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$O(4)$ Symmetry, $N\bar{N}$ Scattering, and the Regge Trajectory of the Pion*

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The $N\bar{N}$ Bethe-Salpeter equation is studied exploiting the $O(4)$ symmetry of the scattering amplitude at $t=0$. Spinor $O(4)$ spherical harmonics are developed and used in the projection of the Bethe-Salpeter equation both at $t=0$ and for small finite t. With the choice of Toller quantum number $M=1$ for the pion trajectory at $t=0$, as seems to be implied by high-energy data, we find that the $M=1$ trajectory must mix with another trajectory near $t=0$ in order to produce a physical pion. In addition, $O(4)$ expansions are given for the $N\bar{N}$ helicity amplitudes for both equal and unequal mass.

I. INTRODUCTION

After the discovery of $O(4)$ or $O(3,1)$ symmetry of the scattering amplitude for equal-mass particles at $t=0$,[1,2] the question of the $O(4)$, or Toller, quantum number M of the pion's Regge trajectory has been considered by various authors. Fits to high energy-nucleon-antinucleon scattering and pion photo-production data seem to support the choice $M=1$,[3,4] as do finite-energy sum rules.[5] Mandelstam has shown that the choice of M for the pion trajectory is related to the hypothesis of partially conserved axial-vector current (PCAC).[6]

In this paper we use an off-mass-shell approach to study the pion, considering it as a bound state of the $N\bar{N}$ system. We use the Bethe-Salpeter equation for $N\bar{N}$ scattering and expand the M-function in $O(4)$ basis functions. Working with others we have already outlined this method for the scattering of particles of arbitrary spins and masses.[7] (Nevertheless, the present paper is largely self-contained.) In a later paper we will use the same method to treat the scattering of unequal-mass spin-0, spin-1 particles (e.g., π-ρ scattering).[8] That paper and this one, both containing a large amount of formalism, have previously been summarized.[9] The results reached in both of these papers, as well as by other workers,[10] is that the pion's Regge trajectory can have $M=1$ (at $t=0$) only if there is trajectory mixing in the region $0<t<m_\pi^2$ such that the trajectory becomes pure $M=0$ at $J=0$. If the trajectory remained $M=1$ at $t=m_\pi^2$ it would choose nonsense and therefore not produce a physical particle (the pion) at $J=0$.

In Sec. II we present the Bethe-Salpeter (BS) equation for the M function, the off-shell scattering amplitude without the external spinors, and make the Wick rotation. This puts the momentum vectors in a four-dimensional Euclidean space. In Sec. III we find expressions for $O(4)$ spinor spherical harmonics, which are the projection on $O(4)$ basis functions of objects which transform like a direct product of a 4-vector times two Dirac spinors. We use these in Sec. IV to expand the M function and get a projected BS equation. At $t=0$ where there is $O(4)$ symmetry the equation is nearly diagonal and is easy to study. We find there are six types of solutions at $t=0$.[11] In Sec. VI we find how a solution of the projected BS equation will contribute to a particular partial-wave amplitude. To do this we use a group representation definition of the Dirac spinors. From this we find that three of the six types of solutions do not contribute to any partial wave amplitudes at $t=0$. We also reproduce the results of Freedman and Wang,[12] including the fact that if the pion were massless its trajectory would choose nonsense at $J=0$ for $M=1$ or sense for $M=0$.

After establishing these results at $t=0$ we show that the same methods apply to study the properties of the trajectory for t greater than zero. We start in Sec. VII by calculating the symmetry breaking terms in the BS equation and recalculating for $t\neq 0$ the projections from $O(4)$ basis to $O(3)$ helicity partial-wave states. From an examination of these we find an $O(4)$ analog to sense and nonsense states in $O(3)$. This allows an easy proof that the finite-mass pion's trajectory would choose nonsense at $J=0$ if it had $M=1$ at $t=0$ and if it could be calculated from simple perturbation theory. Since we want to keep $M=1$ at $t=0$ to agree with the high-energy data but not have nonsense at $J=0$ (pions exist) we study in Sec. VIII forms of the solutions of the BS equation which come from degenerate perturbation

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theory. Here we show that we can get a trajectory with $M=1$ at $t=0$ which does not choose nonsense at $J=0$. Included in this section is the justification of a 2×2 matrix model used in the summary paper.[7] Appendix A gives group representation definitions for Dirac spinors, propagators, and γ matrices. Appendix B contains the generalization to unequal-mass "$N\bar{N}$" scattering.

