Proposed Test of Toller ω Dependence of Two-Reggeon–One-Particle Vertex Functions

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(Received 29 July 1968; revised manuscript received 9 September 1968)

It is argued that a particular distribution, in the Toller ω variable, of data from two- to three-particle reactions plausibly described by the double-Regge model, covering a wide range of energies, is *flat* if the two-Reggeon–one-particle vertex function in the amplitude is independent of ω. A prescription for selecting events useful for the test is given, and an estimate of the fraction of events thus selected is made.

I. INTRODUCTION

In recent years increasing attention has been given to the phenomenological multi-Regge model of production amplitudes—a model which was formulated independently by numerous authors in terms of Mandelstam invariants: Koba[1] from a conjecture involving the successive application of single-Reggeon exchange, Ter-Martirosyan,[2] Kibble,[3] and Zachariasen and Zweig[4] on the basis of multiple Sommerfeld-Watson transformations, and Polkinghorne[5] from a study of the high-energy limit of certain Feynman amplitudes. Subsequently Bali *et al.*[6] have shown how the multi-Regge model might emerge from a generalization of Toller's[7] group-theoretical analysis of scattering amplitudes, in terms of a special set of variables ("Toller variables"). Most recently two specific models based on Feynman diagrams have been suggested: that of Blankenbecler and Sugar[8] derived using the Reggeization techniques of Van Hove[9] and of Durand[10] and that of Drummond[11] derived using Gribov's[12] Reggeon calculus.

Detailed studies and applications of the model have also been extensive. In the special case of double-Reggeon exchange a few predictions[2,13] have been made, while some comparisons or fits[14–17] to experimental data have shown quantitative agreement. Of particular importance is the thorough discussion of kinematic-constraint effects and of the applicability of the double-Regge model by Chan Hong-Mo *et al.*[15] For exchange of more than two Reggeons, the model has been used to predict cross sections, multiplicities, and final-state momentum distributions[4,6,18] with good qualitative agreement with experiment where data are available.

Most of these calculations were done by assuming a multi-Regge form for the absolute square of the amplitude, summed over final spins and averaged over initial spins, with the ranges of the invariants restricted so that one diagram was dominant. Furthermore, they all neglected any dependence of the two-Reggeon–one-particle vertex function on the Toller ω variable[19] (called φ in Refs. 15 and 16), the resulting reasonable agreement with experiment lending support to the notion that this dependence is weak. In any case until recently there was no indication as to the form of this dependence. Now the two available Feynman-diagram models[8,11] each suggest a somewhat similar form for this ω dependence. Since this question is of some importance, it is the purpose of the present paper to present a way of determining experimentally whether the ω dependence of these vertex functions is actually severe or not. Off hand, it might be thought that a simple determination of the differential cross section in ω would be a good test. Unfortunately, both the theoretical distribution with neglect of ω dependence and

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[9] L. Van Hove, Phys. Letters **24B**, 183 (1967).
[10] L. Durand, III, Phys. Rev. Letters **18**, 58 (1967); Phys. Rev. **154**, 1537 (1967).
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[12] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **53**, 654 (1967) [English transl.: Soviet Phys.—JETP **26**, 414 (1968)].
[13] M. S. K. Razmi, Nuovo Cimento **31**, 615 (1964); J. A. Verdiyev, Nucl. Phys. **68**, 673 (1965); I. G. Ivanter, A. M. Popova, and K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **46**, 568 (1964) [English transl.: Soviet Phys.—JETP **19**, 387 (1964)].
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[17] W. E. Ellis, D. J. Miller, T. W. Morris, R. S. Panvini, A. M. Thorndike, and E. L. Berger, paper (from Brookhaven Bubble Chamber Group) submitted to the Fourteenth International Conference on High-Energy Physics, Vienna, 1968 (unpublished).
[18] I. A. Verdiev, A. M. Popova, and K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **46**, 1295 (1964) [English transl.: Soviet Phys.—JETP **19**, 878 (1964)]; I. A. Verdiev, O. V. Kancheli, S. G. Martinyan, A. M. Popova, and K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **46**, 1700 (1964) [English transl.: Soviet Phys.—JETP **19**, 1148 (1964)]; K. Kajantie, Nuovo Cimento **53A**, 424 (1968); J. Finkelstein and K. Kajantie, CERN Report No. Th.857, 1967 (unpublished); Chan Hong-Mo, J. Loskiewicz, and W. W. M. Allison, CERN Report No. Th.866, 1968 (unpublished); J. C. Polkinghorne, Cambridge Report No. DAMTP 68/16, 1968 (unpublished).
[19] The angle ω is defined in Refs. 6 and 15 (actually called φ in this latter reference). It is redefined in Sec. II of the present paper.

the experimental distribution seem to be strongly peaked at $\omega=\pi$,[16,17] thus rendering any conclusions regarding ω dependence difficult to make. In the test proposed below based on the *double*-Regge model, however, a particular distribution (using data at many incident energies) will be *flat* if the vertex function is independent of ω; thus the test can be quite sensitive.

In Sec. II the double-Regge model is reviewed and an expression for the cross section using Mandelstam variables as kinematic variables is determined. In Sec. III the proposed test for ω dependence of the model is discussed and in Sec. IV an estimate is made of the fraction of events at each energy that is useful for the test. This latter calculation is intended to show that an unreasonable number of events at each energy is not required for the test.

