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# $K \to 2\pi$ Phenomenology with Small $CP$ Nonconservation

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The theoretical expressions for experimentally determined parameters for $K \to 2\pi$ decays are used to determine solutions for various theoretical parameters. This is accomplished by first assuming $CP$ conservation and then treating the $CP$-nonconserving parts of the theoretical expressions as small corrections. We find two types of solutions: one type for which $|\Delta I| = \frac{1}{2}$ dominance is valid, and one type for which $|\Delta I| = \frac{1}{2}$ dominance is not valid. There are two solutions of each type, but only one solution ($|\Delta I| = \frac{1}{2}$ type) yields the original data when substituted in the exact equations. All four solutions are used as input in a least-squares fit to the data, and are thereby transformed into exact solutions. No other exact solutions were found in a large number of random-input searches. Two of the four solutions agree with values of Re$\epsilon$ determined in leptonic decays; however, they give different predictions for the phase of $\eta_{00}$.

## I. INTRODUCTION

Recent calculations[1] of $K \to 2\pi$ decay parameters have involved several approximations: e.g., $(|\Delta I| = \frac{1}{2}) \gg (\Delta I = \frac{3}{2}) \gg (\Delta I = \frac{5}{2})$. Other calculations[2] have used information from other $K$ decays to aid in determining the $K \to 2\pi$ decay parameters. In the calculation reported here, we use only $K \to 2\pi$ decay experimental information and the approximation that $CP$ violation is a small effect for all possible $|\Delta I|$ values.

We begin with Kenny's[3] formulation of $K \to 2\pi$ decays in terms of reduced matrix elements for $|\Delta I| = \frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ in the decays. By means of the experimental values of various decay ratios, we are able to derive values for the ratio ($\bar{b}_3$) of the complex $|\Delta I| = \frac{3}{2}$ amplitude to the $|\Delta I| = \frac{1}{2}$ amplitude, the ratio ($\bar{b}_5$) of the complex $|\Delta I| = \frac{5}{2}$ amplitude to the $|\Delta I| = \frac{1}{2}$ amplitude, and the complex $K^0 - \bar{K}^0$ mixing parameter ($\epsilon$).

The $CP$-conserving parameters are the ratio of the *real* part of the $|\Delta I| = \frac{3}{2}$ amplitude to the $|\Delta I| = \frac{1}{2}$ amplitude, Re$\bar{b}_3$, and the ratio of the *real* part of the $|\Delta I| = \frac{5}{2}$ amplitude to the $|\Delta I| = \frac{1}{2}$ amplitude, Re$\bar{b}_5$. Since we assume that violation of $CP$ conservation is a small effect, the $CP$-conserving parameters are first calculated assuming $CP$ conservation. Then they are used to calculate the $CP$-nonconserving parameters. Finally, these parameters are used as input in a least-squares fit to the data by means of the exact equations.

In the first, or $CP$-invariant, part of the calculation we obtain two solutions for these two parameters; one is approximately unity in magnitude and the other is approximately $10^{-2}$ in magnitude. The latter solution satisfies the $|\Delta I| = \frac{1}{2}$ dominance rule while the former does not. For each of these two solutions for the $CP$-conserving parameters, we obtain two solutions for the $CP$-nonconserving parameters, Im$\bar{b}_3$, Im$\bar{b}_5$, Re$\epsilon$, and Im$\epsilon$, for a total of four solutions. One each of the $|\Delta I| = \frac{1}{2}$ dominance satisfying and nonsatisfying solutions agrees with values of Re$\epsilon$ determined elsewhere.[4]

Random numbers between $-100$ and $+100$ were used as input in a least-squares fitting of the data to the exact equations in search of other solutions. Approximately 100 such random-input searches yielded only the four solutions mentioned above.

## II. THEORY AND DATA

We begin with the formulation developed by Kenny[3] with two changes. Instead of writing, for example, the reduced matrix element for $|\Delta I| = \frac{3}{2}$ as

$$b_3 e^{-i\phi_3'} \text{ where } b_3 \text{ is real,}$$

we write it as

$$b_3 = \text{Re}b_3 + i\,\text{Im}b_3 .$$

Likewise, the reduced matrix element for $|\Delta I| = \frac{5}{2}$ is

$$b_5 = \text{Re}b_5 + i\,\text{Im}b_5 .$$

If time-reversal ($T$) invariance holds,[3] Im$b_3$=Im$b_5$=0. We assume $CPT$ invariance, and, therefore, $T$ invariance implies $CP$ invariance. Also, instead of using the complex parameter $r$ in

$$|K_{S,L}\rangle = (1+|r|^2)^{-1/2}(|K^0\rangle \pm r|\bar{K}^0\rangle),$$

we use the complex parameter $\epsilon$ in

$$|K_{S,L}\rangle = [2(1+|\epsilon|^2)]^{-1/2} \times [|K^0\rangle \pm |\bar{K}^0\rangle + \epsilon(|K^0\rangle \mp |\bar{K}^0\rangle)].$$

