for three-point functions.[13] Since our results do not depend upon the subtraction constant, we may take, for simplicity, $\nu_0=0$. With this choice, from Eqs. (2) and (4) it at once follows that $F_3(\nu,q^2)$ satisfies an unsubtracted dispersion relation and that F_5 leads to a superconvergent relation.

To sum up, we have shown that the current-algebra results for virtual-photon–pion scattering are consistent with dispersion relations, provided that we write OSDR for the amplitudes F_1 and F_2. In other words, we have shown that hard-pion results for a four-point function in the forward direction can be easily obtained by writing OSDR, and by determining the subtraction constant from the soft-pion current algebra[14] for the four-point function and the dispersion integral from the hard-pion three-point function. The only anomaly is that with the neglect of the Regge cuts, it is expected[5] that F_1 and F_2 should satisfy unsubtracted dispersion relations. Thus, the algebra of currents and Regge pole theory (without cuts) lead us to different results. Similar calculations are in progress for πA_1 and $\pi\rho$ scattering amplitudes, and perhaps these will help in reaching a more definite conclusion.

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[13] For the first three sum rules, see papers quoted in Ref. 3. For the last sum rule, see Ref. 8 and D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967).

[14] The first successful application of current algebra to determine the subtraction constant in OSDR was made by S. Okubo *et al.*, Phys. Rev. Letters **19**, 407 (1967) for the decay $K\rightarrow 2\pi$. Later, a similar approach in an N/D formalism was used by S. C. Bhargava, S. N. Biswas, K. C. Gupta, and K. Datta, *ibid.* **20**, 558 (1968) to calculate unitarity corrections to current-algebra results for s-wave πN scattering.

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Compton Scattering of Spin-One Particles

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Kinematic singularity- and zero-free invariant amplitudes for Compton scattering of spin-one particles are derived. Gauge invariance is automatically satisfied. A complete set of low-energy theorems is obtained. In addition to the well-known first-order theorems, higher-order theorems also exist. It is shown that up to second order in the photon energy the 12 independent amplitudes are determined by the static moments of the target particle plus four dynamic structure-dependent constants; up to third order they are determined by the static moments plus eight additional constants. Dispersion relations may be written down for the invariant amplitudes without any *ad hoc* subtractions. The asymptotic behavior of these amplitudes is carefully examined, leading to a systematic derivation of possible sum rules and superconvergence relations. Generalizations of previously known sum rules as well as new ones are obtained.

I. INTRODUCTION

RECENTLY, there has been much interest in studying the analytic structure of the scattering amplitude due to Lorentz transformation properties of the *external* particle states involving spin. Detailed analysis of these *kinematic* singularities and zeros of the scattering amplitude has led to physically interesting results. In particular, for processes involving photons these analytic properties are intimately related to gauge invariance. Detailed analysis of these analytic properties has helped in clarifying the meaning of the well-known low-energy theorems as well as in deriving new ones for various photon processes.[1]

The kinematic structure of the two-body helicity amplitudes has been extensively studied recently.[2] It is shown that regularized helicity amplitudes can be defined such that they are free of kinematic singularities. However, they must satisfy certain kinematic constraints which make their use in certain applications rather cumbersome. For example, if one wants to write down dispersion relations for these amplitudes, one must make *ad hoc* subtractions in order to ensure that these constraints are satisfied.

On the other hand, there exist invariant amplitudes which, if properly chosen, are free of both kinematic singularities and zeros[3] (constraints). In obtaining these

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[1] H. Abarbanel and M. Goldberger, Phys. Rev. **165**, 1594 (1968); K. Bardakci and H. Pagels, *ibid.* **166**, 1783 (1968); S. Choudhury and D. Z. Freedman, *ibid.* **168**, 1739 (1968).

[2] Y. Hara, Phys. Rev. **136**, B507 (1964); L. L. C. Wang, *ibid.* **142**, 1187 (1965); H. P. Stapp, *ibid.* **160**, 1251 (1967); G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968); J. P. Ader, M. Capdeville, and H. Navelet, Nuovo Cimento **55A**, 315 (1968). The last reference deals with two-body scattering amplitudes involving at least one massless particle.

[3] A. C. Hearn, Nuovo Cimento **21**, 333 (1961); K. Hepp, Helv. Phys. Acta **36**, 355 (1963); D. N. Williams (unpublished); D. Zwanziger, in *Lectures in Theoretical Physics* (The University of Colorado Press, Boulder, Colorado, 1965), Vol. VII A.

invariant amplitudes, all requirements imposed by Lorentz invariance, crossing, and, for photon processes, gauge invariance are explicitly taken care of from the beginning. For many purposes, therefore, the invariant amplitudes are the most convenient ones to use.

A general procedure for obtaining kinematic singularity- and zero-free invariant amplitudes for photon processes was proposed in a recent publication.[4] In the present paper, we apply this method to the case of Compton scattering on a spin-one particle. In Sec. II, we derive these kinematic singularity- and zero-free amplitudes. In Sec. III, we derive the complete set of low-energy theorems for this process. It is shown that in addition to the familiar first-order low-energy theorems there exist eight new low energy theorems to second order and four new ones to third order in the photon energy. In Sec. IV we write down single and double spectral representations for the invariant amplitudes. By a careful examination of the asymptotic behavior of these amplitudes, we also discuss possible sum rules and superconvergence relations that can be derived.

The notation we adopt in this paper is the same as in Ref. 4, which we shall refer to henceforth as **A**.

