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Crystalline Order in Two Dimensions*

N. D. Mermin†
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York

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If N classical particles in two dimensions interacting through a pair potential $\Phi(\vec{r})$ are in equilibrium in a parallelogram box, it is proved that every $\vec{k}\neq 0$ Fourier component of the density must vanish in the thermodynamic limit, provided that $\Phi(\vec{r}) - \lambda r^2 |\nabla^2\Phi(\vec{r})|$ is integrable at $r=\infty$ and positive and nonintegrable at $r=0$, both for $\lambda=0$ and for some positive λ. This result excludes conventional crystalline long-range order in two dimensions for power-law potentials of the Lennard-Jones type, but is inconclusive for hard-core potentials. The corresponding analysis for the quantum case is outlined. Similar results hold in one dimension.

Over thirty years ago Peierls[1] and Landau[2] concluded that there can be no one- or two-dimensional crystalline long-range order. Computer experiments, however, have since indicated a transition to a two-dimensional crystalline ordered state,[3] casting some doubt on the validity of these results in two dimensions. The arguments are not unassailable. Peierls gave a qualitative argument for the general one-dimensional case, and a quantitative argument (only in one dimension, but the extension to two is obvious) based on the harmonic approximation. Landau, on the other hand, used his general theory of second-order phase transitions, which is known to be misleading near the critical point.

Within the past two years a series of exact arguments have appeared, extending Hohenberg's[4] use of Bogoliubov's[5] inequality to exclude ordering in a variety of one- and two-dimensional systems under very general conditions. In this paper such techniques will be used to place the Peierls-Landau argument on a rigorous basis for a large class of interparticle interactions without making either the harmonic approximation or an order-parameter expansion.

The application of the Bogoliubov inequality to crystalline ordering follows the general pattern of earlier applications, but the extension is not completely straightforward. This is because earlier applications have either been to lattice models in which the structure of the Hamiltonian insures that a given spin interacts only with a finite number of others (for finite-range forces), or to types of ordering (superfluid, superconducting, itinerant electron magnetic, or excitonic) in which the energies of order-destroying fluctuations are purely kinetic and therefore independent of the number of particles interacting with a given one. In the crystalline case the crucial energy is potential, and there is no guarantee that a given particle will not diffuse through the crystal, eventually interacting with all the others. This possibility makes a rigorous analysis more difficult, and requires the pair potential to satisfy restrictions that have not arisen in other applications.[6]

Another complication in the crystalline case is the variety of not necessarily equivalent choices one has in selecting a criterion for the presence of ordering.

For these reasons it is worth recording one form of the full argument, in spite of its many similarities to other applications of Bogoliubov's inequality. To keep algebraic complications to a minimum we consider in detail only classical crystals. The generalization to the quantum case is briefly described in the Appendix.

Consider, then, classical particles interacting through a pair potential $\Phi(\vec{r})$. The form of the argument depends on whether one uses periodic or impenetrable wall boundary conditions. Impenetrable wall conditions will be used here for two reasons: existence theorems for the free energy will be required which have only been proved in the impenetrable wall case; furthermore enclosing the system within fixed walls provides a convenient mechanism for localizing and orienting the supposed crystal, which must be supplied by suitable fictitious forces in the periodic case.[7] Thus if the particles are supposed to form a crystal with Bravais lattice generated by $\vec{a}_1$, $\vec{a}_2$, we take the box spanned by $N_1\vec{a}_1$, $N_2\vec{a}_2$ (i.e., the set of points $\vec{r}=x_1N_1\vec{a}_1+x_2N_2\vec{a}_2$, $0\leq x_1$, $x_2\leq 1$) and fill it with $N=nN_1N_2$ particles, where n is the number of particles per unit cell.