The relation between $r$ and $\epsilon$ is $\epsilon = (1-r)/(1+r)$. It is obvious that we are using the convention $CP|K^0\rangle = |\bar{K}^0\rangle$. If $CP$ invariance holds, $\epsilon = 0$.

The isotopic-spin amplitudes are[3]

$$\langle 0,0|A|K^0\rangle = \beta_0 e^{i\delta_0}, \qquad \langle 0,0|A|\bar{K}^0\rangle = \beta_0 e^{i\delta_0},$$
$$\langle 2,0|A|K^0\rangle = \beta_2 e^{i\delta_2}, \qquad \langle 2,0|A|\bar{K}^0\rangle = \beta_2^* e^{i\delta_2},$$
$$\langle 2,1|A|K^+\rangle = \beta_1 e^{i\delta_2}, \quad \langle 2,-1|A|K^-\rangle = \beta_1^* e^{i\delta_2},$$

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[1] For example, see B. G. Kenny, Ann. Phys. (N. Y.) **43**, 25 (1967), and S. L. Glashow, Phys. Rev. Letters **18**, 524 (1967).
[2] B. R. Martin and E. de Rafael, Phys. Rev. **162**, 1453 (1967).
[3] B. G. Kenny, Ann. Phys. (N. Y.) **43**, 25 (1967).
[4] D. Dorfan *et al.*, in *Proceedings of the 1967 International Symposium on Electron and Photon Interactions at High Energies* (Stanford Linear Accelerator Center, Stanford, Calif., 1968), p. 484; S. Bennett *et al.*, in *Proceedings of the 1967 International Symposium on Electron and Photon Interactions at High Energies* (Stanford Linear Accelerator Center, Stanford, Calif., 1968), p. 494.

where

$$\beta_0=b_1/\sqrt{2},\quad \beta_2=(b_3+b_5)/\sqrt{2},\quad \text{and}\quad \beta_1=\sqrt{\tfrac{3}{4}}[b_3-(\tfrac{2}{3})b_5].$$

Here $b_n$ is the reduced matrix element for $|\Delta I|=\frac{1}{2}n$, $\delta_0$ is the $I=0$ $\pi$-$\pi$ scattering phase shift at $\sim$500 MeV total c.m. energy, and $\delta_2$ is the corresponding $I=2$ phase shift. The arbitrary phase has been chosen such that $\beta_0$ (or $b_1$) is real. $CPT$ conservation is assumed. For $CP$ conservation, $\beta_2$ and $\beta_1$ (or $b_3$ and $b_5$) are real.[3] That is, $\beta_0$, $\mathrm{Re}\beta_1$, and $\mathrm{Re}\beta_2$ are $CP$-conserving parameters, while $\mathrm{Im}\beta_1$ and $\mathrm{Im}\beta_2$ are $CP$-*non*conserving parameters. The parameters $\mathrm{Re}\epsilon$ and $\mathrm{Im}\epsilon$ are also $CP$-*non*conserving parameters.

We define:

$$\bar{b}_3=b_3/b_1,\quad \bar{b}_5=b_5/b_1,$$

$$\bar{\beta}_2=\sqrt{2}\beta_2/b_1=\bar{b}_3+\bar{b}_5,\quad \bar{\beta}_1=\sqrt{\tfrac{4}{3}}\beta_1/b_1=\bar{b}_3-(\tfrac{2}{3})\bar{b}_5;$$

therefore,

$$\bar{b}_3=(2\bar{\beta}_2+3\bar{\beta}_1)/5\quad \text{and}\quad \bar{b}_5=3(\bar{\beta}_2-\bar{\beta}_1)/5.$$