II. INVARIANT AMPLITUDES

We consider Compton scattering on a target particle of spin one and mass m. Let the momenta of the initial and final photon be denoted by k, k' and those of the target particle by p, p', respectively. The physical scattering amplitude can be written

$$\langle k',\lambda';p',\sigma'|T|k,\lambda;p,\sigma\rangle = \xi_{\mu'}{}^{(\lambda')*}\xi_{\nu'}{}^{(\sigma')*}M^{\mu'\nu',\mu\nu}\xi_\mu{}^{(\lambda)}\xi_\nu{}^{(\sigma)}, \quad (1)$$

where the ξ's are the polarization vectors for the various particles involved and $M^{\mu'\nu',\mu\nu}$ is the M function for this process. M is an analytic function of the four-momentum components except for dynamical singularities.[5] There are three independent four-vectors in this problem; we shall take them to be

$$\Delta = k-k' = p'-p, \quad (2)$$
$$K=\tfrac{1}{2}(k+k'), \quad P=\tfrac{1}{2}(p+p').$$

A set of invariant amplitudes can be obtained by expanding M in terms of a tensor basis $\{\mathfrak{L}_i\}$ with $\mathfrak{L}_i$ satisfying the correct Lorentz transformation properties (including space- and time-reversal invariance) as well as gauge invariance,

$$M^{\mu'\nu',\mu\nu}(\Delta,K,P)=\sum_i \mathfrak{L}_i{}^{\mu'\nu',\mu\nu}(\Delta,K,P)A_i(s,t). \quad (3)$$

The A_i's are free from kinematic singularities if $\{\mathfrak{L}_i\}$ corresponds to a *minimal polynomial basis*.[6]

To find the appropriate $\{\mathfrak{L}_i\}$, we follow the procedure of **A**. We first expand M in terms of a simple polynomial basis $\{l_i\}$ which transforms properly under Lorentz transformations (including space and time reversal):

$$M^{\mu'\nu',\mu\nu}=\sum_i l_i{}^{\mu'\nu',\mu\nu}B_i(s,t). \quad (4)$$

We then impose gauge invariance by means of a simple projection operator $I^{\mu\mu'}$, defined as

$$I^{\mu\mu'}=g^{\mu\mu'}-k^{\mu'}k'^{\mu}/k\cdot k'. \quad (5)$$

It has the following properties:

$$k'_{\mu'}I^{\mu'\mu}=I^{\mu'\mu}k_\mu=0, \quad (6)$$
$$I^{\mu'}{}_{\tau'}M^{\tau'\nu',\tau\nu}I_\tau{}^\mu=M^{\mu'\nu',\mu\nu}. \quad (7)$$

Equation (7) follows from the gauge-invariance condition on the M function:

$$k'_{\mu'}M^{\mu'\nu',\mu\nu}=M^{\mu'\nu',\mu\nu}k_\mu=0. \quad (8)$$

From Eqs. (4) and (7) we obtain

$$M=IMI=\textstyle\sum_i(Il_iI)B_i, \quad (9)$$

where the vector indices are suppressed and $(IMI)=I^{\mu'}{}_{\tau'}M^{\tau'\nu',\tau\nu}I_\tau{}^\mu$. On the right-hand side of Eq. (9), the new tensor basis $\{Il_iI\}$ is now explicitly gauge-invariant because of Eq. (6). However, because of the momentum-dependent denominator appearing in (5), this basis is not a *polynomial basis*. In **A**, a procedure to obtain a minimal polynomial basis from $\{Il_iI\}$ by taking appropriate linear combinations is outlined. We follow this procedure here. First we write down the simple tensors[7] $\{l_i\}$:

$$\begin{aligned}
l_1&=g^{\mu'\mu}g^{\nu'\nu},\\
l_2&=g^{\mu'\mu}P^{\nu'}P^{\nu},\\
l_3&=g^{\mu'\mu}K^{\nu'}K^{\nu},\\
l_4&=g^{\mu'\mu}(P^{\nu'}K^{\nu}+K^{\nu'}P^{\nu}),\\
l_5&=P^{\mu'}P^{\mu}g^{\nu'\nu},\\
l_6&=P^{\mu'}P^{\mu}P^{\nu'}P^{\nu},\\
l_7&=P^{\mu'}P^{\mu}K^{\nu'}K^{\nu},\\
l_8&=P^{\mu'}P^{\mu}(P^{\nu'}K^{\nu}+K^{\nu'}P^{\nu}),\\
l_9&=g^{\mu'\nu'}g^{\mu\nu},\\
l_{10}&=g^{\mu'\nu}g^{\mu\nu'},\\
l_{11}&=g^{\mu'\nu'}P^{\mu}P^{\nu}+g^{\mu\nu}P^{\mu'}P^{\nu'},\\
l_{12}&=g^{\mu'\nu'}P^{\mu}K^{\nu}+g^{\mu\nu}P^{\mu'}K^{\nu'},\\
l_{13}&=g^{\mu'\nu}P^{\mu}P^{\nu'}+g^{\mu\nu'}P^{\mu'}P^{\nu},\\
l_{14}&=g^{\mu'\nu}P^{\mu}K^{\nu'}+g^{\mu\nu'}P^{\mu'}K^{\nu}.
\end{aligned} \quad (10)$$

[4] W. A. Bardeen and Wu-Ki Tung, Phys. Rev. **173**, 1423 (1968).
[5] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).
[6] A polynomial tensor basis is said to be minimal if any other such basis can be expressed in terms of this without introducing kinematic singularities. For more precise discussions see Ref. 4.

[7] There are 27 simple tensors satisfying the required Lorentz transformation and space, time-reversal properties. We have not written down the ones containing $k^{\mu'}$ or k'^{μ} because they automatically drop out when the new basis $\{Il_iI\}$ is formed as a consequence of (6).

Forming $\{Il_iI\}$ and using the procedure of A, we obtain the following set of minimal polynomial tensors:

$$\begin{aligned}
\mathcal{K}_i&=K^2Il_iI\,,\quad i=1,2,3,4\\
\mathcal{K}_5&=I[K^2(l_9-l_{10})+l_4]I\,,\\
\mathcal{K}_6&=I[K^2(l_{11}-l_{13})+2(P\cdot K)l_2]I\,,\\
\mathcal{K}_7&=I[K^2(l_{12}-l_{14})+(P\cdot K)l_4]I\,,\\
\mathcal{K}_8&=I[K^2l_5+\tfrac{1}{2}(P\cdot K)^2l_1]I\,,\\
\mathcal{K}_9&=I[(l_6+l_7)+\tfrac{1}{2}(P\cdot K)^2(l_9+l_{10})-\tfrac{1}{2}(P\cdot K)(l_{11}-l_{13})\\
&\qquad-\tfrac{1}{2}(P\cdot K)(l_{12}+l_{14})]I\,,\qquad(11)\\
\mathcal{K}_{10}&=I[l_8+\tfrac{1}{2}(P\cdot K)^2(l_9-l_{10})-\tfrac{1}{2}(P\cdot K)(l_{11}+l_{13})\\
&\qquad-\tfrac{1}{2}(P\cdot K)(l_{12}-l_{14})]I\,,\\
\mathcal{K}_{11}&=I[K^2(l_9+l_{10})+(l_2+l_3)]I\,,\\
\mathcal{K}_{12}&=I[K^2(l_{11}+l_{13})+(P\cdot K)l_4]I\,,\\
\mathcal{K}_{13}&=I[K^2(l_{12}+l_{14})+2(P\cdot K)l_3]I\,,\\
\mathcal{K}_{14}&=I[K^2l_6+\tfrac{1}{2}(P\cdot K)^2l_2]I\,.
\end{aligned}$$

