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Helium Drag in the Theory of Impurity Mobility in Liquid Helium*

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A "helium-drag" effect is calculated for the mobility of charged impurities in He^3, and He^3-He^4 mixtures. For low enough temperatures, the momentum imparted by the impurities to the helium may be partly trapped in the impurity-helium system, thereby raising the conductivity. This effect gives rise to a term in the mobility proportional to the concentration of impurities, and varying as T^{-2} for the lowest temperatures. It is estimated that the impurity concentration for this effect to be observed is easily within experimental reach for He^3-He^4 mixtures, and possibly even for pure He^3.

I. INTRODUCTION

The purpose of the present paper is to demonstrate that at low temperatures there may exist a "drag" effect in the mobility of charged impurities in He^3 or He^3-He^4 mixtures. A number of papers both theoretical[1–4] and experimental[5] have appeared in which the mobility of carriers in He^3 or He^4 has been discussed, but none of these have considered the possibility of an effect arising from the host material, helium, not being able to get rid of all of the momentum delivered to it by the accelerating impurities. If the helium-helium interaction, or helium-boundary interaction is not strong enough, then the momentum increase in the impurities caused by the electric field is partially trapped in the impurity-helium system. The mobility increases when this happens. In this paper we discuss the consequences of this effect, which resembles closely the corresponding behavior of an electron-phonon system.[7–10]

The drag effect will obviously become more important when the number of scatterers decreases, since then the interaction between them gets smaller. If we wish to enhance the drag effect, we can consider He^3-He^4 mixtures, in which the He^3 percentage is small (say 5% or less, as has been considered by some workers[11]). The reason for this is that the mobility in a He^3-He^4 mixture is primarily due to the scattering off of He^3 alone. This feature of the scattering can be inferred from experiments[5] in which the mobility of charged impurities in He^4 is four orders of magnitude larger than in He^3 (the former being a superfluid at low enough temperatures). In a mixture of scatterers the resistances add and we have

$$\sigma_{\text{mixture}} = \left(\frac{1}{\sigma_{He^3}} + \frac{1}{\sigma_{He^4}}\right)^{-1} \cong \sigma_{He^3} \ .$$

Thus, in a mixture, it is still only the He^3 that does any effective scattering. By making the percentage of He^3 small, we can enhance the drag effect.

The calculations that have been made generally rely on some simplifying assumptions about the impurity-He^3 scattering, such as a constant cross section. If the impurities are described by a distribution function $N(\vec{p})$, $\vec{p}$ being momentum, $\vec{v}_{\vec{p}}$ velocity and $E_{\vec{p}}$ energy, then a constant cross section corresponds to choosing the parameter a_0, a constant in the following expansion:

$$N(\vec{p}) = N_0(\vec{p}) - \frac{\partial N_0(\vec{p})}{\partial E_{\vec{p}}}\psi(\vec{p}) \ , \tag{1.1}$$

$$\psi(\vec{p}) = \mathcal{E}_x v_{\vec{p},x} a_0 \ . \tag{1.2}$$

Here $\mathcal{E}_x$ is the electric field, and N_0 the equilibrium Boltzmann distribution. Schappert[2] has taken into account a polarization effect in the He^3, but this just changes the constant a_0 at low temperatures. In these calculations, the ultimate low-temperature temperature dependence in the mobility is T^{-2}. (He^3 is a fermion system; in the boson He^4 system, the corresponding dependence[1] is T^{-4}.) In the present paper we shall also use the assumption that a_0 is a constant, and we shall determine its value by a variational principle (as Ref. 1 did in the nondrag case).

When the drag effect comes into play, the helium itself is not in equilibrium. If its distribution function is called $f(\vec{k})$, the equilibrium Fermi distribution being $f_0(\vec{k})$, Fermi level E_F, then we write analogously to Eqs. (1.1) and (1.2)

$$f(\vec{k}) = f_0(\vec{k}) - \frac{\partial f_0}{\partial E_{\vec{k}}}\varphi(\vec{k}) \ , \tag{1.3}$$

$$\varphi(\vec{k}) = \mathcal{E}_x v_{\vec{k},x} b_0 \tag{1.4}$$

Again we shall choose b_0 as a constant and determine it variationally.