The physical amplitudes are

$$A_{+-}{}^{(S)}=\langle\pi^+\pi^-|A|K_S\rangle=\alpha[\sqrt{2}\beta_0+e^{i(\delta_2-\delta_0)}\,\mathrm{Re}\beta_2+\epsilon(ie^{i(\delta_2-\delta_0)}\,\mathrm{Im}\beta_2)],$$

$$A_{+-}{}^{(L)}=\langle\pi^+\pi^-|A|K_L\rangle=\alpha[ie^{i(\delta_2-\delta_0)}\,\mathrm{Im}\beta_2+\epsilon(\sqrt{2}\beta_0+e^{i(\delta_2-\delta_0)}\,\mathrm{Re}\beta_2)],$$

$$A_{00}{}^{(S)}=\langle\pi^0\pi^0|A|K_S\rangle=\alpha[\beta_0-\sqrt{2}e^{i(\delta_2-\delta_0)}\,\mathrm{Re}\beta_2+\epsilon(-\sqrt{2}ie^{i(\delta_2-\delta_0)}\,\mathrm{Im}\beta_2)],$$

$$A_{00}{}^{(L)}=\langle\pi^0\pi^0|A|K_L\rangle=\alpha[-\sqrt{2}ie^{i(\delta_2-\delta_0)}\,\mathrm{Im}\beta_2+\epsilon(\beta_0-\sqrt{2}e^{i(\delta_2-\delta_0)}\,\mathrm{Re}\beta_2)],$$

and

$$A_{+0}=\beta_1e^{i\delta_2},$$

where

$$\alpha=2e^{i\delta_0}[6(1+|\epsilon|^2)]^{-1/2}.$$

The experimental quantities are:

(i) the $K_S$ branching ratio[5]

$$R=\frac{|A_{+-}{}^{(S)}|^2\rho_{+-}{}^{(S)}}{|A_{00}{}^{(S)}|^2\rho_{00}{}^{(S)}}=2.17\pm0.08,$$

where the phase-space ratio[6] is

$$\frac{\rho_{00}{}^{(S)}}{\rho_{+-}{}^{(S)}}=\left(\frac{M_S{}^2-4m_0{}^2}{M_S{}^2-4m_+{}^2}\right)^{1/2}=1.015\pm0.00006$$

in terms of the $K_S$ mass $M_S$, the $\pi^+$ mass $m_+$, and the $\pi^0$ mass $m_0$;

(ii) the $K_S\to\pi^++\pi^-$ to $K^+\to\pi^++\pi^0$ partial-decay-rate ratio[5]

$$X=\frac{|A_{+-}{}^{(S)}|^2\rho_{+-}{}^{(S)}}{|A_{+0}|^2\rho_{+0}}=462\pm17,$$

where the phase-space ratio is

$$\frac{\rho_{+0}}{\rho_{+-}{}^{(S)}}=\frac{M_S}{M_+{}^2}\left(\frac{[M_+{}^2-(m_+-m_0)^2][M_+{}^2-(m_++m_0)^2]}{M_S{}^2-4m_+{}^2}\right)^{1/2}=1.004\pm0.073,$$

$M_+$ being the $K^+$ mass;

(iii) the $K_L\to\pi^++\pi^-$ to $K_S\to\pi^++\pi^-$ ratio[7]

$$|\eta_{+-}|=|A_{+-}{}^{(L)}|/|A_{+-}{}^{(S)}|=(1.91\pm0.09)\times10^{-3};$$

(iv) the $K_L\to\pi^0+\pi^0$ to $K_S\to\pi^0+\pi^0$ ratio[8]

$$|\eta_{00}|=|A_{00}{}^{(L)}|/|A_{00}{}^{(S)}|=(4.17\pm0.30)\times10^{-3};$$

(v) the phase[9] of $\eta_{+-}$,

$$\theta_{+-}=\tan^{-1}\left(\frac{\mathrm{Im}\eta_{+-}}{\mathrm{Re}\eta_{+-}}\right)=\tan^{-1}\left(\frac{\mathrm{Im}A_{+-}{}^{(L)}\,\mathrm{Re}A_{+-}{}^{(S)}-\mathrm{Re}A_{+-}{}^{(L)}\,\mathrm{Im}A_{+-}{}^{(S)}}{\mathrm{Re}A_{+-}{}^{(L)}\,\mathrm{Re}A_{+-}{}^{(S)}+\mathrm{Im}A_{+-}{}^{(L)}\,\mathrm{Im}A_{+-}{}^{(S)}}\right)=(65\pm11)^\circ.$$