The $\mathcal{K}_i$'s are not completely independent because from a helicity count one finds there should be only 12 independent amplitudes. Indeed, there exist two linear relations among the $\mathcal{K}_i$.[8] They are

$$2(P\cdot K)\mathcal{K}_2-P^2\mathcal{K}_4+[P^2K^2-(P\cdot K)^2]\mathcal{K}_5-K^2\mathcal{K}_6+(P\cdot K)\mathcal{K}_7=0\qquad(12)$$

and

$$[2P^2K^2-(P\cdot K)^2]\mathcal{K}_1-(P^2+2K^2)\mathcal{K}_2-P^2\mathcal{K}_3+(P\cdot K)\mathcal{K}_4+(P\cdot K)\mathcal{K}_6-2K^2\mathcal{K}_8+2K^2\mathcal{K}_9-P^2K^2\mathcal{K}_{11}+K^2\mathcal{K}_{12}=0.\qquad(13)$$

These relations allow us to eliminate two redundant linear combinations of $\mathcal{K}_i$. These must be chosen in a way such that all the 14 $\mathcal{K}_i$'s can be expressed in terms of the remaining 12 independent ones with polynomial coefficients. This then ensures that the final tensor basis, called $\{\mathcal{L}_i\}$, is a *minimal polynomial basis*. The result is[9,10]

$$\begin{aligned}
&\mathcal{L}_1=\mathcal{K}_1\,,\quad \mathcal{L}_2=\mathcal{K}_5\,,\quad \mathcal{L}_3=\mathcal{K}_2\,,\\
&\mathcal{L}_4=\mathcal{K}_3+2\mathcal{K}_8-2\mathcal{K}_9+P^2\mathcal{K}_{11}-\mathcal{K}_{12}-P^2\mathcal{K}_1\,,\\
&\mathcal{L}_5=\mathcal{K}_4+\mathcal{K}_6\,,\quad \mathcal{L}_6=\mathcal{K}_7-(P\cdot K)\mathcal{K}_5\,,\\
&\mathcal{L}_7=\mathcal{K}_{11}-\mathcal{K}_1\,,\quad \mathcal{L}_8=\mathcal{K}_8-\tfrac{1}{2}P^2\mathcal{K}_1\,,\qquad(14)\\
&\mathcal{L}_9=\mathcal{K}_{10}-\tfrac{1}{2}P^2\mathcal{K}_5+\tfrac{1}{2}\mathcal{K}_6\,,\\
&\mathcal{L}_{10}=\mathcal{K}_{12}-2\mathcal{K}_2\,,\\
&\mathcal{L}_{11}=\mathcal{K}_{13}-2(P\cdot K)\mathcal{K}_{11}+2(P\cdot K)\mathcal{K}_1-\mathcal{K}_4\,,\\
&\mathcal{L}_{12}=\mathcal{K}_{14}-\tfrac{1}{2}P^2\mathcal{K}_2\,.
\end{aligned}$$

In constructing this basis $\{\mathcal{L}_i\}$, we have taken into account Lorentz invariance, space- and time-reversal invariance, and gauge invariance. A final requirement is that of *s*-*u* crossing symmetry or, equivalently, Bose symmetry of the two-photon states in the *t* channel. This demands

$$M^{\mu'\nu',\mu\nu}(\Delta,K,P)=M^{\mu\nu',\mu'\nu}(\Delta,-K,P).\qquad(15)$$

A quick look at definitions (10), (11), and (14) reveals that $\mathcal{L}_2$, $\mathcal{L}_5$, $\mathcal{L}_9$, and $\mathcal{L}_{11}$ are odd under *s*-*u* crossing while the rest are even. To satisfy (15), the usual practice would demand that the invariant amplitudes associated with the odd $\mathcal{L}_i$ be odd under *s*-*u* crossing. It is much more natural, however, to have crossing symmetry built in from the beginning by using a tensor basis satisfying (15) and otherwise still *minimal*. Such a basis, called $\{\mathcal{L}_i'\}$, consists of

$$\begin{aligned}
&\mathcal{L}_i'=(P\cdot K)\mathcal{L}_i\,,\quad i=2,5,9,11\\
&\mathcal{L}_i'=\mathcal{L}_i\,,\quad \text{otherwise.}
\end{aligned}\qquad(16)$$

The fact that $\{\mathcal{L}_i'\}$ is, indeed, the right choice can be seen in another way. As mentioned before, if $\{\mathcal{L}_i\}$ were used, the invariant amplitudes corresponding to $i=2$, 5, 9, and 11 would be odd under *s*-*u* crossing. Hence these amplitudes must vanish when $s=u$ [or $P\cdot K=\frac{1}{4}(s-u)=0$]. In other words, they must have a *kinematic zero* when $P\cdot K=0$. Removing this kinematic zero amounts to choosing $\{\mathcal{L}_i'\}$ as our tensor basis.

If we now define the invariant functions by the expansion

$$M^{\mu'\nu',\mu\nu}=\sum_{i=1}^{12}\mathcal{L}_i'^{\mu'\nu',\mu\nu}A_i(s,t)\,,\qquad(17)$$

the A_i's are free of all kinematic singularities and zeros.[4] In particular, all the A_i's are even under *s*-*u* crossing.