We define the kth Fourier component of the density by

$$\rho_{\vec{k}} = N^{-1}\langle\hat{\rho}_{\vec{k}}\rangle\,, \tag{1}$$

where

$$\hat{\rho}_{\vec{k}} = \int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}\hat{\rho}(\vec{r}) = \sum_{i=1}^{N} e^{-i\vec{k}\cdot\vec{r}_i}\,, \tag{2}$$

$$\hat{\rho}(\vec{r}) = \sum_{i=1}^{N}\delta(\vec{r}-\vec{r}_i)\,, \tag{3}$$

$$\langle f\rangle = \int d\vec{r}_1\cdots d\vec{r}_N\, e^{-\beta U(\vec{r}_1\cdots\vec{r}_N)} f(\vec{r}_1\cdots\vec{r}_N)/Q,$$

$$Q = \int d\vec{r}_1\cdots d\vec{r}_N\, e^{-\beta U(\vec{r}_1\cdots\vec{r}_N)}, \tag{4}$$

and

$$U(\vec{r}_1\cdots\vec{r}_N) = \frac{1}{2}\sum_{i\neq j}\Phi(\vec{r}_i-\vec{r}_j)\,. \tag{5}$$

The integrations are all over the interior of the box.

If $\vec{k}$ is a vector of the lattice reciprocal to that generated by $\vec{a}_1$, $\vec{a}_2$, then $\rho_{\vec{k}}$ will in general be nonzero, whereas it will vanish in the thermodynamic limit if $\vec{k}$ is not a reciprocal-lattice vector. (This is simply the statement that in the thermodynamic limit the density is a periodic function with the periodicity of the direct lattice; the assumption is that the effects of deviations from periodicity at the surface become of negligible significance in the thermodynamic limit.) We therefore adopt as our criterion for crystallinity the following:

$$\begin{aligned}&\lim \rho_{\vec{k}} = 0, \ \vec{k} \text{ not a reciprocal-lattice vector;}\\ &\lim \rho_{\vec{k}} \neq 0, \ \text{for at least one nonzero reciprocal-lattice vector } \vec{k}, \end{aligned} \tag{6}$$

where lim denotes the thermodynamic limit $N_1, N_2 \to \infty$.

We now prove that (6) cannot be satisfied in two (and by the obvious analogous argument, one) dimensions. The proof is based on the following inequality, which holds for continuous differentiable functions $\psi(\vec{r})$ and $\varphi(\vec{r})$, provided $\varphi(\vec{r})$ vanishes when $\vec{r}$ is on the surface of the box

$$\langle |\sum \psi_i|^2 \rangle \geqslant \frac{k_B T |\sum \langle \varphi_i \vec{\nabla} \psi_i \rangle|^2}{\langle \frac{1}{2} \sum \nabla^2 \Phi_{ij} |\varphi_i - \varphi_j|^2 + k_B T \sum |\vec{\nabla} \varphi_i|^2 \rangle} . \tag{7}$$

Here $\varphi_i = \varphi(\vec{r}_i)$, $\psi_i = \psi(\vec{r}_i)$, $\Phi_{ij} = \Phi(\vec{r}_i - \vec{r}_j)$, and all indices are summed from 1 to N.

The derivation of (7), which is closely related to Bogoliubov's inequality, is elementary and has been given elsewhere.[8] Briefly, (7) is just the Schwartz inequality

$$\langle |A|^2 \rangle \geqslant |\langle A^* \vec{B} \rangle|^2 / \langle |\vec{B}|^2 \rangle \tag{8}$$

in which

$$A(\vec{r}_1 \cdots \vec{r}_N) = \sum \psi_i ,$$

$$\vec{B}(\vec{r}_1 \cdots \vec{r}_N) = -k_B T e^{\beta U} \sum_{i=1}^{N} \vec{\nabla}_i (\varphi_i e^{-\beta U}) .$$

An integration by parts in the numerator on the right side of (8), and two integrations by parts in the denominator (in all of which the surface terms vanish due to the vanishing of φ on the surface) lead directly to (7). (For details see Ref. 8.)