In the nondrag problem, the helium atoms interact with one another to such an extent that the distribution function does not have a chance to get out of equilibrium. In this case the φ term in Eq. (1.3) is very small, and $f \cong f_0$. This is the usual assumption. However, the collision time of helium-helium interactions gets large at low temperatures (Baym and Ebner[11] and Wheatley[12] give this time varying as T^{-2}), and the helium system does not have a chance to reach equilibrium when continually bombarded by the impurities. Then the φ term in Eq. (1.3) becomes important. The problem is to solve for the ψ in Eq. (1.1) and the φ in Eq. (1.3) simultaneously.

In Sec. 2 we set up and solve the variational problem for these functions, and in Sec. 3 we discuss the results.

II. VARIATIONAL SOLUTION TO THE COUPLED ION-He^3 SYSTEM

The problem is to solve simultaneously the Boltzmann equation for the distribution function $N(\vec{p})$ for the Boltzmann system of impurities of charge and wave vector $\vec{p}$, which scatter off of Fermi He^3 particles with a distribution function $f(\vec{k})$, and wave vector $\vec{k}$. The Fermi particles interact not only with the impurities, but with each other and with the He^4 particles, in some fashion, and all these other interactions are to be described by a collision time $\tau(\vec{k})$ which is supposedly known. The intrinsic scattering probability will be called $P(\vec{p}'\vec{k}'|\vec{p}\vec{k})$ for a transition $\vec{p}'\vec{k}' \rightarrow \vec{p}\vec{k}$. Momentum and energy are conserved in the interaction

$$\vec{p}'+\vec{k}'=\vec{p}+\vec{k}, \qquad (2.1)$$

$$E_{\vec{p}'}+E_{\vec{k}'}=E_{\vec{p}}+E_{\vec{k}}. \qquad (2.2)$$

We do not need to know the detailed form of P in this paper, as it will appear ultimately only in a term which corresponds to the nondrag problem, and which has been calculated already in the papers cited.[1–4] We shall use, however, the symmetry property

$$P(\vec{p}'\vec{k}'|\vec{p}\vec{k})=P(\vec{p}\vec{k}|\vec{p}'\vec{k}'). \qquad (2.3)$$

The Boltzmann equation for the impurities of charge e is then

$$0=\left(\frac{\partial N(\vec{p})}{\partial t}\right)_{\text{drift}}+\left(\frac{\partial N(\vec{p})}{\partial t}\right)_{\text{coll.}}$$

$$=-e\,\frac{\partial N(\vec{p})}{\partial \vec{p}}\cdot\mathcal{E}+\sum_{\vec{k}\vec{k}'\vec{p}'}P(\vec{p}'\vec{k}'|\vec{p}\vec{k})$$

$$\times\{N(\vec{p}')f(\vec{k}')[1-f(\vec{k})]-N(\vec{p})f(\vec{k})[1-f(\vec{k}')]\}$$

$$\cong -e\,\frac{\partial N_0(\vec{p})}{\partial E_{\vec{p}}}v_{\vec{p},x}\,\mathcal{E}_x-\sum_{\vec{k}\vec{k}'\vec{p}'}Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})$$

$$\times[\psi(\vec{p})+\varphi(\vec{k})-\psi(\vec{p}')-\varphi(\vec{k}')]\,. \qquad (2.4)$$

The last form for the collision sum has been linearized in the electric field using Eqs. (1.1) and (1.3). Here

$$Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})=Q(\vec{p}\vec{k}|\vec{p}'\vec{k}')=(k_BT)^{-1}P(\vec{p}\vec{k}|\vec{p}'\vec{k}')$$

$$\times N_0(\vec{p})f_0(\vec{k})[1-f_0(\vec{k}')]\,. \qquad (2.5)$$

The symmetry statement in Eq. (2.5) just means detailed balancing when the electric field is off. The drag effects in Eq. (2.4) appear through the φ terms.

To find the effect of these φ terms, we must obtain the Boltzmann equation for the He^3 particles. Using the $\tau(\vec{k})$ factor for all collisions not involving the charged impurities, and linearizing using Eqs. (1.3) and (1.1) again, we get

$$0=\left(\frac{\partial f(\vec{k})}{\partial t}\right)_{\text{drift}}+\left(\frac{\partial f(\vec{k})}{\partial t}\right)_{\text{coll.}}$$

$$=0+\sum_{\vec{p}\vec{p}'\vec{k}'}P(\vec{p}'\vec{k}'|\vec{p}\vec{k})\{N(\vec{p}')f(\vec{k}')[1-f(\vec{k})]$$

$$-N(\vec{p})f(\vec{k})[1-f(\vec{k}')]\}-\frac{f(\vec{k})-f_0(\vec{k})}{\tau(\vec{k})}$$

$$=-\sum_{\vec{p}\vec{p}'\vec{k}'}Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})[\psi(\vec{p})+\varphi(\vec{k})$$

$$-\psi(\vec{p}')-\varphi(\vec{k}')]+\frac{\partial f_0}{\partial E_{\vec{k}}}\frac{\varphi(\vec{k})}{\tau(\vec{k})} \qquad (2.6)$$