It is straightforward, although somewhat tedious, to compare our results with recent analyses of the kinematic-singularity structure of the helicity amplitudes. It suffices to write down the relation between the A_i and the *t*-channel helicity amplitudes:

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[8] This has its origin in the fact that the $\{l_i\}$ are not all independent. Among the 27 simple tensors that one can write down, there exist two complicated linear relations. For our purpose it is necessary to keep all the l_i until the end because all of them are needed in constructing $\{\mathcal{K}_i\}$ to satisfy the requirement of gauge invariance. Only after arriving at $\{\mathcal{K}_i\}$ can we decide which two particular combinations of l_i can be eliminated to yield a *minimal* gauge-invariant polynomial basis, as we shall demonstrate in the following.

[9] The linear combinations $\mathcal{L}_4$ and $\mathcal{L}_5$ are the essential ones formed to effect the elimination of redundant tensors as explained in the text. The other combinations $\mathcal{L}_6$, $\mathcal{L}_7$, …, $\mathcal{L}_{12}$ are formed for later convenience.

[10] See the last reference in Ref. 2.

$$f_{11,11}{}^t+f_{-1-1,11}{}^t=(1/p^2)\{-2P^2K^2A_1-(P^2+K^2)[P^2K^2-(P\cdot K)^2]A_4\},$$
$$f_{11,11}{}^t-f_{-1-1,11}{}^t=2(k/p)\{-(P\cdot K)^2A_2+P^2K^2A_6\},$$
$$(\sin\psi)^{-1}[f_{10,11}{}^t+f_{-10,11}{}^t]=(\sqrt{2}/m)k(P\cdot K)\{-K^2A_2-P^2A_4-P^2K^2A_5\},$$
$$(\sin\psi)^{-1}[f_{10,11}{}^t-f_{-10,11}{}^t]=(\sqrt{2}/m)(k^2/p)(P\cdot K)\{-P^2A_2-K^2A_4-P^2K^2A_5\},$$
$$(\sin\psi)^{-2}f_{1-1,11}{}^t=-\tfrac{1}{2}m^2k^2\{A_4\},$$
$$f_{00,11}{}^t=(1/m^2p^2)\{P^2K^2(P^2+K^2)A_1+2P^2K^2(P\cdot K)^2A_2+P^4K^4A_3+[(P^4+K^4)(P\cdot K)^2-P^4K^4]A_4+2P^2K^2(P^2+K^2)(P\cdot K)^2A_5\},$$
$$(\sin\psi)^{-1}\left[\frac{f_{10,1-1}{}^t}{1+\cos\psi}+\frac{f_{-10,1-1}{}^t}{1-\cos\psi}\right]=(\sqrt{2}/m)k^3\{-A_7-P^2A_{10}\}, \tag{18}$$
$$(\sin\psi)^{-1}\left[\frac{f_{10,1-1}{}^t}{1+\cos\psi}-\frac{f_{-10,1-1}{}^t}{1-\cos\psi}\right]=(1/\sqrt{2}m)pk^2(P\cdot K)\{-P^2A_9-2K^2A_{11}\},$$
$$\frac{f_{1-1,1-1}{}^t}{(1+\cos\psi)^2}+\frac{f_{-11,1-1}{}^t}{(1-\cos\psi)^2}=k^2\{-A_7\},$$
$$\frac{f_{1-1,1-1}{}^t}{(1+\cos\psi)^2}-\frac{f_{-11,1-1}{}^t}{(1-\cos\psi)^2}=2k^3p(P\cdot K)\{A_{11}\},$$
$$(\sin\psi)^{-2}f_{11,1-1}{}^t=k^2\{-\tfrac{1}{2}A_7+\tfrac{1}{2}P^2A_8\},$$
$$(\sin\psi)^{-2}f_{00,1-1}{}^t=(k^2/m^2)\{-K^2A_7-\tfrac{1}{2}P^2(P^2+K^2)A_8-2P^2K^2A_{10}-\tfrac{1}{2}P^4K^2A_{12}\}.$$

In the above equations ψ denotes the t-channel c.m. scattering angle and k and p are the magnitudes of the 3-momenta of the photon and the target particle, respectively. The curly brackets on the right-hand side of these equations correspond to the regularized helicity amplitudes.[11] Notice that with our choice of $\{\mathcal{L}_i'\}$ the first six A_i are related only to t-channel amplitudes with photon helicity $(\lambda,\lambda')=(1,1)$, while the last six are related to those with photon helicity $(1,-1)$. In the s channel, the first group corresponds to photon helicity-flip amplitudes and the second group corresponds to photon helicity-nonflip amplitudes.

III. LOW-ENERGY THEOREMS

The fact that the invariant amplitudes A_i are free of kinematic singularities and zeros immediately implies a complete set of low-energy theorems[4] for Compton scattering of a spin-one particle. These are obtained by observing that in the limit of zero photon energy (at fixed scattering angle) the only singularities in A_i are the dynamical singularities due to the single-particle intermediate states in the s and u channels.[12] The contribution of these *pole terms* to A_i can be calculated unambiguously in terms of the static moments of the target particles involved. We shall label these pole terms by $A_i{}^p$ and refer to $A_i{}^c=A_i-A_i{}^p$ as the *continuum contribution* to A_i. The $A_i{}^c$ are finite in the low-energy limit because both kinematic and dynamical singularities are absent. If we express the physical scattering amplitudes (for instance, the helicity amplitudes) in terms of $\{A_i\}$, $A_i{}^c$ only contributes to terms of higher order in the photon energy. The lower-order terms are completely determined by $A_i{}^p$ and thus by the static moments.

Although the explicit form of the low-energy theorems is most useful when expressed in terms of the s-channel physical scattering amplitudes, the main features of these theorems can be seen more easily from the t-channel helicity amplitudes given in the previous section. We note that in the low-energy limit $\sin\psi$, $\cos\psi$, p, and P^2 remain constant while $k=\sqrt{(-K^2)}$ and $P\cdot K$ go to zero as single powers of the photon energy (in the s channel). Inspection of (18) reveals that $A_1{}^c$, $A_4{}^c$, $A_7{}^c$, and $A_8{}^c$ contribute to terms of second or higher orders in the photon energy, $A_2{}^c$, $A_6{}^c$, $A_9{}^c$, and $A_{10}{}^c$ contribute to terms of third or higher orders while $A_3{}^c$, $A_5{}^c$, $A_{11}{}^c$, and $A_{12}{}^c$ contribute to terms of fourth or higher orders. This means that, to first order in the (s-channel) photon energy, all the 12 scattering amplitudes are determined by the static moments of the target particle; to second order, eight linear combinations of the scattering amplitudes depend only on the static moments, and to third order, four linear combinations are

[11] Note that the kinematic zero at $P\cdot K=0$ due to $s-u$ crossing is not considered in the analysis of Ref. 2.