Throughout the paper the reaction $pp\rightarrow p\pi^-\Delta^{++}(1238)$ is used as an illustrative example and all numerical calculations are based on it. Needless to say, most remarks will apply to any reaction that can plasuibly be described by a double-Regge model.

II. DOUBLE-REGGE MODEL

Depicted in Fig. 1 is the double-Regge diagram under consideration with Mandelstam variables defined as $s_0=(p_1+p_2)^2$, $t_i=(p_i-q_i)^2$, and $s_i=(q_i+q_3)^2$ for $i=1$, 2. It is usually conjectured that the amplitude for this diagram describes the whole reaction if the kinematic variables are restricted to a certain domain. This domain is established as follows:

(i) The total energy s_0 is large.

(ii) The momentum transfers t_1 and t_2 are small; otherwise diagrams with final particles permuted could be important.

(iii) The subenergy s_1 is large enough so that the *leading* trajectory on the left side of the diagram is dominant; in the present case this is the Pomeranchukon.

(iv) Because of (iii) quantum numbers allow only a limited number of trajectories on the right. Because of the smallness of the pion's mass and its strong coupling to the nucleon the pion trajectory may be expected to dominate, at least for moderate values of s_2.[20–22] Then by Dolen-Horn-Schmid duality[20,23,24] it is reasonable to leave s_2 unrestricted[25] if semilocal average effects and no detailed structures in this variable are of concern. Such is the case in the present work.

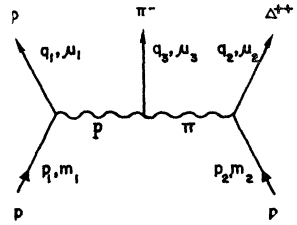

FIG. 1. Double-Regge diagram for $pp\rightarrow p\pi^-\Delta^{++}$ (1238) with Pomeranchukon (P) and pion (π) exchange. p_i and q_i denote four-momenta, while m_i and μ_i denote masses. The usual Mandelstam invariants are defined in the text.

From Bali *et al.*[6] each helicity amplitude corresponding to Fig. 1 has the parametrization, neglecting helicity labels,

$$M\sim f_1(t_1)(\cosh 2\zeta_1)^{\alpha_P(t_1)}f_3(t_1,t_2,\omega)(\cosh 2\zeta_2)^{\alpha_\pi(t_2)}f_2(t_2) \quad (1)$$

for large values of the Toller variables $\cosh 2\zeta_i$. In fact, is it reasonable to assume this form to hold for all values of the variables in the domain described by (i)–(iv) above.[20,24] The definition of the variable $\cosh 2\zeta_i$ may be found in Ref. 6. It sufficies here to note for interest's sake that $\cosh 2\zeta_i$ is linearly dependent on s_i, so that, for large s_i, Eq. (1) is essentially the form of the amplitude used in most of the studies referred to in the Introduction. The one remaining undefined variable ω is conveniently described[19] as the angle between the plane of $\mathbf{p}_1$, $\mathbf{q}_1$ and the plane of $\mathbf{p}_2$, $\mathbf{q}_2$ in the frame $\mathbf{q}_3=0$. Specifically,

$$\cos\omega=\frac{(\mathbf{p}_1\times\mathbf{q}_1)\cdot(\mathbf{p}_2\times\mathbf{q}_2)}{|\mathbf{p}_1\times\mathbf{q}_1|\,|\mathbf{p}_2\times\mathbf{q}_2|}\text{ in the frame } \mathbf{q}_3=0.$$

It may be put in terms of s_0, s_i, and t_i.[6,15] Thus s_0 and ω may in a sense be viewed as complementary variables; it is convenient to think of either s_0 or ω along with s_1, s_2, t_1, and t_2 as a complete set of kinematic invariants.

For present purposes it is important to note that (1) is given most conveniently in terms of the second set of invariants, in which s_0 does not appear explicitly. The same remains true of the absolute square of the amplitude summed over final and averaged over initial spins. It is the ω dependence of this latter quantity, $\sum|M|^2$, that the test will study. That is, in a sense, the average ω dependence of the middle vertex of Fig. 1 will be measured by the test.[26] As mentioned in the Introduction, two different Feynman-diagram models[8,11] of multi-Regge behavior each give an indication of the ω dependence of $f_3(t_1,t_2,\omega)$ in (1). For the reaction depicted in Fig. 1 the form is (taking the minimal-derivative-coupling case in Ref. 8)

$$f_3(t_1,t_2,\omega)\approx g(t_1,t_2)W^{\alpha_\pi(t_2)},$$

with

$$W=2\mu_3{}^2[(\mu_3{}^2-t_1-t_2)+2(t_1t_2)^{1/2}\cos\omega]/\lambda(t_1,t_2,\mu_3{}^2), \quad (2)$$

where $\lambda(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx$.

[20] G. F. Chew and A. Pignotti, Phys. Rev. Letters **20**, 1078 (1968).

[21] B. Haber, U. Maor, G. Yekutieli, and E. Gotsman, Phys. Rev. **168**, 1773 (1968).

[22] The question of a pion conspirator is ignored here. If one *is* necessary, then Ref. 26 suggests the modified viewpoint required.

[23] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967); Phys. Rev. **166**, 1768 (1968).