The problem is to solve Eqs. (2.4) and (2.6) simultaneously. We do this by a variational principle which says that if we maximize M with respect to variations of φ and ψ, where

$$M\equiv\tfrac{1}{4}\sum_{\vec{p}\vec{p}'\vec{k}\vec{k}'}Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})[\psi(\vec{p})+\varphi(\vec{k})$$

$$-\psi(\vec{p}')-\varphi(\vec{k}')]^2-\tfrac{1}{2}\frac{\partial f_0}{\partial E_{\vec{k}}}\frac{\varphi(\vec{k})^2}{\tau(\vec{k})}\,, \qquad (2.7)$$

subject to the side condition

$$\lambda\Big[M+\tfrac{1}{2}\sum_{\vec{p}}e\,\frac{\partial N_0(\vec{p})}{\partial E_{\vec{p}}}\,v_{\vec{p},x}\,\mathcal{E}_x\psi(\vec{p})\Big]=0\,, \qquad (2.8)$$

then the two equations, (2.4) and (2.6), emerge if $\lambda=-2$. In verifying this, notice for example that

$$\delta_{\varphi}M=\sum_{\vec{k}}\delta\varphi(\vec{k})\Bigg(\sum_{\vec{p}\vec{p}'\vec{k}'}Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})[\psi(\vec{p})+\varphi(\vec{k})$$

$$-\psi(\vec{p}')-\varphi(\vec{k}')]-\frac{\partial f_0}{\partial E_{\vec{k}}}\frac{\varphi(\vec{k})}{\tau(\vec{k})}\Bigg). \qquad (2.9)$$

The quantity therefore upon which variations are taken is

$$Z=-(1+\lambda)M-\tfrac{1}{2}\lambda e\sum_{\vec{p}}\frac{\partial N_0}{\partial E_{\vec{p}}}v_{\vec{p},x}\,\psi(\vec{p})\mathcal{E}_x$$

$$=\tfrac{1}{4}\sum Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})[\psi(\vec{p})+\varphi(\vec{k})-\psi(\vec{p}')-\varphi(\vec{k}')]^2$$

$$-\tfrac{1}{2}\sum_{\vec{k}}\frac{\partial f_0}{\partial E_{\vec{k}}}\frac{\varphi(\vec{k})^2}{\tau(\vec{k})}+e\sum_{\vec{p}}\frac{\partial N_0}{\partial E_{\vec{p}}}v_{\vec{p},x}\,\psi(\vec{p})\mathcal{E}_x\,. \qquad (2.10)$$

The current is then

$$J_x=e\sum_{\vec{p}}v_{\vec{p},x}N(\vec{p})=-e\sum_{\vec{p}}\frac{\partial N_0(\vec{p})}{\partial E_{\vec{p}}}v_{\vec{p},x}\,\psi(\vec{p})=\frac{2M}{\mathcal{E}_x}\,, \qquad (2.11)$$

using Eq. (2.8). Thus the quantity that gets maximized is the conductivity. Anything we find by an approximation will constitute a lower bound to the conductivity.

To solve the variational problem approximately, we shall use Eqs. (1.2) and (1.4). The factor in square brackets in Z becomes, upon using the conservation of momentum, Eq. (2.1):

$$\mathcal{E}_x(v_{\vec{k},x}-v_{\vec{k}',x})[b_0-(m/M)a_0]\,, \qquad (2.12)$$

where m is the mass of the helium atom, and M

of the impurity, and $\vec{v}_{\vec{k}} = \hbar\vec{k}/m$, $\vec{v}_{\vec{p}} = \hbar\vec{p}/M$. Thus we can write

$$Z = [b_0 - (m/M)a_0]^2 d + b_0^2\gamma - a_0\alpha \, , \tag{2.13}$$

where

$$d = \tfrac{1}{4}\sum Q(\vec{p}'\vec{k}'|\vec{p}\vec{k})(v_{\vec{k},x} - v_{\vec{k}',x})^2 \, , \tag{2.14}$$

$$\gamma = -\tfrac{1}{2}\sum_{\vec{k}} (\partial f_0/\partial E_{\vec{k}}) v_{\vec{k},x}{}^2 \tau(\vec{k})^{-1} \, , \tag{2.15}$$

$$\alpha = -e\sum_{\vec{p}} (\partial N_0/\partial E_{\vec{p}}) v_{\vec{p},x}{}^2 \, . \tag{2.16}$$