[12] This statement is true only if intermediate states involving photon are excluded or, in other words, electromagnetic interaction are treated only to the lowest order.

completely specified by the static moments. These are *all* the low-energy theorems (for fixed scattering angle) that one can derive from Lorentz invariance, crossing symmetry, and the massless nature of the photon (gauge invariance[13]) without invoking any *dynamical* assumptions.

In order to specify all the 12 independent amplitudes to second order in the photon energy, one needs to know, in addition to the static moments, four dynamical structure-dependent constants, namely, A_1^c, A_4^c, A_7^c, and A_8^c evaluated at threshold. To determine completely the scattering amplitudes to third order in the photon energy, one has to know the first derivatives of the above amplitudes and also the value of A_2^c, A_6^c, A_9^c, and A_{10}^c at threshold. The detailed expressions for the expansion of the s-channel helicity amplitudes to third order in the photon energy are rather involved, and we shall not write them down here. In the Appendix we give the results for the six photon helicity-nonflip amplitudes. These will be used in the next section to derive various sum rules.

The low-energy theorems discussed above can be applied to Compton scattering by any spin-one particle. The deuteron is one good example. In this case, as well as in those involving other nuclear targets, it would be of much interest to investigate the connection between the dynamical structure-dependent constants which enter the low-energy theorems and known nuclear wave functions of the target particle. In this respect, these low-energy theorems seem to be of more physical interest than those for nucleon Compton scattering. In the latter case, the corresponding unknown constants[14] cannot be related to any well-formulated dynamical structure.

IV. DISPERSION RELATIONS AND SUM RULES

The invariant amplitudes A_i, being free of kinematic singularities and zeros, individually have the analytic structure of a scattering amplitude for spinless particles. As such, they are the most natural ones to satisfy a Mandelstam representation without any *ad hoc* subtractions. In contrast to these, other sets of kinematic singularity-free amplitudes, notably the regularized helicity amplitudes, must satisfy certain constraint equations which require the imposition of such subtractions. In other words, with all the appropriate kinematic factors explicitly separated out, the A_i have more convergent asymptotic behavior than the other amplitudes. To get some idea about the asymptotic behavior of the A_i, let us assume that the helicity amplitudes satisfy a certain upper bound:

$$f_{\{\lambda\}}\sim C(s) \quad \text{as} \quad s\to\infty \quad \text{for fixed } t. \tag{19}$$

We then find

$$\begin{gathered} A_1,\, A_3,\, A_6\sim C(s)\,, \\ A_2,\, A_4,\, A_5,\, A_7,\, A_8,\, A_{10},\, A_{12}\sim C(s)s^{-2}\,, \\ A_9,\, A_{11}\sim C(s)s^{-3}. \end{gathered} \tag{20}$$

Similarly, if

$$f_{\{\lambda\}}\sim C(t) \quad \text{as} \quad t\to\infty \quad \text{for fixed } s\,, \tag{21}$$

we obtain

$$\begin{gathered} A_1,\, A_4,\, A_6,\, A_7\sim C(t)\,, \\ A_2-A_4,\, A_3+5A_4,\, A_5,\, A_8,\, A_{10}\sim C(t)t^{-1}\,, \\ A_9,\, A_{11},\, A_{12}\sim C(t)t^{-2}. \end{gathered} \tag{22}$$

It is possible to obtain better bounds in the latter case $(t\to\infty)$ by taking further combinations; however, we shall not be concerned with it here.

The double spectral representation of A_i can be written in the following form:

$$\begin{aligned} A_i(s,t,u)=&\frac{R_i^{su}}{(s-m^2)(u-m^2)}+\frac{R_i^t}{t-m_t^2} \\ &+\frac{1}{\pi^2}\int\int ds'dt'\frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} \\ &+\frac{1}{\pi^2}\int\int dt'du'\frac{\rho_i^{tu}(t',u')}{(t'-t)(u'-u)} \\ &+\frac{1}{\pi^2}\int\int du'ds'\frac{\rho_i^{us}(u',s')}{(u'-u)(s'-s)}\,, \end{aligned} \tag{23}$$

where m_t denotes the mass of possible meson poles in the t channel. The residue functions R_i^{su} can be expressed in terms of the electromagnetic static moments of the target particles while R_i^t are products of meson-photon and meson–target-particle coupling constants. Because of s-u crossing symmetry of the A_i, we have $\rho^{st}(s,t)=\rho^{tu}(t,s)$ and $\rho^{us}(u,s)=\rho^{us}(s,u)$. From Eq. (23) one can deduce various kinds of single-dispersion relations for A_i. The fixed-s and fixed-scattering-angle (in the s channel) dispersion relations[15] assume the usual form; we shall not discuss them explicitly in the following. For the fixed-t dispersion relations, it is desirable to make use of the s-u crossing symmetry of the A_i. In terms of the variables t and $\nu\equiv P\cdot K=\frac{1}{4}(s\text{-}u)$, the invariant amplitudes satisfy

$$A_i(\nu,t)=A_i(-\nu,\,t)\,. \tag{24}$$

The fixed-t dispersion relations, therefore, take the simple form

$$A_i(\nu,t)=\frac{R_i^{su}}{(\frac{1}{2}t)^2-(2\nu)^2}+\frac{1}{\pi}\int_{\nu_0^2}^{\infty}d\nu'^2\frac{\text{Im}A_i(\nu',t)}{\nu'^2-\nu^2}\,, \tag{25}$$

[13] That gauge invariance in the on-shell s-matrix formalism is equivalent to the correct Lorentz transformation law for massless particles is advocated by S. Weinberg, Phys. Rev. **135**, B1049 (1964); **138**, B988 (1965).

[14] Two unknown constants enter the second-order low-energy theorem for nucleon Compton scattering. Choudhury and Freedman (Ref. 1) termed these *dynamical electric and magnetic* polarizabilities of the nucleon. For explanation and more references, see their paper.