[24] C. Schmid, Phys. Rev. Letters **20**, 689 (1968). See, however, P. D. B. Collins, R. C. Johnson, and E. J. Squires, Phys. Letters **27B**, 23 (1968); V. A. Alessandrini and E. J. Squires, Phys. Letters **27B**, 300 (1968).

[25] Acutally for large s_2 other trajectories, e.g., A_2, are probably important. However, for present purposes they may be neglected, since most of the data are at small s_2.

[26] The presence of more than one contributing trajectory on the right is not likely to alter the ability of the test to determine the *existence* of ω dependence. Information on how much and what form this dependence takes will be unavailable for the most part, however.

Thus the models predict an enhancement of events at $\omega=\pi$, since $\alpha_\pi(t_2)<0$ for $t_2<0$. The purpose of the test is to detect any such effect experimentally.

The total cross section for the general reaction of Fig. 1 with amplitude (1) is[27]

$$\sigma(s_0)=\frac{(2\pi)^4}{4m_2p_{1L}}\left(\prod_{i=1}^{3}\int\frac{d^3q_i}{2\omega_i(2\pi)^3}\right)(\sum|M|^2)\times\delta^4(p_1+p_2-q_1-q_2-q_3)\,,$$

written for spinless distinguishable particles (any spin and/or statistical factors can be included in $\sum|N|^2$). p_{1L} is the magnitude of the laboratory three-momentum of p_1 and depends on s_0 according to

$$4m_2^2p_{1L}^2=\lambda(s_0,m_1^2,m_2^2)\,,$$

where $\lambda(x,y,z)$ is defined in (2).

Changing the integration variables to invariant (Mandelstam) ones, there results[28]

$$\sigma(s_0)=\frac{1}{2^6(2\pi)^4m_2^2p_{1L}^2}\int ds_1\int dt_2\int dt_1\int ds_2\times[\sum|M(s_1,s_2,t_1,t_2,\omega)|^2](-L)^{-1/2}\,,\quad(3)$$

with[29]

$$L\equiv\begin{vmatrix}2s_1 & -s_1-t_2+m_1^2 & s_0-s_1-\mu_2^2 & s_1+\mu_3^2-\mu_1^2\\ -s_1-t_2+m_1^2 & 2t_2 & t_2-m_2^2+\mu_2^2 & t_1-t_2-\mu_3^2\\ s_0-s_1-\mu_2^2 & t_2-m_2^2+\mu_2^2 & 2\mu_2^2 & s_2-\mu_2^2-\mu_3^2\\ s_1+\mu_3^2-\mu_1^2 & t_1-t_2-\mu_3^2 & s_2-\mu_2^2-\mu_3^2 & 2\mu_3^2\end{vmatrix}.\quad(4)$$

The integration limits $s_1(\pm)$, $t_2(\pm)$, and $t_1(\pm)$ in (3) are given in Appendix A, to which must be added the restrictions (ii) and (iii) discussed at the beginning of this section. The (unrestricted) limits on s_2 are easily found from the requirement, set out in Appendix A, that $L\leq0$, Indeed, regarding L as a symmetric determinant, whose elements a_{ij} are defined by (4), with $L_{ij\cdots k}$ as its principal minor obtained by deleting rows and columns i, j, $\cdots$, k and V_{ij} as the cofactor (signed minor) of the element a_{ij}, it follows from determinantal identities listed in I that

$$s_2(\pm)=A_{34}(\pm)\equiv\mu_2^2+\mu_3^2+[-V_{34_0}\pm(L_3L_4)^{1/2}]/\lambda(t_2,s_1,m_1^2)\,,\quad(5)$$

where V_{34_0} is defined to be V_{34} evaluated at $a_{43}=0$. Furthermore, there is the useful relation $L=\lambda(t_2,s_1,m_1^2)\times[s_2-s_2(+)][s_2-s_2(-)]$.

III. ω-DEPENDENCY TEST

To arrive at an expression that seeks out the ω dependence of the middle vertex function of (1) consider the quantity

$$\Xi\equiv\int_{(m_1+m_2)^2}^{\infty}ds_0F(s_0)\sigma(s_0)=\int ds_1\int dt_2\int dt_1\int ds_2\int ds_0\times(\sum|M|^2)(-L)^{-1/2}\,,\quad(6)$$

with

$$F(s_0)\equiv2^6(2\pi)^4m_2^2p_{1L}^2(s_0)\quad(7)$$

and with integration limits as given in Appendix B subject, of course, to restrictions (ii) and (iii) given earlier.

Now on the right-hand side of (6), with s_1, s_2, t_1, t_2, and ω as the independent variables and the matrix element of the form (1), the dependent variable s_0 appears only in L and as an integration variable. It is easily put in terms of the independent variables. To this end, it is possible to show, from Ref. 6, that

$$\cos\omega=V_{13}/(L_1L_3)^{1/2}$$

and

$$s_0\equiv a_{13}+s_1+\mu_2^2=s_1+\mu_2^2+[-V_{13_0}+\cos\omega\,(L_1L_3)^{1/2}]/\lambda(t_1,t_2,\mu_3^2)\,,\quad(8)$$

where the symbols have been defined above. It then follows from determinantal identities in I that

$$L=\frac{L_1L_3-V_{13}^2}{L_{13}}=\sin^2\omega\frac{L_1L_3}{L_{13}}=-\sin^2\omega\frac{L_1L_3}{\lambda(t_1,t_2,\mu_3^2)}\,,\quad(9)$$

with L_1 and L_3 independent of ω (and s_0).