The theoretical expressions for the experimental quantities are:

$$\bar{R}=\frac{R\rho_{00}{}^{(S)}}{\rho_{+-}{}^{(S)}}=\frac{A-2\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2-2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon+f\,\mathrm{Im}\epsilon)+|\epsilon|^2(\mathrm{Im}\beta_2)^2}{B-4\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2+2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon+f\,\mathrm{Im}\epsilon)+2|\epsilon|^2(\mathrm{Im}\beta_2)^2},\tag{1}$$

where

$$f=\cos(\delta_2-\delta_0),\quad g=\sin(\delta_2-\delta_0),\quad A=2\beta_0{}^2+(\mathrm{Re}\beta_2)^2+2\sqrt{2}f\beta_0\,\mathrm{Re}\beta_2,$$

and

$$B=\beta_0{}^2+2(\mathrm{Re}\beta_2)^2-2\sqrt{2}f\beta_0\,\mathrm{Re}\beta_2;$$

$$\bar{X}=\frac{X\rho_{+0}}{\rho_{+-}{}^{(S)}}=\frac{2}{3}\,\frac{A-2\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2-2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon+f\,\mathrm{Im}\epsilon)+|\epsilon|^2(\mathrm{Im}\beta_2)^2}{(1+|\epsilon|^2)|\beta_1|^2};\tag{2}$$

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[5] The world average is used as given by A. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[6] W. Koch and O. Skjeggestad, in Proceedings of the 1964 Easter School for Physics, CERN, Geneva (unpublished).

[7] V. L. Fitch, R. F. Roth, J. Russ, and W. Vernon, Phys. Rev. **164**, 1711 (1967).

[8] J. W. Cronin *et al.*, in Proceedings of the Rochester Conference on Elementary Particles and Fields, 1967 (unpublished). Recent measurements of $|\eta_{00}|$ seem to indicate a smaller value (Cronin, private communication). We use $(3.0\pm0.30)\times10^{-3}$ in our calculations as a representative smaller $|\eta_{00}|$ to indicate how it would change our solutions.

[9] Since the few values available vary between 25° and 84° with large errors, we use the world average given in Ref. 5.

$$|\eta_{+-}|^2=\frac{|\epsilon|^2A+(\mathrm{Im}\beta_2)^2-2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon-f\,\mathrm{Im}\epsilon)+2\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2}{A-2\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2-2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon+f\,\mathrm{Im}\epsilon)+|\epsilon|^2(\mathrm{Im}\beta_2)^2}; \tag{3}$$

$$|\eta_{00}|^2=\frac{|\epsilon|^2B+2(\mathrm{Im}\beta_2)^2+2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon-f\,\mathrm{Im}\epsilon)+4\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2}{B-4\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2+2\sqrt{2}\beta_0\,\mathrm{Im}\beta_2(g\,\mathrm{Re}\epsilon+f\,\mathrm{Im}\epsilon)+2|\epsilon|^2(\mathrm{Im}\beta_2)^2}; \tag{4}$$

$$\theta_{+-}=\tan^{-1}\left(\frac{(1-|\epsilon|^2)\,\mathrm{Im}\beta_2(\mathrm{Re}\beta_2+\sqrt{2}f\beta_0)+A\,\mathrm{Im}\epsilon-(\mathrm{Im}\beta_2)^2\,\mathrm{Im}\epsilon}{-(1+|\epsilon|^2)\sqrt{2}g\beta_0\,\mathrm{Im}\beta_2+A\,\mathrm{Re}\epsilon-(\mathrm{Im}\beta_2)^2\,\mathrm{Re}\epsilon}\right). \tag{5}$$

When using Eq. (5) to determine the parameters, one would finally check that the parameters put $\theta_{+-}$ in the correct quadrant.

Note that $\mathrm{Re}\beta_1$ and $\mathrm{Im}\beta_1$ occur in Eq. (2) only, and therefore cannot be determined separately. For this reason we use the parameter $|\beta_1|$.

All five equations above can be divided top and bottom by $b_1$ (or $\beta_0$), which leaves six parameters to be determined: two *CP*-conserving parameters ($\mathrm{Re}\bar{\beta}_2$, $\mathrm{Re}\bar{\beta}_1$) and four *CP*-nonconserving parameters ($\mathrm{Im}\bar{\beta}_2$, $\mathrm{Im}\bar{\beta}_1$, $\mathrm{Re}\epsilon$, $\mathrm{Im}\epsilon$). However, we have $|\bar{\beta}_1|^2=(\mathrm{Re}\bar{\beta}_1)^2+(\mathrm{Im}\bar{\beta}_1)^2$, a mixture of the two parameter types. So there are five parameters and five equations. In principle, one could solve for the five parameters. A more feasible procedure is to arrive at values for the parameters by some reasonable approximation and then do a least-squares fit with these values as input.