Now choose

$$\psi(\vec{r}) = e^{-i(\vec{k} + \vec{K}) \cdot \vec{r}}, \quad \varphi(r) = \sin \vec{k} \cdot \vec{r} , \tag{9}$$

where $\vec{K}$ is the reciprocal-lattice vector for which $\rho_{\vec{K}}$ is claimed not to vanish, and $\vec{k}$ is of the form

$$\vec{k} = n_1 \vec{b}_1 / N_1 + n_2 \vec{b}_2 / N_2, \quad \vec{b}_i \cdot \vec{a}_j = 2\pi \delta_{ij} ,$$

so that φ vanishes on the surface as required. Eq. (7) then becomes

$$\begin{aligned}\langle \rho_{\vec{K}+\vec{k}} \rho_{-\vec{K}-\vec{k}} \rangle \geqslant{}& \tfrac{1}{4} k_B T (\vec{k} + \vec{K})^2 |\langle \rho_{\vec{K}} - \rho_{\vec{K}+2\vec{k}} \rangle|^2 \\ &\times [\tfrac{1}{2} \sum \langle \nabla^2 \Phi_{ij} (\sin \vec{k} \cdot \vec{r}_i - \sin \vec{k} \cdot \vec{r}_j)^2 \rangle \\ &+ k_B T k^2 \langle \sum \cos^2 \vec{k} \cdot \vec{r}_i \rangle]^{-1} \end{aligned} \tag{10}$$

The denominator on the right side of (10) is increased by replacing $\cos^2 \vec{k} \cdot \vec{r}_i$ by 1, and using the inequality

$$(\sin \vec{k} \cdot \vec{r}_i - \sin \vec{k} \cdot \vec{r}_j)^2 \leqslant k^2 (\vec{r}_i - \vec{r}_j)^2 ,$$

which leads to

$$\begin{aligned}&(1/N) \langle \rho_{\vec{K}+\vec{k}} \rho_{-\vec{K}-\vec{k}} \rangle \\ &\qquad \geqslant \frac{\frac{1}{4} k_B T (\vec{k} + \vec{K})^2 (\rho_{\vec{K}} - \rho_{\vec{K}+2\vec{k}})^2}{k^2 (k_B T + (1/2N) \sum \langle |\nabla^2 \Phi_{ij}| r_{ij}^2 \rangle)} .\end{aligned} \tag{11}$$

To complete the argument we must establish that the coefficient of k^2 in the denominator on the right side of (11) is bounded in the thermodynamic limit. (This is the problem peculiar to the crystalline case referred to at the start of this paper.) The following argument[9] yields a fairly general condition on Φ (which is, however, probably too restrictive at small r) which insures this boundedness:

Let F_λ be the configurational contribution to the free energy when the pair potential is taken to be

$$\Phi_\lambda(\vec{r}) = \Phi(\vec{r}) - \lambda r^2 |\nabla^2 \Phi(\vec{r})| ; \tag{12}$$

i.e.,

$$-\beta F_\lambda = \ln \int d\vec{r}_1 \cdots d\vec{r}_N e^{-\beta(U - \lambda \Delta)} , \tag{13}$$

where

$$\Delta = \frac{1}{2} \sum_{i \neq j} |\vec{r}_i - \vec{r}_j|^2 |\nabla^2 \Phi(\vec{r}_i - \vec{r}_j)| . \tag{14}$$

Then

$$-\partial F / \partial \lambda = \langle \Delta \rangle_\lambda \geqslant 0 , \tag{15}$$

$$\text{and } (\partial/\partial\lambda) \langle \Delta \rangle_\lambda = \beta \langle (\Delta - \langle \Delta \rangle_\lambda)^2 \rangle_\lambda \geqslant 0 , \tag{16}$$

where the subscript λ indicates the average is in an ensemble interacting through Φ_λ. Equations (15) and (16) imply

$$F_0 - F_\lambda = \int_0^\lambda \langle \Delta \rangle_\mu d\mu \geqslant \lambda \langle \Delta \rangle_0 \geqslant 0 , \tag{17}$$

or, equivalently,

$$(F_0 - F_\lambda)/N\lambda \geqslant (1/2N) \sum \langle |\nabla^2 \Phi_{ij}| r_{ij}^2 \rangle \geqslant 0 . \tag{18}$$