The change of variable s_0 to ω in (6) is thus easily effected with (8) and (9):

$$\Xi=\int ds_1\int dt_2\int dt_1\int ds_2\int_0^\pi d\omega\times(\sum|M|^2)\lambda(t_1,t_2,\mu_3^2)^{-1/2}\,,\quad(10)$$

where the range of ω follows from a comparison of (8) with (B5). This unrestricted range is as expected[6] and allows a "differential distribution" in ω to be straightforwardly defined:

$$\frac{d\Xi}{d\omega}=\int ds_1\int dt_2\int dt_1\int ds_2\,(\sum|M|^2)\lambda(t_1,t_2,\mu_3^2)^{-1/2}\,,\quad(11)$$

[27] Compare J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), Appendix B.

[28] N. Byers and C. N. Yang, Rev. Mod. Phys. **36**, 595 (1964); H. S. Song, W. Kernan, and R. H. Good, Jr., Phys. Rev. **140**, B914 (1965).

[29] The determinant L is $-2^4\Delta_4'$ in the notation of Byers and Yang (Ref. 28). It is also the determinant of the same name in R. A. Morrow [J. Math. Phys. **7**, 844 (1966); hereafter referred to as I], with $s=s_0$, $t=t_2$, $u=s_2$, $v=s_1$, $w=t_1$, $m_1=m_1$, $m_2=m_2$, $m_3=\mu_2$, $m_4=\mu_3$, and $m_5=\mu_1$. In the notation of I it also has the value $2^4\Delta_4$.

with limits (B1)–(B4), independent of ω (and s_0). Note the significant fact that the only ω dependence is in the matrix element which is of the form (1). The severity of this dependence is thus indicated in an average way by the distribution (11).

Unfortunately, (11) is not in a form amenable to experimental application, since data for all values of the variables, which implies for all energies s_0, are needed. Consequently it is necessary to find a way of restricting the ranges of the integration variables in (11) in such a way that (a) only data at pertinent available energies are used ,while (b) no ω (or s_0) dependence is introduced by the restrictions. The procedure will be to find restrictions on the range of s_2 so that in (10) ω will still have range 0–π, while in (6) the range of s_0 will be contained in a given, fixed, extreme range s_a–s_b. That is, for all allowed values of the integration variables in (10), subject to the proposed restriction on s_2, the value of s_0 calculated from (8) will lie between s_a and s_b.

A convenient starting point is to look at the physical ranges of s_2 and s_0 in (6) for given physical values of the other invariants. This region in the s_0-versus-s_2 plane is bounded by $L=0$ as discussed in Appendix B. Equation (B5) then shows that this bounding curve lies in the region where $L_1\geq 0$ and has as tangents the lines $s_2=A_{34}(1,\pm)$ which solve $L_1=0$. Additional information follows from (5), with

$$L_4=-2t_2[s_0-A_{13}(4,+)][s_0-A_{13}(4,-)]$$

and

$$A_{13}(4,\pm)=s_1+\mu_2^2+[(-s_1-t_2+m_1^2)(t_2-m_2^2+\mu_2^2)\pm\lambda(t_2,m_2^2,\mu_2^2)^{1/2}\lambda(s_1,t_2,m_1^2)^{1/2}]/2t_2. \quad (12)$$

Thus, since $L_3\geq 0$, the curve $L=0$ as solved by (5) exists only where $L_4\geq 0$, and in fact the lines $s_0=A_{13}(4,\pm)$ which solve $L_4=0$ are tangent to $L=0$.

The desired region may now be exhibited in the s_0-versus-s_2 plane. Two distinct cases occur according as $t_2>0$ or $t_2\leq 0$; these are shown in Figs. 2(a) and 2(b), respectively. In both cases, for fixed s_2, $\omega=0$ (π) is the largest (smallest) value of s_0 on $L=0$. The dashed straight line, shown for purposes of orientation, is the curve $\omega=\frac{1}{2}\pi$ obtained from (8). Furthermore, it follows from (8) and (9) that all curves for ω fixed ($\neq 0, \frac{1}{2}\pi, \pi$) intersect $L=0$ only at the points W_+ and W_-, where, in fact, they are tangent to the latter curve. Letting $W_\pm$ denote also the s_0 coordinate of this point, it follows from (B5) that

$$W_\pm=s_1+\mu_2^2-\lambda(t_1,t_2,\mu_3^2)^{-1}V_{130}|_{s_2=A_{34}(1,\pm)}. \quad (13)$$

Also of interest are the following points deduced from (5) and shown in Fig. 2:

$$s_2(b,\pm)=A_{34}(\pm)|_{s_0=s_b},$$
$$s_2(a,\pm)=A_{34}(\pm)|_{s_0=s_a}. \quad (14)$$

Now for given s_a and s_b it is desired to find the range of s_2 such that for s_2 in this range ω can vary over the full range 0–π without s_0 leaving the extreme range s_a–s_b. The cases (a) $t_2>0$ and (b) $t_2<0$ will be remarked upon separately and then a comprehensive set of limits on s_2 will be written down.