II. BETHE-SALPETER EQUATION FOR NUCLEON-ANTINUCLEON SCATTERING

In this paper we consider only the ladder approximation of the BS equation although we believe that the results hold without this approximation. By the "ladder approximation" we mean that the kernel (Born term) is a single-particle exchange or a sum of single-particle exchanges. We also neglect isospin although this could easily be added.

To simplify the equation we use the fact that nucleon-antinucleon scattering is equal to nucleon-nucleon scattering if we change the sign of Born terms for the exchange of odd C parity particles (see Appendix A). We write the scattering as

$$T_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(p',p,k)=\bar{u}_{\alpha'}(\tfrac{1}{2}k+p',\lambda_1')\bar{u}_{\beta'}(\tfrac{1}{2}k-p',\lambda_2')\times M_{\alpha'\beta',\alpha\beta}(p',p,k)u_\alpha(\tfrac{1}{2}k+p,\lambda_1)u_\beta(\tfrac{1}{2}k-p,\lambda_2)\,, \tag{2.1}$$

where λ_1, λ_2, and p are the initial-state nucleon helicity, antinucleon helicity, and relative 4-momentum, respectively, primed quantities are final-state variables, and k is the total 4-momentum. We restrict our attention to the center-of-mass system where $k=(\sqrt{t},0,0,0)$. $M_{\alpha'\beta',\alpha\beta}(p',p,k)$ obeys the following BS equation:

$$M_{\alpha'\beta',\alpha\beta}(p',p,k)=B_{\alpha'\beta',\alpha\beta}(p',p,k)-\frac{1}{(2\pi)^4}\int d^4q\, M_{\alpha'\beta',\gamma\delta}(p',q,k)\times S_{\gamma\gamma'}(q+\tfrac{1}{2}k)S_{\delta\delta'}(\tfrac{1}{2}k-q)B_{\gamma'\delta',\alpha\beta}(q,p,k)\,. \tag{2.2}$$

Let

$$M_{\alpha'\beta',\gamma\delta}(p',p,t)S_{\gamma\alpha}(\tfrac{1}{2}k+p)S_{\delta\beta}(\tfrac{1}{2}k-p)=R_{\alpha'\beta',\alpha\beta}(p',p,t)\,,\quad k=(\sqrt{t},0,0,0)\,. \tag{2.3}$$

Then R obeys the following BS equation:

$$R_{\alpha'\beta',\gamma\delta}(p',p,t)S^{-1}{}_{\gamma\alpha}(\tfrac{1}{2}k+p)S^{-1}{}_{\delta\beta}(\tfrac{1}{2}k-p)=B_{\alpha'\beta',\alpha\beta}(p',p,t)-\frac{1}{(2\pi)^4}\int d^4q\times R_{\alpha'\beta',\gamma\delta}(p',q,t)B_{\gamma\delta,\alpha\beta}(q,p,t)\,, \tag{2.4}$$

where $S^{-1}(p)=(ip\!\!\!/+m)$ (see Appendix A) and for scalar exchange

$$B_{\alpha'\beta',\alpha\beta}(p',p)=\frac{i\eta_c g^2\delta_{\alpha'\alpha}\delta_{\beta'\beta}}{(p-p')^2+\mu^2-i\epsilon}\,,$$

where μ and η_c are the mass and charge-conjugation parity of the exchanged particle.