Another possible experimental quantity is the phase of $\eta_{00}$,

$$\theta_{00}=\tan^{-1}\left(\frac{(1-|\epsilon|^2)\,\mathrm{Im}\beta_2(2\,\mathrm{Re}\beta_2-\sqrt{2}f\beta_0)+B\,\mathrm{Im}\epsilon-2(\mathrm{Im}\beta_2)^2\,\mathrm{Im}\epsilon}{(1+|\epsilon|^2)\sqrt{2}g\beta_0\,\mathrm{Im}\beta_2+B\,\mathrm{Re}\epsilon-2(\mathrm{Im}\beta_2)^2\,\mathrm{Re}\epsilon}\right). \tag{6}$$

No experimental data are available for $\theta_{00}$ yet. We shall calculate $\theta_{00}$ using our final parameters.

## III. APPROXIMATION OF SMALL *CP* NONCONSERVATION

It is known[2,10,11] that $\mathrm{Re}\epsilon$ and $|\epsilon|$ are $\sim 10^{-3}$ and, since $|\eta_{+-}|\sim|\eta_{00}|\sim 10^{-3}$, most probably $\mathrm{Re}\epsilon$, $\mathrm{Im}\epsilon$, and $\mathrm{Im}\bar{\beta}_2$ are all $\sim 10^{-3}$. [See Eqs. (3) and (4) above.] Because $\bar{R}-2=0.20\pm0.08\gg 10^{-3}$, we surmise that most probably $\mathrm{Re}\bar{\beta}_2\gg 10^{-3}$ or $\mathrm{Re}\bar{\beta}_2\gg\mathrm{Im}\epsilon$ and $\mathrm{Re}\bar{\beta}_2\gg\mathrm{Im}\bar{\beta}_2$. [See Eq. (1) above.] The latter inequality states that *T* noninvariance, and therefore *CP* noninvariance, is a small effect for $|\Delta I|>\frac{1}{2}$. Using the inequalities above, we can make certain approximations and solve for the theoretical parameters.

The first approximation is

$$\bar{R}\cong\bar{A}/\bar{B}, \tag{1'}$$

where

$$\bar{A}=2+(\mathrm{Re}\bar{\beta}_2)^2+2\sqrt{2}f\,\mathrm{Re}\bar{\beta}_2$$

and

$$\bar{B}=1+2(\mathrm{Re}\bar{\beta}_2)^2-2\sqrt{2}f\,\mathrm{Re}\bar{\beta}_2.$$

Then

$$\mathrm{Re}\bar{\beta}_2=\{\sqrt{2}f(\bar{R}+1)\pm[2f^2(\bar{R}+1)^2-(2\bar{R}-1)(\bar{R}-2)]^{1/2}\}/(2\bar{R}-1)=1.60\pm0.40\text{ or }0.037\pm0.017.$$

We have used[12] $\delta_2-\delta_0=(-52\pm11)°$. Thus, we have one solution (the latter) which agrees with the $|\Delta I|=\frac{1}{2}$ dominance rule, and one which does not. The requirement that $\mathrm{Re}\bar{\beta}_2$ be real restricts $\delta_2-\delta_0$ to the ranges $(-79.4\pm2.1)°\leqslant(\delta_2-\delta_0)\leqslant(79.4\pm2.1)°$ and $(100.6\pm2.1)°\leqslant(\delta_2-\delta_0)\leqslant(259.4\pm2.1)°$.

The second approximation is

$$\bar{X}\cong 4\bar{A}/9|\bar{\beta}_1|^2. \tag{2'}$$

Then

$$|\bar{\beta}_1|=\tfrac{2}{3}\bar{A}/\bar{X}=0.084\pm0.015\text{ or }0.0445\pm0.0009.$$

The third approximation is

$$\theta\equiv\tan\theta_{+-}\cong[\mathrm{Im}\bar{\beta}_2(\mathrm{Re}\bar{\beta}_2+\sqrt{2}f)+\bar{A}\,\mathrm{Im}\epsilon]/[-\sqrt{2}g\,\mathrm{Im}\bar{\beta}_2+\bar{A}\,\mathrm{Re}\epsilon], \tag{5'}$$

or

$$\mathrm{Im}\bar{\beta}_2=Z(\theta\,\mathrm{Re}\epsilon-\mathrm{Im}\epsilon),$$

where

$$Z=\bar{A}/[\mathrm{Re}\bar{\beta}_2+\sqrt{2}(f+g\theta)].$$

The next is

$$|\eta_{+-}|^2\cong[|\epsilon|^2\bar{A}+(\mathrm{Im}\bar{\beta}_2)^2-2\sqrt{2}\,\mathrm{Im}\bar{\beta}_2(g\,\mathrm{Re}\epsilon-f\,\mathrm{Im}\epsilon)+2\,\mathrm{Im}\epsilon\,\mathrm{Re}\bar{\beta}_2\,\mathrm{Im}\bar{\beta}_2]/\bar{A}. \tag{3'}$$