[15] See, for instance, A. Hearn and E. Leader, Phys. Rev. **126**, 879 (1962), for the case of nucleon Compton scattering.

where ν_0 denotes the inelastic threshold. We remark that provided the helicity amplitudes do not grow like ν^2 or faster, the unsubtracted form of the dispersion relations (25) is justified for all A_i except for $i=1$, 3, and 6 [see Eqs. (19) and (20)]. In the rest of this section, we shall only use Eq. (25) for $A_7, A_8, \cdots, A_{12}$.

We now turn to discussions of possible sum rules that can be derived for Compton scattering of spin-one particles. The conventional method[16] for deriving sum rules depends on two inputs. The first is a low-energy theorem for a certain scattering amplitude. (This may follow from considerations similar to those discussed above or from some dynamical model like current algebra.[16]) The second ingredient is the assumption of an unsubtracted dispersion relation for the same amplitude. We have seen in the last section that when the Compton scattering is described in terms of the invariant amplitudes A_i, the low-energy theorems are immediate consequences of the fact that these amplitudes are free of kinematic singularities.[17] The only assumption needed for the derivation of sum rules is, therefore, that concerning the asymptotic behavior of the A_i. Here, an unsubtracted dispersion relation is not sufficient as the value of A_i^c is not known anywhere *a priori*. *In order to satisfy a sum rule, A_i must be superconvergent.* This condition is equivalent to those necessary for the derivation of sum rules in the conventional method mentioned above. The reason is obvious; when the physical scattering amplitudes are expressed in terms of A_i, there must be momentum factors which cancel the singularity of the pole term in (23) or (25) at threshold (thus giving rise to low-energy theorems for these amplitudes); these same momentum factors make the relevent A_i superconvergent when the physical amplitude satisfies an unsubtracted dispersion relation and vice versa. The advantage of using the A_i in deriving sum rules lies in the fact that a systematic and complete discussion is made easier because all the kinematical effects are already taken care of in advance.

Instead of arbitrarily assuming a certain asymptotic behavior for the A_i and deriving many sum rules of uncertain significance, we shall try to start with specific models for the asymptotic behavior and see what sum rules are expected to arise. In this way, it will be clear what is being tested when it becomes possible to compare the sum rules with experiment. As it turns out, we are forced to rely heavily on Regge phenomenology which can either be regarded as consequences of a fundamental theory or, more appropriately for our purpose at least, as a convenient parametrization of high-energy scattering phenomenology which is not inconsistent with existing experimental data.

We have a final remark before entering into the actual derivation of sum rules. We note that all A_i in our problem are even functions of ν for fixed t. As a consequence, they satisfy a trivial superconvergence relation in ν. Nontrivial results can be obtained only if the odd function νA_i is superconvergent or, in other words, if A_i converges faster than $\nu^{-2-\epsilon}$ where $\epsilon>0$. Similarly, the superconvergence relation for $\nu^2 A_i$ is trivial; the next nontrivial relation can be obtained only if A_i converges faster than $\nu^{-4-\epsilon}$.

We start from the least restrictive of asymptotic behaviors for the scattering amplitude—the Froissart bound for the helicity amplitudes. (It has been proved[18] from axiomatic field theory that all two-body helicity amplitudes *not involving* massless particles satisfy the Froissart bound. Similar proof does not exist for photon processes. Therefore, we have to take this as an assumption.) We then find, from Eq. (20), that the most convergent of the invariant amplitudes behave like $\nu^{-2}\ln\nu$ [$C(s)=s\ln s$] for fixed t. According to our discussion in the preceding paragraph, this is just short of being superconvergent. Similarly, from Eqs. (22) we find the most convergent amplitudes in the variable t (for fixed s) behave like $t^{-1}\ln t$, again just short of being superconvergent. We conclude then that no sum rule exists if only the Froissart bound is applied.[19]

We therefore look for stronger bounds on the invariant amplitudes by assuming Regge asymptotic behavior. Here one is confronted from the beginning with the problem of the existence of fixed poles[20] in the complex angular momentum plane. If fixed poles exist at nonsense right signature values in the J plane, in particular at $J=1$, then asymptotically all the Compton scattering amplitudes would be dominated by this fixed pole. This means $C(s)\sim s$ in Eq. (19) and again no superconvergence relations can be derived. To go further, therefore, one has to assume that fixed poles do not exist for $J\geq 1$. One then may look for the contributions of the moving (Regge) poles to the various amplitudes. It is not hard to see that among the six photon helicity-nonflip amplitudes A_7, A_8, A_{10}, and A_{12} receive contributions from Regge poles with even charge conjugation and normal parity $[(-1)^J P=1]$, while A_9 and A_{11} receive contributions from Regge poles with even charge conjugation and abnormal parity $[(-1)^J P=-1]$. The leading trajectory in the first group is the Pomeranchuk trajectory with $\alpha(t)\lesssim 1$, $\alpha(0)=1$. There is no well-established trajectory in the second group; all the possible candidates are expected to have $\alpha(t)<0$ (for example, η and π trajectories). Therefore, we can derive the following sum rules for A_9 and A_{11}:

$$\tfrac{1}{4}\pi R_i^{su}=-\int_{\nu_0^2}^{\infty} d\nu'^2\,\mathrm{Im}A_i(\nu'^2,t)\,,\quad i=9,\,11. \quad (26)$$

[16] See, for example, S. Drell and A. Hearn, Phys. Rev. Letters **16**, 908 (1966); and M. A. B. Bég, *ibid.* **17**, 333 (1966).

[17] The fact that A_i are also free of kinematic zeros enabled us to derive more higher-order low-energy theorems than one can obtain from other sets of kinematic singularity-free amplitudes.

[18] G. Mahoux and A. Martin, Phys. Rev. **174**, 2140 (1968).

[19] This agrees with the conclusion of Ref. 18.

[20] This sum rule has previously been obtained by H. Pagels, Phys. Rev. **158**, 1566 (1967); and A. Pais, Nuovo Cimento **53A**, 433 (1968).