(a) $t_2>0$. A study of Fig. 2(a) shows that, depending on the relative ordering of the horizontal lines, there may be as many as three ranges of s_2, which will be denoted by $s_2(1,-)$ to $s_2(1,+)$, $s_2(2,-)$ to $s_2(2,+)$, and $s_2(3,-)$ to $s_2(3,+)$, with $s_2(1,+)\leq s_2(2,-)$ and $s_2(2,+)\leq s_2(3,-)$. It is also convenient to use this notation even when there are fewer than three ranges. The actual limits may be written by inspection of Fig. 2(a) and in order to obtain a set compatible with (b) it is proper to interpret $s_b\gtrless A_{13}(4,+)$ subject to $s_b>A_{13}(4,-)$ or $s_b>W_-$ as $L_4|_{s_0=s_b}\lesseqgtr 0$.

(b) $t_2<0$. The limits here are found by inspection of Fig. 2(b). There are at most two ranges of s_2, and so $s_2(3,-)=s_2(3,+)$ must be taken. Also, a range exists only if $s_b>W_-$ [$>A_{13}(4,-)$], which implies $L_4|_{s_0=s_b}>0$.

In accordance with these instructions a comprehensive set of limits is

$$\begin{aligned}
s_2(3,+)&=A_{34}(1,+), && \text{if } s_a<W_+<s_b\\
&=s_2(3,-), && \text{if } s_a\geq W_+,\ s_b\leq W_+, \text{ or } t_2\leq 0\\
s_2(3,-)&=s_2(b,+), && \text{if } s_a\leq A_{13}(4,-),\ s_b\leq A_{13}(4,+), \text{ and } s_a<W_+<s_b\\
&=s_2(1,+), && \text{if } s_a\leq A_{13}(4,-),\ s_b>A_{13}(4,+), \text{ and } s_a<W_+<s_b\\
&=\max[s_2(a,+),s_2(b,+)], && \text{if } s_a>A_{13}(4,-),\ s_b\leq A_{13}(4,+), \text{ and } s_a<W_+<s_b\\
&=s_2(a,+), && \text{if } s_a>A_{13}(4,-),\ s_b>A_{13}(4,+), \text{ and } s_a<W_+<s_b\\
&=s_2(3,+), && \text{if } s_a\geq W_+,\ s_b\leq W_+, \text{ or } t_2\leq 0\\
s_2(2,+)&=s_2(b,-), && \text{if } s_a>A_{13}(4,-),\ L_4|_{s_0=s_b}>0, \text{ and } s_2(a,+)<s_2(b,-)\\
&=s_2(2,-), && \text{otherwise}\\
s_2(2,-)&=s_2(a,+), && \text{if } s_a>A_{13}(4,-),\ L_4|_{s_0=s_b}>0, \text{ and } s_2(a,+)<s_2(b,-)\\
&=s_2(2,+), && \text{otherwise}\\
s_2(1,+)&=s_2(b,-), && \text{if } s_a\leq A_{13}(4,-),\ L_4|_{s_0=s_b}\geq 0, \text{ and } s_a<W_-<s_b\\
&=s_2(3,-), && \text{if } s_a\leq A_{13}(4,-),\ L_4|_{s_0=s_b}<0, \text{ and } s_a<W_-<s_b\\
&=\min[s_2(a,-),s_2(b,-)], && \text{if } s_a>A_{13}(4,-),\ L_4|_{s_0=s_b}\geq 0, \text{ and } s_a<W_-<s_b\\
&=s_2(a,-), && \text{if } s_a>A_{13}(4,-),\ L_4|_{s_0=s_b}<0, \text{ and } s_a<W_-<s_b\\
&=s_2(1,-), && \text{if } s_a\geq W_- \text{ or } s_b\leq W_-\\
s_2(1,-)&=A_{34}(1,-), && \text{if } s_a<W_-<s_b,\\
&=s_2(1,+), && \text{if } s_a\geq W_- \text{ or } s_b\leq W_-
\end{aligned} \quad (15)$$

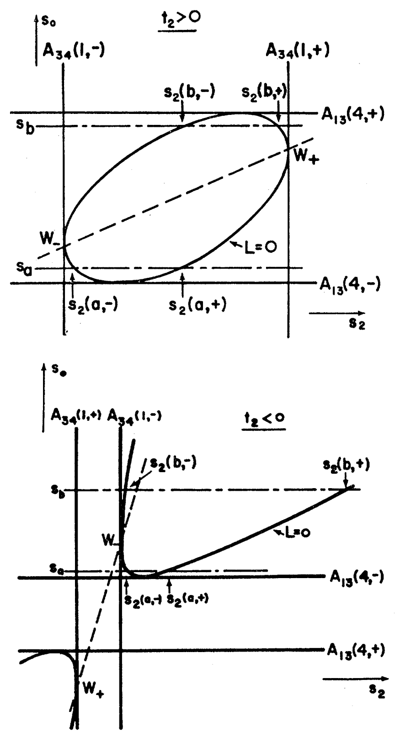

FIG. 2. Physical region in the s_0-versus-s_2 plane for fixed s_1, t_2, and t_1, with $t_2>0$ in (a) and $t_2<0$ in (b). In (a) the allowed region is the interior of the ellipse $L=0$, while in (b) it is the interior of the upper-right branch of the hyperbola $L=0$. The dashed straight line is the curve $\omega=\frac{1}{2}\pi$ obtained from (8). These curves are *not* drawn to scale.

where $A_{34}(1,\pm)$ is defined in (B6), L_4 and $A_{13}(4,\pm)$ in (12), $W_\pm$ in (13), and $s_2(a,\pm)$ and $s_2(b,\pm)$ in (14).