Upon using the expression for $\mathrm{Im}\bar{\beta}_2$ in (5′) above we can rewrite this expression as

$$A_R(\mathrm{Re}\epsilon)^2+A_I(\mathrm{Im}\epsilon)^2+A_M\,\mathrm{Re}\epsilon\,\mathrm{Im}\epsilon=|\eta_{+-}|^2\bar{A}=N_1,$$

where

$$A_R=\bar{A}+Z\theta(Z\theta-2\sqrt{2}g),$$

$$A_I=\bar{A}+Z(Z-2\,\mathrm{Re}\bar{\beta}_2-2\sqrt{2}f),$$

[10] V. L. Fitch, in Proceedings of the Second Hawaii Topical Conference, 1967 (unpublished).

[11] H. Primakoff, Orsay Report No. TH/204, 1967 (unpublished).

[12] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters 18, 630 (1967); W. D. Walker, Rev. Mod. Phys. 39, 695 (1967).

TABLE I. Calculated and measured experimental parameters for $K\to 2\pi$ decay.

| Parameter \ Solution | A. Approximate solutions $\lvert\Delta I\rvert=\frac{1}{2}$ dominance | | $\lvert\Delta I\rvert\neq\frac{1}{2}$ | | Input data | B. Final solutions obtained by least-squares fits to the data[a] $\lvert\Delta I\rvert=\frac{1}{2}$ dominance | | $\lvert\Delta I\rvert\neq\frac{1}{2}$ | |
|---|---|---|---|---|---|---|---|---|---|
| $10^3\times\mathrm{Re}\bar\beta_2$ | 37.1 | 37.1 | 1601 | 1601 | | 37 ±17 | 37 ±17 | 1600 ±400 | 1600 ±400 |
| $10^3\times\lvert\bar\beta_1\rvert$ | 44.5 | 44.5 | 83.9 | 83.9 | | 44. 5± 0.9 | 44.5 ± 0.9 | 84 ± 15 | 84 ± 15 |
| $10^3\times\mathrm{Im}\bar\beta_2$ | −1.19 | 3.22 | −18.2 | 11.2 | | −1.11± 0.20 | 2.67± 0.25 | −7 ± 12 | 5. 5± 4.0 |
| $10^3\times\mathrm{Re}\epsilon$ | 1.45 | −0.929 | 3.57 | −0.894 | | 1.41± 0.40 | −0.63± 0.40 | 1.8± 1.2 | −0.03± 0.30 |
| $10^3\times\mathrm{Im}\epsilon$ | 2.26 | 0.317 | 7.86 | −2.04 | | 2.22± 0.25 | 0.56± 0.50 | 4.0± 4.0 | −0.12± 1.4 |
| $10^3\times\lvert\epsilon\rvert$ | 2.68 | 0.982 | 8.63 | 2.46 | | 2.63± 0.15 | 0.84± 0.50 | 4.4± 4.5 | 0.13± 1.5 |
| $\theta_\epsilon$ (deg) | 57.3 | 161.2 | 65.6 | −113.6 | | 58 ±10 | 139 ±25 | 65 ± 30 | −103 ±175 |
| $\theta_{00}$ (deg) | 48.7 | −153.1 | −26.8 | 128.7 | | 49 ±10 | −156 ±20 | −11 ± 20 | 117 ± 10 |
| $\theta_{+-}$ (deg) | 65 | 65 | 65 | 65 | 65 ±11 | 65 | 65 | 65 | 65 |
| $10^3\times\lvert\eta_{+-}\rvert$ | 1.91 | 1.91 | 1.91 | 1.91 | 1.91± 0.09 | 1.91 | 1.91 | 1.91 | 1.91 |
| $10^3\times\lvert\eta_{00}\rvert$ | 4.35 | 5.33 | 10.8 | 7.43 | 4.17± 0.30 | 4.17 | 4.17 | 4.17 | 4.17 |
| $R$ | 2.17 | 2.17 | 2.17 | 2.17 | 2.17± 0.08 | 2.17 | 2.17 | 2.17 | 2.17 |
| $X$ | 462 | 462 | 462 | 462 | 462 ± 17 | 462 | 462 | 462 | 462 |
| | 1 | 2 | 3 | 4 | Solution No. | 1 | 2 | 3 | 4 |