In Eq. (2.15), τ^{-1} can be removed at its value averaged over the Fermi surface:

$$\begin{aligned}\gamma &\cong \langle \tau(\vec{k})^{-1}\rangle N_{\mathrm{He}^3}/(2m) \\ &\equiv \langle \tau_{\mathrm{He}^3}{}^{-1}\rangle N_{\mathrm{He}^3}/(2m),\end{aligned} \tag{2.17}$$

which defines τ_{He^3}. Here N_{He^3} is the number of He^3 atoms in a unit of volume. The integral in Eq. (2.16) can also be evaluated:

$$\alpha = (e/M)N_i \, , \tag{2.18}$$

where the number of impurities is N_i, the function $N(\vec{p})$ being normalized to N_i per unit volume.

The solution to the variational problem in this approximation is obtained by maximizing Eq. (2.13) with respect to a_0 and then with respect to b_0:

$$-2(m/M)b_0 + 2(m/M)^2 a_0 - \alpha/d = 0 \, , \tag{2.19}$$

$$b_0 - (m/M)a_0 + b_0\gamma/d = 0 \, . \tag{2.20}$$

To get the current in Eq. (2.11) we need ψ, i.e., a_0. From Eqs. (2.19), (2.20), and (2.11)

$$\sigma = J_x/\mathcal{E} = \alpha a_0 = \tfrac{1}{2}(M/m)^2\alpha^2(d^{-1} + \gamma^{-1}) \, . \tag{2.21}$$

Using Eqs. (2.17) and (2.18) and writing d in terms of the collision time τ_{ion} for the ions not including the drag effect, this result becomes

$$\sigma = N_i\left[\frac{e^2}{M}\langle \tau_{\mathrm{ion}}{}^{-1}\rangle^{-1} + \frac{e^2}{m}\langle \tau_{\mathrm{He}^3}{}^{-1}\rangle^{-1}\frac{N_i}{N_{\mathrm{He}^3}}\right] \, . \tag{2.21}$$

The first term is the "normal" term for the mobility of the ions, the second is the drag effect.

III. DISCUSSIONS AND RESULTS

Before discussing the results, notice that the conductivity has separated into two terms: One is the ordinary term and the other is the drag term. The latter looks very much like a Drude conductivity, but for the helium. It is as if the helium becomes endowed with an effective charge e (it has a mass m to begin with) and relaxes through τ_{He^3}, i.e., through collisions not involving the impurities.

In comparing the two terms in Eq. (2.21) we shall first consider the pure He^3 case, then the mixture. For pure He^3, Schappert[2] has shown that at low temperatures τ_{ion} can be written

$$\begin{aligned}\lim_{T\to 0} \langle \tau_{\mathrm{ion}}{}^{-1}\rangle^{-1} &= \frac{0.41\hbar^3}{(k_B T)^2\sigma_0}\,\frac{M}{(M+m)^2}\,\frac{\sigma_0}{\sigma_{\mathrm{scatt}}} \\ &= A_1 T^{-2} \, .\end{aligned} \tag{3.1}$$

Here we have inserted, above and below, the cross section σ_0 which gives the hard-sphere value 10^{-14} cm^2. The ratio $\sigma_0/\sigma_{\mathrm{scatt}}$ gives the correction from better scattering calculations. Schappert's calculation of the polarization effect provides $\sigma_0/\sigma_{\mathrm{scatt}} = 5\times 10^{-4}$, a tremendous effect. The value for A_1 is computed using $M/m = 40$, and m the bare He^3 mass:

$$\begin{aligned}A_1 &\cong 10^{-13}\ \mathrm{sec}\ (^\circ\mathrm{K})^2\sigma_0/\sigma_{\mathrm{scatt}} \\ &\cong 5\times 10^{-17}\ \mathrm{sec}\ (^\circ\mathrm{K})^2 \, .\end{aligned} \tag{3.2}$$