Therefore provided

$$\lim (F_\lambda / N) = f_\lambda < \infty \tag{19}$$

for $\lambda = 0$ and for some $\lambda > 0$, we can replace the denominator on the right side of (11) by a larger constant which will, for sufficiently large N, be very close to

$$k^2[\beta^{-1}+(f_0-f_\lambda)/\lambda]. \qquad (20)$$

Now (19) is just the condition that the free energy per particle for the pair potential Φ_λ be intensive in the thermodynamic limit. A sufficient condition for this is that the potential be integrable at infinity and repulsive and nonintegrable at the origin.[10] Thus (19) will hold, provided

$$\Phi_\lambda(r)\sim 1/r^{2+|\epsilon|}, \quad r\to\infty;$$

$$\Phi_\lambda(r)>|A|/r^{2+|\epsilon|}, \quad r\to 0\ . \qquad (21)$$

Most potentials one is likely to consider will satisfy (21) for $\lambda=0$, so the pertinent conditions for the absence of crystalline ordering in two dimensions are

$$\nabla^2\Phi\sim 1/r^{4+|\epsilon|}, \quad r\to\infty; \qquad (22)$$

$$\Phi(\vec{r})-\lambda r^2|\nabla^2\varphi(\vec{r})|>|A|/r^{2+|\epsilon|},$$

$$r\approx 0, \text{ for some } \lambda>0\ . \qquad (23)$$

If Φ satisfies these conditions, the argument that $\rho_{\vec{K}}$ vanishes in the thermodynamic limit is completed as follows:

Multiply both sides of (11) by $g(\vec{k}+\vec{K})$, where $g(\vec{q})$ is a positive Gaussian centered at $q=0$, divide by the volume of the system, and sum over k. We can replace the right side, which is term by term positive, by a summation restricted to values of $\vec{k}$ less than half the smallest reciprocal-lattice vector $\vec{K}_0$; for these $\rho_{\vec{K}+2\vec{k}}$ vanishes in the thermodynamic limit by the first criterion (6). We thus have

$$\frac{1}{V}\sum_{\vec{q}} g(q)\langle\rho_{\vec{q}}\rho_{-\vec{q}}\rangle/N$$

$$\geqslant\frac{\frac{1}{16}k_BTK^2g(\frac{1}{2}K_0)\rho_{\vec{K}}^2}{k_BT+(f_0-f_\lambda)/\lambda}\ \frac{1}{V}\sum_{k<\frac{1}{2}K_0}\frac{1}{k^2} \qquad (24)$$

Now if

$$\Delta\Phi(r)=\int[d\vec{q}/(2\pi)^2]g(\vec{q})e^{i\vec{q}\cdot\vec{r}}\ ,$$

then the right side of (24) is bounded in the thermodynamic limit by

$$\Delta\Phi(0)+\frac{1}{N}\sum_{i\neq j}\langle\Delta\Phi(\vec{r}_i-\vec{r}_j)\rangle \qquad (25)$$

But[11] the second term in (25) is bounded by the difference in free energies per particle for the pair potentials Φ and $\Phi-\Delta\Phi$. Since if Φ satisfies the condition for the existence of a thermodynamic limit so will Φ minus a Gaussian, (25) is bounded in the thermodynamic limit.

However the sum over k on the right of (24) goes to an integral that diverges logarithmically at $k=0$ in the thermodynamic limit. Thus (24) can be satisfied only if $\rho_{\vec{K}}$ vanishes in this limit.