The next step is to make the Wick rotation.[13] For $0\leq t<$ threshold, the integrand has no poles in the first and third quadrants of the complex q^0 plane. Assuming the integrand vanishes as $|q^0|\to\infty$ gives the integral along the imaginary q^0 axis equal to the integral along the real axis. Next the integral is continued in p^0 to the imaginary p^0 axis (no poles of the integrand cross the new path of integration in the q^0 plane during this) giving an integral equation for R as a function of imaginary p^0. Now we make the change of variables $p^0=ip^4$, $q^0=iq^4$. Since $-(p^0)^2=(p^4)^2$ we may use Euclidean 4-space notation if we define $k=(k_1,k_2,k_3,k_4)=(0,0,0,-i\sqrt{t})$ and

$$\gamma^4=i\gamma^0=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix}$$

[see Eq. (A6)]. Now, of course

$$p\!\!\!/=p^1\gamma^1+p^2\gamma^2+p^3\gamma^3+p^4\gamma^4.$$

After redefining R and B to absorb some i's, the BS equation becomes

$$RS^{-1}S^{-1}=-B+\frac{1}{(2\pi)^4}\int d^4q\, RB\,,$$

with the integral over a Euclidean 4-space and B for a scalar exchange given by

$$B_{\alpha'\beta',\alpha\beta}(p',p)=\frac{\eta_c g^2\delta_{\alpha'\alpha}\delta_{\beta'\beta}}{(p-p')^2+\mu^2}\,. \tag{2.5}$$

The relation between Cartesian and spherical components in $O(4)$ is

$$\begin{aligned}p^4&=P\cos\psi\,, & p^3&=P\sin\psi\cos\theta\,,\\ p^2&=P\sin\psi\sin\theta\sin\varphi\,, & p^1&=P\sin\psi\sin\theta\cos\varphi\,,\end{aligned} \tag{2.6}$$

and

$$\int_0^\infty d^4p$$

becomes

$$\int_0^\infty P^3dP\int_0^\pi \sin^2\psi\, d\psi\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi} d\varphi\,. \tag{2.7}$$

III. O(4) SPINOR SPHERICAL HARMONICS

A. States for Two Dirac Field Fermions

Considering R before the Wick rotation, its Lorentz transformation properties were

$$R_{\gamma\delta,\alpha\beta}(p',p,k)=\mathfrak{D}_{\gamma\gamma'}(\Lambda^{-1})\mathfrak{D}_{\delta\delta'}(\Lambda^{-1})\times R_{\gamma'\delta',\alpha'\beta'}(\Lambda p',\Lambda p,\Lambda k)\mathfrak{D}_{\alpha'\alpha}(\Lambda)\mathfrak{D}_{\beta'\beta}(\Lambda)\,, \tag{3.1}$$

[13] G. C. Wick, Phys. Rev. 127, 2266 (1962).

where $\mathfrak{D}_{\alpha'\alpha}(\Lambda)$ is the representation (nonunitary) of the Lorentz group to which the nucleon field belongs. When α is replaced by the set (r,ρ), $r=\pm1$, $\rho=\pm\frac{1}{2}$, then (see Appendix A)

$$\mathfrak{D}_{\alpha'\alpha}(\Lambda)=\begin{pmatrix}\mathfrak{D}_{\rho'\rho}{}^{\frac{1}{2}0}(\Lambda) & 0\\ 0 & \mathfrak{D}_{\rho'\rho}{}^{0\frac{1}{2}}(\Lambda)\end{pmatrix}_{r'r}. \tag{3.2}$$

When Eq. (3.1) is continued to the complex Lorentz transformations and then restricted to transformations which keep p^0 imaginary and p^i ($i=1,2,3$) real, then the new symmetry group is $O(4)$ and the above representations become the unitary representations of $O(4)$ with the same eigenvalues. (These eigenvalues are explained in the next section.)