The residue functions R_i^{su} are given in the Appendix. Of special interest is the $t=0$ limit of (26) because in that limit the integrand may be related to total cross sections through the optical theorem. A glance at Eq. (A2), however, reveals that A_{11} decouples from the physical scattering amplitudes in the forward direction. The only total cross-section sum rule that can be derived under the assumed asymptotic behavior for A_i is therefore the $t=0$ limit of (26) for A_9. From results given in the Appendix, it is easy to verify that this sum rule takes the form[20]

$$\frac{\pi^2\alpha}{m^2}(2m\mu-2)^2=\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma_p(\nu)-\sigma_a(\nu)], \qquad (27)$$

where σ_p and σ_a are the total photoabsorption cross sections for polarized photons on a polarized target with their spins parallel and antiparallel to each other, respectively. Also, in the above equation, α stands for the fine-structure constant and μ is the magnetic moment of the target particle. It is quite obvious that Eq. (27) is the analog of the familiar Drell-Hearn sum rule[16] for nucleon Compton scattering. We note in particular that the quadrupole moment Q cannot satisfy a total cross-section sum rule. In the forward direction Q only enters A_{11} and A_{12}, but neither of these amplitudes contributes to the physical scattering amplitudes.[21] Of course, Q does enter the general sum rules (26) for both A_9 and A_{11}; however, they are rather hard to test experimentally away from the forward direction.

Adhering to Regge asymptotic behavior, the only possible way to obtain additional sum rules is when the target particle carries isotopic spin. Let us consider the case of $I=1$. Each amplitude can be decomposed into three parts corresponding to $I=0$, 1, 2 in the t channel. The dominant trajectory in the $I=0$ and 1 pieces of the amplitudes corresponding to normal parity (A_7, A_8, A_{10}, and A_{12}) are the Pomeranchuk and the A_2 trajectories, respectively. There is no known trajectory with $I=2$. Asymptotically, these amplitudes behave like $\nu^{-2+\alpha(t)}$ for fixed t. We only obtain sum rules if $\alpha(t)<0$. This presumably is true for the $I=2$ amplitudes; it may also be true for the $I=1$ amplitudes at large momentum transfers $[-t\gtrsim 0.5\ (\mathrm{GeV}/c)^2]$. Let us denote by $\mathcal{A}_i^I$ ($I=0$, 1, 2) the t-channel amplitudes with isotopic spin I and by $A_i^{I_3}$ ($I_3=+$, 0, $-$) the amplitudes for scattering of photon on a target with given I_3 in the s channel; then

$$\mathcal{A}_i^0=(\sqrt{\tfrac{1}{3}})A_i^+-(\sqrt{\tfrac{1}{3}})A_i^0+(\sqrt{\tfrac{1}{3}})A_i^-,$$
$$\mathcal{A}_i^1=(\sqrt{\tfrac{1}{2}})A_i^+-(\sqrt{\tfrac{1}{2}})A_i^-,$$
$$\mathcal{A}_i^2=(\sqrt{\tfrac{1}{6}})A_i^++(\sqrt{\tfrac{2}{3}})A_i^0+(\sqrt{\tfrac{1}{6}})A_i^-.$$

[21] Bardakci and Pagels (Ref. 1) obtained a sum rule for Q in terms of an amplitude which decouples from the physical scattering amplitudes at forward direction. A closer examination of their result reveals that the amplitudes they used are related to A_{12}. Since the Pomeranchuk pole contributes to A_{12}, their assumption about the asymptotic behavior does not seem to be justified.

The sum rules that one obtains for $\mathcal{A}_i^1$ and $\mathcal{A}_i^2$ can therefore be expressed in terms of linear combinations of Compton amplitudes on targets in different charge states. In particular, in taking the limit $t\rightarrow 0$ in the sum rules for $\mathcal{A}_7^2$ and $\mathcal{A}_8^2$, we obtain the following total cross-section sum rules:

$$\int_{\nu_0}^{\infty}d\nu[(\sigma_p+\sigma_a-\sigma_0)^++(\sigma_p+\sigma_a-\sigma_0)^- -2(\sigma_p+\sigma_a-\sigma_0')^0]=0, \qquad (28)$$

$$\int_{\nu_0}^{\infty}d\nu[(\sigma_p+\sigma_a)^++(\sigma_p+\sigma_a)^--2(\sigma_p+\sigma_a)^0]=8\pi^2\alpha, \quad (29)$$

where the subscripts p, a, and 0 denote the spin states of the target particle as before and the superscripts $+$, $-$, 0 denote the charge states of the target particle. We shall not write down the explicit form of the more general sum rules involving the nonforward scattering amplitudes. They are much less likely to be susceptible to experimental test than the total cross-section sum rules given above.

So far we have only discussed sum rules with fixed momentum transfer in the s channel. Similar considerations apply to possible sum rules with fixed s for scattering amplitudes evaluated in the t-channel physical region. As a function of t, the invariant functions do not have definite symmetry properties. The invariant amplitudes become superconvergent if they behave like $t^{-1-\epsilon}$ with $\epsilon>0$. This, together with the fact that there is no moving pole with $\alpha(t)=1$ in this channel, means many sum rules can be derived if the assumption of no fixed pole at $J=1$ is made. Although it is a straightforward matter to write them down, we shall not do it here because it is rather unlikely that these sum rules will become experimentally testable in the near future.

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APPENDIX

With our choice of minimal tensor basis $\{\mathcal{L}_i'\}$, the connection between the invariant amplitudes A_i and the s-channel helicity amplitudes is rather involved. In this appendix we give the s-channel photon helicity-nonflip amplitudes in terms of the A_i. We also write down the pole terms of these amplitudes in terms of the electromagnetic static moments of the target particle.