Evidently if N_1, N_2 are both of order $N^{1/2}$ one can also conclude for the large but finite system that

$$\rho_{\vec{K}}\lesssim 1/(\ln N)^{1/2}\ . \qquad (26)$$

The following further points are worth noting:

(a) Evidently a similar argument can be produced in one dimension. Conditions (22) and (23) become

$$\nabla^2\Phi\sim 1/r^{3+|\epsilon|}, \quad r\to\infty; \qquad (27)$$

$$\Phi(\vec{r})-\lambda r^2|\nabla^2\varphi(\vec{r})|>|A|/r^{1+|\epsilon|},$$

$$r\approx 0, \text{ for some } \lambda>0\ . \qquad (28)$$

Equation (26) becomes

$$\rho_{\vec{K}}\lesssim 1/\sqrt{N}\ . \qquad (29)$$

(b) The bounds (26) and (29) are weaker than the bounds one finds for a harmonic model:

$$\rho_{\vec{K}}\sim e^{-N\alpha K^2}, \quad 1 \text{ dimension}; \qquad (30)$$

$$\rho_{\vec{K}}\sim N^{-K^2\alpha}, \quad 2 \text{ dimensions}. \qquad (31)$$

(c) Note that the exact bound (26) may well be so weak as to allow two-dimensional systems of less than astronomic size to display crystalline order. This property is maintained in the harmonic case [Eq. (31)], where for reasonable temperatures and force constants αK^2 may well be of order 10^{-2} for the smallest reciprocal-lattice vectors.

(d) The weakness of the instability suggests that some kind of ordering may still be present. An example of this is provided by the two-dimensional harmonic lattice. If the supposed equilibrium sites are $\vec{R}=n_1\vec{a}_1+n_2\vec{a}_2$ and the actual instantaneous position of the ions $\vec{r}(\vec{R})=\vec{R}+\vec{u}(\vec{R})$, then the absence of long-range crystalline order is reflected in the divergence of the displacement autocorrelation function:

$$\langle[\vec{u}(\vec{R})-\vec{u}(\vec{R}')]^2\rangle\sim\ln|\vec{R}-\vec{R}'|, \quad |\vec{R}-\vec{R}'|\to\infty\ . \qquad (32)$$

This reveals that positional long-range order does not exist. However *directional* long-range order is transmitted infinitely far, as revealed by a calculation of

$$\langle[\vec{r}(\vec{R}+\vec{a}_1)-\vec{r}(\vec{R})]\cdot[\vec{r}(\vec{R}'+\vec{a}_1)-\vec{r}(\vec{R}')]\rangle\ . \qquad (33)$$

If the particles were frozen to their equilibrium sites this would just be a_1^2. If, due to thermal fluctuations there were no long-range propagation of the local crystalline orientation, it would vanish for large $|\vec{R}-\vec{R}'|$. In fact it approaches a_1^2 as

$|\vec{R}-\vec{R}'|\to\infty$, revealing that there is long-range orientational order in the two-dimensional harmonic net.

(e) Note that the conditions (22) and (23) are satisfied by potentials of the form

$$\Phi(\vec{r}) = |B|/r^m - |C|/r^n, \quad m > n > 2 + |\epsilon| \ . \tag{34}$$

The conditions also have the appealing feature that if one tries to help the crystal by putting a periodicity into the pair potential, e.g.,

$$\Phi(\vec{r}) \sim \cos\vec{K}\cdot\vec{r}/r^n \ , \tag{35}$$

then a more rapid falling off at large r, ($n > 4 + |\epsilon|$), is required to exclude the crystal. However the restriction (23) at small r is probably too stringent, excluding, for example, behavior of the of the form

$$\Phi(\vec{r}) \sim e^{|B|/r}, \quad r \sim 0 \ .$$

Also, of course, no conclusion can be drawn from these conditions for hard-core potentials.

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M. E. Fisher's insistence on the difficulty of specifying a criterion for crystalline ordering led me to discard several earlier versions of the argument. He is not to blame for the present one. I am also grateful to O. Penrose for showing me a way around a major difficulty in the analysis – a condition on Φ sufficient to guarantee that (11) have a finite denominator.

