)/k^2$, i.e.,[2] σ_l is independent of the magnetic field strength if $\vec{k}$ is parallel to the direction of the magnetic field. It is also possible to obtain this result, without recourse to any approximation, from the fact that the flow of electric current is not affected by imposing a steady magnetic field in the direction of the flow of current.[1] In the present paper, we wish to show by using Kubo formalism that even when $\vec{k}$ is not parallel to the magnetic field[3] the invariant $k_\mu k_\nu \sigma_{\mu\nu}(\vec{k},\omega)/k^2$ is independent of the magnetic field.

II. DERIVATION OF THE RESULT

We consider a fully ionized homogeneous plasma with electrons moving against a fixed, neutralizing, smeared-out, positive-ion background. The conductivity tensor $\sigma_{\mu\nu}(\vec{k},\omega)$ is the Fourier transform of the response function $K_{\mu\nu}(\vec{x},\omega)$, defined by the nonlocal relation