Let us denote by θ, q, and E the scattering angle, the magnitude of the 3-momentum, and the energy of the target particle, respectively, in the c.m. system of the

s channel. We have

$$f_{11,11}{}^s = q^2\{2\sin^2(\tfrac{1}{2}\theta)A_7+\tfrac{1}{2}s\cos^4(\tfrac{1}{2}\theta)A_8+\tfrac{1}{2}(\sqrt{s})q\cos^2(\tfrac{1}{2}\theta)[E+q\cos^2(\tfrac{1}{2}\theta)][2E\sin^2(\tfrac{1}{2}\theta)+(\sqrt{s})\cos^2(\tfrac{1}{2}\theta)]A_9 -\tfrac{1}{2}(\sqrt{s})q\sin^2\theta\, A_{10}+\tfrac{1}{16}sq^2\sin^2\theta\cos^2(\tfrac{1}{2}\theta)A_{12} +2q^2[E+q\cos^2(\tfrac{1}{2}\theta)][2E\sin^2(\tfrac{1}{2}\theta)+(\sqrt{s})\cos^2(\tfrac{1}{2}\theta)]\sin^2(\tfrac{1}{2}\theta)A_{11}\},$$

$$f_{10,11}{}^s = (1/\sqrt{2}m)q^2\sin\theta\{(\sqrt{s})A_7-\tfrac{1}{2}sE\cos^2(\tfrac{1}{2}\theta)A_8-\tfrac{1}{16}sEq^2\sin^2\theta\, A_{12} -\tfrac{1}{4}q[E+q\cos^2(\tfrac{1}{2}\theta)][2(\sqrt{s})E^2+m^2q\cos^2(\tfrac{1}{2}\theta)]A_9+\tfrac{1}{2}(\sqrt{s})q[2E\sin^2(\tfrac{1}{2}\theta)-(\sqrt{s})\cos^2(\tfrac{1}{2}\theta)]A_{10} -\tfrac{1}{2}(\sqrt{s})q^2[E+q\cos^2(\tfrac{1}{2}\theta)][(4E-2q)\sin^2(\tfrac{1}{2}\theta)+(\sqrt{s})\cos^2(\tfrac{1}{2}\theta)]A_{11}\},$$

$$f_{1-1,11}{}^s = \tfrac{1}{4}(\sqrt{s})q^2\sin^2\theta\{\tfrac{1}{2}(\sqrt{s})A_8-\tfrac{1}{2}Eq[E+q\cos^2(\tfrac{1}{2}\theta)]A_9-qA_{10} +q^2[E+q\cos^2(\tfrac{1}{2}\theta)]A_{11}-\tfrac{1}{4}(\sqrt{s})q^2\cos^2(\tfrac{1}{2}\theta)A_{12}\}, \tag{A1}$$

$$f_{10,10}{}^s = (1/m^2)(\sqrt{s})q^2\cos^2(\tfrac{1}{2}\theta)\{-(\sqrt{s})A_7+(\sqrt{s})[-q^2\sin^2(\tfrac{1}{2}\theta)+m^2\cos^2(\tfrac{1}{2}\theta)]A_8 +Eq[E+q\cos^2(\tfrac{1}{2}\theta)]\sin^2(\tfrac{1}{2}\theta)A_9-2(\sqrt{s})Eq\sin^2(\tfrac{1}{2}\theta)A_{10} +2m^2q^2[E+q\cos^2(\tfrac{1}{2}\theta)]\sin^2(\tfrac{1}{2}\theta)A_{11}-\tfrac{1}{2}(\sqrt{s})E^2q^2\sin^4(\tfrac{1}{2}\theta)A_{12}\},$$

$$f_{1-1,10}{}^s = (1/2\sqrt{2}m)sq^2\sin\theta\cos(\tfrac{1}{2}\theta)\{-EA_8+\tfrac{1}{2}q(2E-q)[E+q\cos^2(\tfrac{1}{2}\theta)]A_9 -qA_{10}+q^2[E+q\cos^2(\tfrac{1}{2}\theta)]A_{11}-\tfrac{1}{2}Eq^2\sin^2(\tfrac{1}{2}\theta)A_{12}\},$$

$$f_{1-1,1-1}{}^s = \tfrac{1}{2}sq^2\cos^4(\tfrac{1}{2}\theta)\{A_8-q[E+q\cos^2(\tfrac{1}{2}\theta)]A_9+\tfrac{1}{2}q^2\sin^2(\tfrac{1}{2}\theta)A_{12}\}.$$

In the forward direction, there are three nonvanishing spin-nonflip amplitudes. The above equations reduce to

$$f_{11,11}{}^s(s,\cos\theta=1)=\tfrac{1}{2}sq^2[A_8+(\sqrt{s})qA_9],$$
$$f_{10,10}{}^s(s,\cos\theta=1)=(1/m^2)sq^2(-A_7+m^2A_8), \tag{A2}$$
$$f_{1-1,1-1}{}^s(s,\cos\theta=1)=\tfrac{1}{2}sq^2[A_8-(\sqrt{s})qA_9].$$

Inverting these relations, we get

$$A_7=(m^2/sq^2)(f_{11,11}{}^s+f_{1-1,1-1}{}^s-f_{10,10}{}^s),\quad A_8=(1/sq^2)(f_{11,11}{}^s+f_{1-1,1-1}{}^s),\quad A_9=(1/s^{3/2}q^3)(f_{11,11}{}^s-f_{1-1,1-1}{}^s). \tag{A3}$$

Next, we write down the pole-term contribution to these amplitudes. We have

$$A_i{}^p=e^2R_i/[(s-m^2)(u-m^2)],$$

where $R_i=(1/e^2)R_i{}^{su}$; then

$$R_7=2t(-m^2\mu^2+m^2Q)+\frac{t^2}{4m^2}[b^2+2a(m\mu+m^2Q)]+\frac{t^3}{8m^4}b^2,$$

$$R_8=16+\frac{4t}{m^2}(m^2\mu^2+m^2Q)+\frac{t^2}{2m^4}b^2,$$

$$R_9=\frac{4}{m^2}a^2+\frac{2t}{m^4}b^2, \tag{A4}$$

$$R_{10}=-16m\mu+\frac{t}{m^2}a(1+m^2Q)-\frac{t^2}{2m^4}b^2,$$

$$R_{11}=\frac{4}{m^2}a(m\mu+m^2Q)-\frac{t}{m^4}b^2,$$

$$R_{12}=(32/m^2)b.$$

In the above equations, e, μ, and Q are the charge, magnetic dipole moment, and electric quadrupole moment of the target particle, respectively, and

$$a=2m\mu-2,\quad b=-1+2m\mu+m^2Q. \tag{A5}$$

We have not written down possible poles in the t channel corresponding to the pseudoscalar mesons. These pole terms satisfy gauge invariance by themselves and do not enter the low-energy theorems and sum rules discussed in the text.