We take advantage of the above fact to construct a set of "$O(4)$ spinor spherical harmonics" which will be used as basis functions in the expansion of R. We start by considering a ket $|p,\alpha,\beta\rangle$ which has relative momentum p and 2 spinorial indices, α for the nucleon and β for the charge-conjugated antinucleon. Each index transforms according to the representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ of $O(4)$. Thus the transformation property of the ket under an $O(4)$ rotation, g, is

$$U(g)|p,\alpha,\beta\rangle=\sum_{\alpha'\beta'}\mathfrak{D}_{\alpha'\alpha}(g)\mathfrak{D}_{\beta'\beta}(g)|gp,\alpha',\beta'\rangle,$$

where the $\mathfrak{D}$'s are given by Eq. (3.2).

When a general $O(4)$ transformation g is expressed by the standard five rotations and a boost[2,7] [we shall call an $O(4)$ rotation of the fourth direction to any other direction a boost since it was that in the Lorentz metric]

$$g(\varphi,\theta,\psi,\alpha,\beta,\gamma)=R(\varphi,\theta,0)S_3(\psi)R(\alpha,\beta,\gamma), \tag{3.3}$$

the representation matrix $\mathfrak{D}$ can be expressed as (see Appendix A)

$$\mathfrak{D}_{\alpha'\alpha}(g)\equiv\mathfrak{D}_{r'\rho',r\rho}(g)$$
$$=\delta_{r'r}\sum_{m=\pm1/2}\mathrm{D}_{\rho'm}{}^{\frac{1}{2}}(\varphi,\theta,0)e^{-irm\psi}\mathrm{D}_{m\rho}{}^{\frac{1}{2}}(\alpha,\beta,\gamma). \tag{3.4}$$

As in any problem we want our final equation as nearly diagonal as possible and so we consider the operators which commute with M. This leads us through a string of "change of basis." We first construct eigenkets of intrinsic parity (see the last part of Appendix A)

$$|p,\alpha,\beta\rangle\equiv|p,(r\rho),(t\tau)\rangle=\sum_{\eta,\epsilon=\pm1}C_{\eta r}C_{\epsilon t}|p,\eta\rho,\epsilon\tau\rangle, \tag{3.5}$$

where

$$C=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{3.6}$$

such that

$$P|p_4,\eta\rho,\epsilon\tau\rangle=\eta\epsilon|p_4,\eta\rho,\epsilon\tau\rangle. \tag{3.7}$$

Next, we define another basis by combining the two "spins" into a total "spin"

$$|p,\eta\epsilon S\sigma\rangle=\sum_\rho C(\tfrac{1}{2},\tfrac{1}{2},S;\rho,\sigma-\rho)|p,\eta\rho,\epsilon(\sigma-\rho)\rangle \tag{3.8}$$

which transforms according to $D^S(R)$ for an $O(3)$ rotation. Before projecting out $O(4)$ basis function from these kets we will interject a review of $O(4)$ representations.

B. Review of $O(4)$ Representations

Since $O(4)$ is homomorphic to $O(3)\otimes O(3)$ or isomorphic to $SU(2)\otimes SU(2)$, a possible basis for $O(4)$ is the direct product of two $SU(2)$ bases.[14] Then a state is labeled as $|j_1m_1j_2m_2\rangle$, where j and m are the "total angular momentum" and the "z component" in each of the $SU(2)$ groups. Then j_1 and j_2, the Casimir quantum numbers, denote the particular irreducible representation of $O(4)$ and m_1 and m_2 enumerate the basis functions of that representation. Thus, an $O(4)$ rotation g, applied to one of these basis functions, gives

$$U(g)|j_1m_1j_2m_2\rangle=\sum_{m_1'm_2'}\mathfrak{D}_{m_1'm_2',m_1m_2}{}^{j_1j_2}(g)|j_1m_1'j_2m_2'\rangle,$$

where $\mathfrak{D}^{j_1j_2}(g)$ is a function of the $O(3)$ representations D^{j_1} and D^{j_2}, the functional dependence being determined by the homomorphism. (The representation used in Sec. A for a Dirac spinor is a special example of the above. It is a reducible representation, a sum of two irreducible representations, one with $j_1=\frac{1}{2}$, $j_2=0$ and the other with $j_1=0$, $j_2=\frac{1}{2}$. Those "m" subscripts which are equal to zero are suppressed.)