[a] The errors are the usual least-squares-fit errors that cause an increase in $\chi^2$ of 1, except that the $\delta_2-\delta_0$ error is accounted for roughly by doing the calculation with $\delta_2-\delta_0$ at its limits since it does not enter into the $\chi^2$ error calculation. Where the errors are large, they are usually due to the $\delta_2-\delta_0$ error. An error is given for $\theta_{00}$ in order that future measurements of this parameter may be easily compared to these solutions.

and

$$A_M=2Z[\theta(\mathrm{Re}\bar\beta_2+\sqrt{2}f-Z)+\sqrt{2}g].$$

The last is

$$|\eta_{00}|^2\cong[|\epsilon|^2\bar B+2(\mathrm{Im}\bar\beta_2)^2+2\sqrt{2}\,\mathrm{Im}\bar\beta_2(g\,\mathrm{Re}\epsilon-f\,\mathrm{Im}\epsilon)+4\,\mathrm{Im}\epsilon\,\mathrm{Re}\beta_2\,\mathrm{Im}\beta_2]/\bar B. \quad (4')$$

Upon using the expression for $\mathrm{Im}\bar\beta_2$ in Eq. (5′) above we can rewrite this expression as

$$B_R(\mathrm{Re}\epsilon)^2+B_I(\mathrm{Im}\epsilon)^2+B_M\,\mathrm{Re}\epsilon\,\mathrm{Im}\epsilon=|\eta_{00}|^2\bar B=N_0,$$

where

$$B_R=\bar B+Z\theta(Z\theta+2\sqrt{2}g),$$
$$B_I=\bar B+Z(Z-4\,\mathrm{Re}\bar\beta_2+2\sqrt{2}f),$$

and

$$B_M=2Z[\theta(2\,\mathrm{Re}\bar\beta_2-\sqrt{2}f-Z)-\sqrt{2}g].$$

Equation (3′) and (4′) can be combined to yield

$$C(\mathrm{Re}\epsilon)^4+D(\mathrm{Re}\epsilon)^2-E^2=0, \quad (3'')$$

where

$$C=v^2(A_M{}^2-4A_IA_R)-w^2,$$
$$D=4v^2A_IN_1-2wE,$$
$$E=B_IN_1/A_I-N_0,$$
$$v=(B_M-B_IA_M/A_I)/2A_I,$$

and

$$w=B_R+(A_M{}^2B_I/A_I-2A_RB_I-A_MB_M)/2A_I.$$

When using Eq. (3″) and, say, Eq. (3′) to obtain solutions for $\mathrm{Re}\epsilon$ and $\mathrm{Im}\epsilon$, one must check to see which solutions satisfy Eq. (4′) since extraneous solutions have been introduced by squaring in the process of obtaining Eq. (3″).

After obtaining $\mathrm{Re}\epsilon$ and $\mathrm{Im}\epsilon$, we can use Eq. (5′) to obtain $\mathrm{Im}\bar\beta_2$.

All four solutions are then put into the exact equations to calculate values for the experimental quantities. All but one do not agree with the measured value of $|\eta_{00}|$. All four solutions are shown in Table I, along with the calculated and measured experimental numbers. Errors have not been propagated through the calculation because they are much more easily obtained in the least-squares fit described below.

These four solutions are used as input in a least-squares fit to the data. A very fast search procedure[13] developed by Arndt for the scattering phase-shift analyses at Livermore and Virginia Polytechnic Institute was used. The results are shown in Table I.

TABLE II. Comparison between the two final solutions in this work that agree with recent determinations of $\mathrm{Re}\epsilon$, and other solutions obtained elsewhere.

| Parameter \ Solution | No. 1 | No. 3 | Bennett *et al.* | Dorfan *et al.* | Fitch Sol. 1 | Martin and de Rafael Sol. a | Primakoff Sol. 1 |
|---|---|---|---|---|---|---|---|
| $10^3\times\mathrm{Re}\epsilon$ | 1.41± 0.40 | 1.8± 1.2 | 1.11±0.18 | 2.0±0.7 | | 1.90± 0.51 | |
| $10^3\times\lvert\epsilon\rvert$ | 2.63± 0.15 | 4.4± 4.0 | | | | | 2.8 |
| $\theta_\epsilon$ (deg) | 58 ±10 | 65 ±30 | | | 43±15 | 36 ± 9 | 65 |
| $\theta_{00}$ (deg) | 49 ±10 | −11 ±20 | | | 33 | 12 ±25 | 52 |

[13] R. A. Arndt and M. H. MacGregor, Methods Comp. Phys. 6, 253 (1966).