It is then concluded that, in (10) with the limits (15) on s_2 and ω anywhere between 0 and π, the dependent variable s_0 lies between s_a and s_b. This permits the proposed test, discussed with respect to (11), to be reconsidered. In particular, suppose the s_1, t_2, and t_1 limits to be given by (B1)–(B3), subject to restrictions (ii) and (iii) given at the beginning of Sec. II, and the s_2 limits to be given by (15). Denote this set of (ω- and s_0-independent) limits and all quantities defined by integration using them by the subscript R. Then (11) becomes

$$\frac{d\Xi_R}{d\omega}=\int_R ds_1dt_2dt_1ds_2\,(\sum|M|^2)\lambda(t_1,t_2,\mu_3^2)^{-1/2}. \quad (16)$$

Let R' denote the analogous set of limits appropriate to (3), i.e., with s_1, t_2, and t_1 limits given by (A1)–(A3), subject to restrictions (ii) and (iii) of Sec. II, and s_2 limits given by the overlap[30] of (5) and (15). Then (3) becomes

$$\sigma_{R'}(s_0)=F(s_0)^{-1}\int_{R'} ds_1dt_2dt_1ds_2\,(\sum|M|^2)(-L)^{-1/2}, \quad (17)$$

[30] These new limits on s_2 are determined in Sec. IV for the case $t_2\leq 0$.

and so from (6), (10), and (16) there is the relation

$$\Xi_R=\int_{(m_1+m_2)^2}^{\infty} ds_0F(s_0)\sigma_{R'}(s_0)=\int_0^\pi d\omega\frac{d\Xi_R}{d\omega}, \quad (18)$$

where, because of (15), $\sigma_{R'}(s_0)=0$ except for $s_a<s_0<s_b$. It then follows from (18) that

$$\frac{d\Xi_R}{d\omega}=\int_{s_a}^{s_b} ds_0F(s_0)\frac{d\sigma_{R'}(s_0)}{d\omega}, \quad (19)$$

where $d\sigma_{R'}(s_0)/d\omega$ is the obvious differential distribution in ω of events contributing to $\sigma_{R'}(s_0)$.

According to (19) the distribution $d\Xi_R/d\omega$ can be constructed directly from experimental data for those reactions that have been studied at many energies over a reasonable range of energies. In order to be truly a test of the ω dependence of the middle vertex of Fig. 1 only those data plausibly explained by the amplitude of Fig. 1 can be used with restriction (15). As mentioned frequently above, this means restrictions (ii) and (iii) on s_1, t_1, and t_2. The methodical procedure in applying the test is then to select at each energy only those events with

(A) t_1 and t_2 small enough so that only the diagram with the desired ordering of particles in the final state is appreciable. Since the reactions are highly peripheral, these are not stringent restrictions.

(B) s_1 large enough so that the left-hand side of Fig. 1 is dominated by a single Reggeon. In practice, this can mean s_1 above any pronounced resonances. This restriction can eliminate many events.

(C) s_2 selected according to (15). The effect of this restriction is studied in Sec. IV.

It should be stressed again that the above cuts on s_1, t_1, and t_2 must be independent of ω and s_0. The claim is then that *the distribution in (19) is flat for* $0\leq\omega\leq\pi$ *if the middle vertex of Fig. 1 is independent of* ω.

IV. EFFECT OF s_2 RESTRICTIONS

This section is concerned with estimating the fraction of data at each energy eliminated by the restriction (15) on s_2. Certainly the test is feasible only if this fraction is not excessively large. Although the reaction $pp\to p\pi^-\Delta^{++}$ will be considered for definiteness, the general features should be true of any suitable reaction.

At a laboratory momentum $p_{1L}=28.5$ GeV/c the following expression of the form (1) for the spin-averaged square of the matrix element for the reaction $pp\to p\pi^-\Delta^{++}$ was found to give reasonable fits to experimental distributions[17,31]:

$$\begin{aligned}\sum|M|^2=Ce^{8t_1}\{&[s_1-t_2-m_1^2+(\mu_1^2-m_1^2-t_1)\\ &\times(-t_1-t_2+\mu_3^2)/2t_1]s_{01}^{-1}\}^{2\alpha_P(t_1)}\\ &\times\{[s_2-t_1-m_2^2+(\mu_2^2-m_2^2-t_2)\\ &\times(-t_1-t_2+\mu_3^2)/2t_2]s_{02}^{-1}\}^{2\alpha_\pi(t_2)}\\ &\times\pi^2\alpha_\pi'(t_2)^2/[1-\cos\pi\alpha_\pi(t_2)], \quad (20)\end{aligned}$$

[31] E. L. Berger (private communication).

with $m_1=m_2=\mu_1=0.94$ GeV/c^2, $\mu_3=0.14$ GeV/c^2, $\mu_2=1.24$ GeV/c^2, $s_{01}=1.0$ GeV^2, $s_{02}=0.7$ GeV^2, $\alpha_P(t_1)=1$, $\alpha_\pi(t_2)=\alpha_\pi'(t_2)(t_2-0.02)$, and $\alpha_\pi'(t_2)=1.2$ GeV^{-2}, and where C is a scaling factor. Cross sections are in mb if, in addition, s_i and t_i are in GeV^2 and $C=1.1\times10^5$.