For the other term in Eq. (2.21), τ_{He} is of the same form[12]:

$$\lim_{T\to 0} \langle \tau_{\mathrm{He}^3}{}^{-1}\rangle^{-1} = A_2 T^{-2} \, , \tag{3.3}$$

where for A_2 we use the "diffusion" relaxation time value[12]

$$A_2 \cong 3\times 10^{-13}\ \mathrm{sec}\ (^\circ\mathrm{K})^2 \, . \tag{3.4}$$

Using $M/m = 40$, we find that the two terms in Eq. (2.21) are of the same order of magnitude when the number ratio is

$$N_i/N_{\mathrm{He}^3} \cong 0.5\times 10^{-4} \, . \tag{3.5}$$

Such a ratio would seem to be within the reach of experiment, and the drag effect in pure He^3 detectable.

Notice that the temperature dependence is the same for both terms. However, the N_i/N_{He^3} factor in the drag term indicates that such an effect can be determined experimentally through a concentration dependence in the mobility.

For the He^3-He^4 mixture, if we neglect the interaction of the impurities with the He^4, we may use again Schappert's result cited in Eqs. (3.1) and (3.2). The reason for this is that all factors that referred to the density of the He^3 (e.g., the Fermi wave vector k_F) have canceled out in arriving at Eq. (3.1), so that there is in fact no dependence of τ_{ion} on the density of scatterers (in the limit $T\to 0$).

The second term in Eq. (2.21) can be computed for the He^3-He^4 mixture using the results of Baym and Ebner,[11] who found for 5% He^3

$$\langle \tau_{\mathrm{He}^3}{}^{-1}\rangle^{-1} = A_3 T^{-2} \, , \tag{3.6}$$

$$A_3 \cong 2\times 10^{-11}\ \mathrm{sec}\ (^\circ\mathrm{K})^2 \, . \tag{3.7}$$

Again the temperature dependence will not distinguish the drag from the normal term, but the concentration dependence will.

The ratio N_i/N_{He^3} that makes the two terms in Eq. (2.21) equal for the mixture using $M/m = 40$ again is for 5% He^3

$$N_i/N_{\mathrm{He}^3} \cong 0.4\times10^{-7}\,. \tag{3.8}$$

This is certainly within experimental reach, and, in fact, unless care is taken, we should expect that experiments would under ordinary conditions be seeing a drag effect in the mobility.

To sum up then, we find that in the mobility of charged impurities in He^3 or He^3-He^4 mixtures, a drag effect should be found for large enough impurity concentration. The effect would appear to have the same temperature dependence at the lowest temperatures as the normal term, but it contains a concentration dependence which distinguishes it from the normal term. At higher temperatures, a distinction could possibly be made in the temperature dependence, but we have not attempted to analyze that here.

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Concentration-Dependent Polarization in He^3-He^4 Solutions[*]

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The nuclear spin polarization of dilute solutions of He^3 in superfluid He^4 in contact with a polarized pure He^3 bath is calculated for He^3 concentrations between zero and 6.37% for various values of the spin relaxation time and transfer rate of He^3 into solution.

If liquid He^3 is in contact with liquid He^4 at very low temperatures, the former will dissolve into the latter up to a concentration somewhat above 6% ($6.37\pm0.05\%$ according to Ifft *et al.*[1]). The inverse process is negligible. Should the liquid He^3 have polarization P_b, then the polarization P_s of the He^3 in solution will depend on the bath polarization P_b, the He^3 concentration x, the spin relaxation time τ of the He^3 in solution, the transfer rate of He^3 atoms across the phase boundary, and the spin dependence of the He^3 chemical potential $\mu_\sigma(x)$ in solution.

To determine P_s the following approximations are made at the outset: The temperature is sufficiently low so that thermal excitations in the superfluid He^4 can be neglected, the polarization of the He^3 bath P_b remains constant, the polarization of the solution is uniform (i.e., polarization gradients due to local effects near the phase boundary or walls are ignored), the transfer rate of He^3 across the phase boundary is proportional to the difference of chemical potentials $\mu_\sigma - \mu_\sigma(x)$ in each phase, and the spin relaxation time τ is independent of He^3 concentration.

With these approximations the density of up-spin He^3 atoms $n_\uparrow$ in solution obeys the simple rate equation

$$dn_\uparrow/dt = -\tau^{-1}(n_\uparrow - n_\downarrow) + \alpha[\mu_\uparrow - \mu_\uparrow(x)]\,, \tag{1}$$