TABLE III. Solutions for $|\eta_{00}|=(3.0\pm0.3)\times10^{-3}$ as a representative smaller value.

| Parameter \ Solution | $\|\Delta I\|=\frac{1}{2}$ dominance | | $\|\Delta I\|\neq\frac{1}{2}$ | |
|---|---|---|---|---|
| $10^3\times\mathrm{Re}\bar{\beta}_2$ | 37 | 37 | 1600 | 1600 |
| $10^3\times\|\bar{\beta}_1\|$ | 44.5 | 44.5 | 84 | 84 |
| $10^3\times\mathrm{Im}\bar{\beta}_2$ | −0.55 | 2.11 | −4.51 | 3.23 |
| $10^3\times\mathrm{Re}\epsilon$ | 1.10 | − 0.33 | 1.49 | 0.32 |
| $10^3\times\mathrm{Im}\epsilon$ | 1.97 | 0.80 | 3.24 | 0.64 |
| $10^3\times\|\epsilon\|$ | 2.26 | 0.87 | 3.57 | 0.72 |
| $\theta_\epsilon$ (deg) | 61 | 112 | 65 | 64 |
| $\theta_{00}$ (deg) | 54 | −171 | 2 | 105 |
| Solution No. | 1 | 2 | 3 | 4 |

## IV. CONCLUSION

Only solutions 1 and 3 of Table I are within the errors of the two recent determinations of $\mathrm{Re}\epsilon$ by means of kaon leptonic decays. Dorfan *et al.*[4] determined $\mathrm{Re}\epsilon$ in the $K_L\rightarrow\pi+\mu+\nu$ decay and Bennett *et al.*[4] determined $\mathrm{Re}\epsilon$ in the $K_L\rightarrow\pi+e+\nu$ decay. Their values, along with values for $\epsilon$ and $\theta_{00}$ obtained in other calculations, are compared to our solutions 1 and 3 in Table II.

From $K\rightarrow2\pi$ decays alone we are not able to say that $|\Delta I|=\frac{1}{2}$ is dominant. Our preference is for solution 1 because it is the $|\Delta I|=\frac{1}{2}$ dominance solution, because it fitted the data very well before searching, and because it is in better agreement with other calculations than is solution 3. Future measurements of $\theta_{00}$ may help select between the two, depending on the precision of the measurement. An experimental value for $\theta_{00}$ would allow $\delta_2-\delta_0$ to be searched in the least-squares fit.

If $\mathrm{Re}\bar{\beta}_1\gg\mathrm{Im}\beta_1$, the values of $\mathrm{Re}\bar{b}_3$ and $\mathrm{Re}\bar{b}_5$ for solution 1 are:

| Solution | $10^3\times\mathrm{Re}\bar{b}_3$ | $10^3\times\mathrm{Re}\bar{b}_5$ |
|---|---|---|
| 1a | − 4.5 | 41.5 |
| 1b | 48.9 | −11.9 |

Thus, *either* $|\bar{b}_3|<|\bar{b}_5|$ or $|\bar{b}_3|>|\bar{b}_5|$. We cannot state which is the actual case.

Our solutions were obtained by first assuming that *CP* nonconservation is small for $|\Delta I|>\frac{1}{2}$, and then achieving least-squares fits to the data. The fact that one of the approximate solutions (No. 1) was almost an exact fit, and that it agrees with $\mathrm{Re}\epsilon$ measured in kaon leptonic decays, indicates that the assumption of small *CP* nonconservation is possibly correct. Of course, there are many other possible solutions. However, most or all of the other solutions may not be real. We have tried to find other solutions by starting the search from random numbers between −100 and +100. In about one hundred such random-input searches, we always obtained one of the four solutions already found above.

Because recent measurements of $|\eta_{00}|$ seem to indicate a smaller value than we have used above,[8] we list in Table III the values of the parameters for the four solutions for $|\eta_{00}|=(3.0\pm0.30)\times10^{-3}$ as a representative smaller value. The errors are essentially the same as for Table I. It is seen that our conclusions are not altered by this smaller $|\eta_{00}|$.


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