APPENDIX: OUTLINE OF QUANTUM THEOREM

To prove the analogous theorem quantum mechanically, start with the Bogoliubov inequality[12]:

$$\tfrac{1}{2}\langle AA^+ + A^+A\rangle \geqslant k_B T\,|\langle[\vec{C},A]\rangle|^2/\langle[[\vec{C},H],\vec{C}^+]\rangle \ . \tag{36}$$

Let

$$A = \textstyle\sum_i e^{i(\vec{k}+\vec{K})\cdot\vec{r}_i} = \rho_{\vec{k}+\vec{K}} \ , \tag{37}$$

and

$$\vec{C} = \tfrac{1}{2}\textstyle\sum_i [\vec{p}_i \sin\vec{k}\cdot\vec{r}_i + (\sin\vec{k}\cdot\vec{r}_i)\vec{p}_i] \ . \tag{38}$$

The particles are enclosed in a box, as before, which now requires that the N-body wave function vanish when any particle is on the surface; operating on such a wave function with A or $\vec{C}$ maintains this boundary condition, and they are therefore acceptable operators to place in (36).

The left side of (36) and the numerator of the right side have exactly the same form as in the classical inequality (10). The denominator reduces to

$$\begin{aligned}\langle[[\vec{C},H],\vec{C}^+]\rangle = (1/m)\textstyle\sum_i\langle(\vec{k}\cdot\vec{p}_i)^2 &+ k^2\vec{p}_i\circ(\cos\vec{k}\cdot\vec{r}_i)^2\vec{p}_i + \tfrac{1}{4}\hbar^2k^4\sin^2\vec{k}\cdot\vec{r}_i\rangle \\ &+ \tfrac{1}{2}\sum_{ij}\langle(\sin\vec{k}\cdot\vec{r}_i - \sin\vec{k}\cdot\vec{r}_j)^2\,\nabla^2\Phi_{ij}\rangle \\ &+ \tfrac{1}{4}\sum_{ij}\langle(\sin2\vec{k}\cdot\vec{r}_i - \sin2\vec{k}\cdot\vec{r}_j)(\vec{k}\cdot\vec{\nabla})\Phi_{ij}\rangle \ .\end{aligned} \tag{39}$$

The first ensemble average in (39) is bounded by

$$N(4t + \hbar^2k^2/4m)k^2 \ ,$$

where t is the kinetic energy per particle in equilibrium.[13] The second term is also bounded by a quantity of order Nk^2, the argument being the obvious generalization of the one given in the classical case, the generalization of Eq. (17) to the full quantum-mechanical free energy being provided by Peierls' theorem.[14] The final term in (39), which could be dealt with by a simple integration by parts in the classical case, can still be bounded by

$$\tfrac{1}{2}k^2\sum_{ij}\langle\,|\vec{r}_{ij}|\ |\vec{\nabla}\Phi_{ij}|\rangle \tag{40}$$

which can be proved to be bounded by something of order Nk^2 provided we add to (22) and (23) the conditions

$$|\nabla\Phi| \sim 1/r^{3+|\epsilon|}, \quad r\to\infty; \tag{41}$$

$$\Phi(\vec{r}) - \lambda r|\nabla\Phi(\vec{r})| > |A|/r^{2+|\epsilon|}, \quad r\to 0 \ . \tag{42}$$

The rest of the argument is exactly as in the classical case, the convergence of the left side of (34) being proved by another appeal to Peierls' theorem.

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[1] R. E. Peierls, Helv. Phys. Acta 7, 81 (1923); Ann. Inst. Henri Poincaré 5, 177 (1935).

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[3] B. J. Alder and T. E. Wainright, Phys. Rev. 127, 359 (1962).

[4] P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

[5] N. N. Bogoliubov, Phys. Abhandl. S. U. 6, 113 (1962).

[6] An analysis of the crystalline case which does not deal with this aspect of the problem has been constructed independently by G. Meissner.

[7] The reader who feels that we are only proving that walls alone cannot localize a two-dimensional crystal is invited to invent potentials that act through all of space, enforcing the required order locally. The thermodynamic limit of any Fourier component of the density can still be shown to be arbitrarily small for sufficiently weak localizing potentials, a result one feels strongly to be incompatible with genuine crystalline ordering.