Since, as just remarked, this ω-independent expression gives reasonable results, it might be concluded that there is no ω dependence of the middle vertex of Fig. 1. However, some distributions (notably that of the Treiman-Yang isotropy test in the $p\pi^-$ rest frame) are not completely accounted for, and if ω dependence is necessary to fit them, it is desirable to know what form it takes and how severe it is. The proposed test hopefully will shed some light on these questions.

For present purposes it is assumed that (20) describes the reaction at all energies of interest subject to restrictions (A) and (B) of Sec. III, which are taken as (A) $t_1\geq-2.0$ GeV^2, $t_2\geq-0.6$ GeV^2 and (B) $s_1\geq3.5$ GeV^2. Then to study the effect of restriction (C) two calculations are done. The first gives $\sigma(s_0)$ from (3) with the above cuts (A) and (B) but with no s_2 restriction, while the second includes the s_2 restriction (15), giving, in fact, $\sigma_{R'}(s_0)$ of (17). In this latter calculation the precise limits on s_2 are easily determined from the overlap of (5) and (15). They will not be written down for the general case, but for the reaction at hand, where $t_2\leq0$, they are

$$\begin{aligned}
s_2(2,+)&=\min[s_2(b,-),A_{34}(+)], &&\text{if } s_a>A_{13}(4,-) \text{ and } s_2(a,+)<s_2(b,-)\\
&=s_2(2,-), &&\text{otherwise}\\
s_2(2,-)&=\max[s_2(a,+),A_{34}(-)], &&\text{if } s_a>A_{13}(4,-) \text{ and } s_2(a,+)<s_2(b,-)\\
&=s_2(2,+), &&\text{otherwise}\\
s_2(1,+)&=\min[s_2(b,-),A_{34}(+)], &&\text{if } s_a<A_{13}(4,-) \text{ and } s_a<(W_-,s_0)<s_b\\
&=\min[s_2(b,-),s_2(a,-)], &&\text{if } s_a\geq A_{13}(4,-) \text{ and } s_a<(W_-,s_0)<s_b\\
&=s_2(-), &&\text{if } s_a\geq W_-,\ s_a\geq s_0,\ s_b\leq W_-, \text{ or } s_b\leq s_0\\
s_2(1,-)&=A_{34}(-), &&\text{if } s_a<(W_-,s_0)<s_b\\
&=s_2(+), &&\text{if } s_a\geq W_-,\ s_a\geq s_0,\ s_b\leq W_-, \text{ or } s_b\leq s_0.
\end{aligned}$$

The results of the calculations are shown in Fig. 3, with s_a and s_b corresponding to p_{1L} of 10 and 30 GeV/c, respectively, where the lower cutoff is determined more or less by the restriction on s_1 (≥3.5 GeV^2) and the upper cutoff is limited by the highest beam momentum at which the reaction has been studied experimentally, namely, 28.5 GeV/c. Also shown in the figure is the trend of the total cross section,[32] without any restrictions on the kinematic variables. It is apparent from this figure that about 9% of the total number of events in the above beam-momentum range is useful for the test. This is not too unreasonable, especially in view of the fact that distributions in ω at all energies of interest are combined, according to (19), to get a single distribution. The crucial requirement, of course, is that the reaction be studied over a wide range of energies.

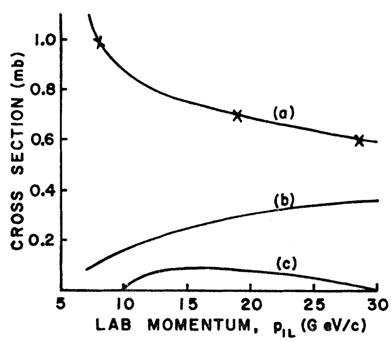

FIG. 3. Cross section $\sigma(s_0)$ for the reaction $pp\to p\pi^-\Delta^{++}$, with (a) no restrictions on the kinematic variables (Ref. 32), (b) $t_1\geq-2.0$ GeV^2, $t_2\geq-0.6$ GeV^2, and $s_1\geq3.5$ GeV^2 and (c) t_1, t_2, and s_2 restricted as in (b) and s_2 restricted as in (21). The ratio of (c) to (a) then indicates the fraction of the total number of events at each energy useful for the test. Error bars are not shown, since only orders of magnitude are of interest here—the curves can safely be viewed as having a maximum possible uncertainty of perhaps 25%.

V. CONCLUSION

An experimental test to detect any ω dependence of two-Reggeon–one-particle vertex functions in the double-Regge model of high-energy production reactions was formulated. It should be especially suitable for those two- to three-particle reactions which can be studied conveniently over a wide range of energies, e.g., $pp\to p\pi\Delta$, $pp\to p\pi n$, $\pi p\to\pi\rho p$, etc. In view of the present lack of definitive theoretical understanding of these vertex functions, this test could be quite useful in suggesting a phenomenological model for them.

ACKNOWLEDGMENTS

The author wishes to thank Professor E. L. Berger for numerous helpful discussions concerning the double-Regge model. He also wishes to thank W. E. Ellis, Professor E. L. Berger, and Dr. R. S. Panvini for making their results available to him before public presentation.

[32] P. L. Connolly, W. E. Ellis, P. V. C. Hough, D. J. Miller, T. W. Morris, C. Ouannes, R. S. Panvini, and A. M. Thorndike, Brookhaven National Laboratory Report No. BNL-11980 (unpublished); report of a talk presented at the Third Topical Conference on Resonant Particles, Athens, Ohio, 1967. References to original works can be found in this paper.

APPENDIX A: INTEGRATION RANGES FOR FIXED s_0

The unrestricted limits of integration in (3) for the order shown there, namely, $\int ds_1 dt_2 dt_1 ds_2$, are easily derived using the results of I, with the substitutions $s=s_2$, $t=s_1$, $u=t_1$, $v=s_0$, $w=t_2$, $m_1=\mu_2$, $m_2=\mu_3$, $m_3=\mu_1$, $m_4=m_1$, and $m_5=m_2$. Indeed, as was shown in the study of the physical region in the s-u plane (which here becomes the s_2-t_1 plane), the ranges of the variables are (i) s_0 such that $L_{23}\leq 0$, (ii) s_1 such that L_{34}, $L_{14}\leq 0$, (iii) t_2 such that $L_3\geq 0$, (iv) t_1 such that $L_1\geq 0$, and (v) s_2 such that $L\leq 0$. Using the techniques explained in the Appendix of I, the limits follow, for $s_0\geq \max[(m_1+m_2)^2, (\mu_1+\mu_2+\mu_3)^2]$, of course,

$$s_1(+)=(s_0^{1/2}-\mu_2)^2\,,\quad s_1(-)=(\mu_1+\mu_3)^2\,, \tag{A1}$$

$$t_2(\pm)=s_1+m_1^2+[(-s_0-s_1+\mu_2^2)(s_0+m_1^2-m_2^2)\pm\lambda(s_0,m_1^2,m_2^2)^{\frac{1}{2}}\lambda(s_0,s_1,\mu_2^2)^{\frac{1}{2}}]/2s_0\,, \tag{A2}$$

$$t_1(\pm)=\mu_1^2+m_1^2+[(s_1-\mu_3^2+\mu_1^2)(t_2-s_1-m_1^2)\pm\lambda(s_1,t_2,m_1^2)^{1/2}\lambda(s_1,\mu_1^2,\mu_3^2)^{1/2}]/2s_1. \tag{A3}$$

The limits on s_2 are given in the text. Note that the symbols L, L_i, etc., have *different* definitions here than in the text.

APPENDIX B: INTEGRATION RANGES FOR FIXED ω

The unrestricted limits of integration in (6) for the order shown there, namely, $\int ds_1 dt_2 dt_1 ds_2 ds_0$, are also derived using the results of I, but with the substitutions $s=s_0$, $t=t_2$, $u=s_2$, $v=s_1$, $w=t_1$, $m_1=m_1$, $m_2=m_2$, $m_3=\mu_2$, $m_4=\mu_3$, and $m_5=\mu_1$. Just as in Appendix A, the ranges of the variables are (i) s_1 such that $L_{23}\leq 0$, (ii) t_2 such that L_{34}, $L_{14}\leq 0$, (iii) t_1 such that $L_3\geq 0$, (iv) s_2 such that $L_1\geq 0$, and (v) s_0 such that $L\leq 0$. Again using the techniques of the Appendix of I, the limits follow:

$$s_1(+)=\infty\,,\quad s_1(-)=(\mu_1+\mu_3)^2\,, \tag{B1}$$

$$\begin{aligned} t_2(+)&=0\,, && \text{if } (s_1^{1/2}-m_1)(m_2-\mu_2)\leq 0\\ &=\min[(m_2-\mu_2)^2,\,(s_1^{1/2}-m_1)^2]\,, && \text{if } (s_1^{1/2}-m_1)(m_2-\mu_2)>0\\ t_2(-)&=-\infty\,, \end{aligned} \tag{B2}$$

$$t_1(\pm)=t_2+\mu_3^2+[(-s_1-t_2+m_1^2)(s_1+\mu_3^2-\mu_1^2)\pm\lambda(s_1,\mu_3^2\,\mu_1^2)^{1/2}\lambda(s_1,t_2,m_1^2)^{1/2}]/2s_1\,, \tag{B3}$$

$$\begin{aligned} s_2(+)&=A_{34}(1,+)\,, && \text{if } t_2>0\\ &=\infty\,, && \text{if } t_2\leq 0\\ s_2(-)&=A_{34}(1,-)\,, \end{aligned} \tag{B4}$$

$$s_0(\pm)=A_{13}(\pm)\equiv s_1+\mu_2^2-[V_{13_0}\mp(L_1L_3)^{1/2}]/\lambda(t_1,t_2,\mu_3^2)\,, \tag{B5}$$

where

$$\begin{aligned} A_{34}(1,\pm)&=\mu_2^2+\mu_3^2+(2t_2)^{-1}[(t_2-m_2^2+\mu_2^2)(t_1-t_2-\mu_3^2)\pm\lambda(t_1,t_2,\mu_3^2)^{1/2}\lambda(t_2,m_2^2,\mu_2^2)^{1/2}]\,,\\ L_1&=-2t_2[s_2-A_{34}(1,+)][s_2-A_{34}(1,-)]\,,\\ L_3&=-2s_1[t_1-t_1(+)][t_1-t_1(-)]. \end{aligned} \tag{B6}$$

The change of variables $s_0\rightarrow\omega$ can be effected as in the text in order to reach (10). The limits on the integrals in the latter expression remain as above. Note that the symbols L, L_i, etc., have the *same* definitions here as in the text.