[8] N. D. Mermin, Phys. Rev. 171, 272 (1968).

[9] This way around the difficulty, leading to conditions (22) and (23), was suggested to me by O. Penrose.

[10] M. E. Fisher, Arch. Rational Mech. Anal. 17,

337 (1964).

[11] This argument has been used by D. C. Mattis and O. Penrose (private communication) to make rigorous some of the magnetic applications of the Bogoliubov inequality.

[12] The angular brackets now denote full canonical ensemble averages, and the vector operators in the denominator of (36) are combined as in a scalar product. The proof of (36) is the obvious generalization of that given by N. D. Mermin and H. Wagner, Phys. Rev. Letters 17, 1133 (1966). The Bogoliubov inequality was first applied to crystals by H. Wagner, Z. Physik 195, 273 (1966).

[13] This is easily proved to be bounded in the thermodynamic limit for any potentials Φ for which the free energy per particle is bounded.

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Superfluid Phase Transitions in One and Two Dimensions

Allan Widom
Northeastern University, Boston, Massachusetts

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A recently proved theorem of statistical physics states that a one- or two-dimensional Bose liquid in homogeneous thermodynamic equilibrium cannot undergo Bose-Einstein condensation. The theorem does not apply to inhomogeneous liquids. This is shown by considering the ideal Bose liquid (i) in the presence of a gravitational field and (ii) in rotational motion. The existence of well-defined one- and two-dimensional phase transitions with Bose-Einstein condensation is proved by these explicit examples.

1. INTRODUCTION

The problem of one- and two-dimensional superfluidity is of considerable importance. Since three-dimensional superfluidity is closely connected to Bose-Einstein condensation,[1] it is important to know whether this connection exists in one and two dimensions. An inequality of Bogoliubov was used by Hohenberg[2] to rule out Bose-Einstein condensation in one- and two-dimensional liquids in homogeneous thermodynamic equilibrium. The question of Bose-Einstein condensation in inhomogeneous liquids was left unresolved. It turns out that an external disturbance which breaks translational invariance may cause Bose-Einstein condensation in one and two dimensions. The purpose of this paper is to give explicit examples of this fact. The ideal Bose liquid is considered (i) in the presence of an external gravitational force and (ii) in rotational motion. In each case, a well-defined one- or two-dimensional phase transition occurs accompanied by macroscopic occupation of a single-particle state.

2. METHOD OF CALCULATION

In the grand canonical ensemble the equations of state are computed from the partition function

$$Z_G=\mathrm{tr}[\exp-\beta(H-\mu N)]\,,\quad \beta=(kT)^{-1}\,. \tag{1}$$

For the ideal Bose liquid, the total Hamiltonian is the sum of single-particle Hamiltonians,

$$H=\sum_{i=1}^{N} h(i)\,, \tag{2}$$

and the energy spectrum is determined by the single-particle eigenvalue problem

$$h\psi_j(\vec{\mathrm{r}})=E_j\psi_j(\vec{\mathrm{r}})\,. \tag{3}$$

The partition function is given by

$$Z_G=\sum_{\{n_j\}}\exp-\beta\Big[\sum_j n_j(E_j-\mu)\Big]\,, \tag{4}$$

and the summation over the occupation numbers is easily performed,

$$\ln Z_G=-\sum_j \ln[1-\exp-\beta(E_j-\mu)]\,. \tag{5}$$

Expanding the right-hand side of the above equation in powers of the activity

$$\zeta=\exp(\beta\mu) \tag{6}$$

yields $$\ln Z_G=\sum_{s=1}^{\infty} s^{-1}\gamma(s\beta)\zeta^{s}\,, \tag{7}$$

where $$\gamma(t)=\sum_j \exp(-E_j t)\,. \tag{8}$$

The crux of the problem is to evaluate the right-hand side of Eq. (8) in the thermodynamic limit. This is accomplished by Green's-function methods.