Another basis is obtained by vectorially adding j_1 and j_2 to form a total angular momentum J:

$$|j_1j_2Jm\rangle=\sum_\mu C(j_1j_2J;\mu,m-\mu)|j_1\mu j_2(m-\mu)\rangle$$

and

$$U(g)|j_1j_2Jm\rangle=\sum_{J'm'}\mathfrak{D}_{J'm',Jm}{}^{j_1j_2}(g)|j_1j_2J'm'\rangle,$$

where[2,7]

$$\mathfrak{D}_{J'm',Jm}{}^{j_1j_2}(\varphi,\theta,\psi,\alpha,\beta,\gamma)$$
$$=\sum_\mu D_{m'\mu}{}^{J'}(\varphi,\theta,0)d_{J'J\mu}{}^{j_1j_2}(\psi)D_{\mu m}{}^{J}(\alpha,\beta,\gamma). \tag{3.9}$$

Here $d_{J'J\mu}{}^{j_1j_2}$ is the "boost" representation in this basis. It is

$$d_{J'J\mu}{}^{j_1j_2}(\psi)=\sum_\nu C(j_1,j_2,J;\nu,\mu-\nu)$$
$$\times C(j_1j_2J';\nu,\mu-\nu)e^{-i(2\nu-\mu)\psi} \tag{3.10}$$

and for any particular set of indices can be conveniently expressed as Gegenbauer polynomials in $\cos\psi$. Reference 7 (App.) summarizes the properties of the boost functions which we shall use. Notice that for a pure

[14] More detailed treatments of the $O(4)$ representations may be found in Refs. 2 and 7.

$O(3)$ rotation R

$$\mathfrak{D}_{J'm',Jm}{}^{j_1j_2}(R)=\delta_{JJ'}D_{m'm}{}^{J}(R).$$

Instead of labeling the representation by j_1 and j_2 it is popular to use n and M, where

$$n=j_1+j_2 \quad\text{and}\quad M=j_1-j_2. \tag{3.11}$$

Obviously, then,

$$n\geqslant J\geqslant |M|. \tag{3.12}$$

After this brief review of $O(4)$ representations, we return to the projection of $O(4)$ basis states, $|nMJm\rangle$, from the ket $|p,\alpha,\beta\rangle$.

C. Construction of the Spherical Harmonics

Besides the $O(4)$ quantum numbers, other quantum numbers will be needed since more than one ket $|p,\alpha,\beta\rangle$ will project onto the same $O(4)$ state. We anticipate this by writing the state as $|n,M,J,m,\eta,\epsilon,\Sigma\rangle$. (Our "$\Sigma$" is the same as Freedman and Wang's[2] "s".)

Using the standard technique, we construct basis functions of the $O(4)$ representation (n,M) by

$$|nMJm\eta\epsilon\Sigma\rangle\propto\int dg \times\mathfrak{D}_{Jm,\Sigma\sigma}{}^{(n,M)*}(g)U(g)|p_4,\eta\epsilon\Sigma\sigma\rangle, \tag{3.13}$$

where $g=g(\varphi,\theta,\psi,\alpha,\beta,\gamma)$ [Eq. (3.3)]. We factor g by $g=\Omega(\varphi,\theta,\psi)R(\alpha,\beta,\gamma)$ so that Ω specifies a general direction in $O(4)$ and R is a general $O(3)$ rotation. Then $dg=d\Omega dR$; $d\Omega$ is over the surface of a 4-sphere [Eq. (2.7)], and dR is over the group $O(3)$. The integrations over dR give

$$\begin{aligned}|nMJm\eta\epsilon\Sigma\rangle&=N_\Sigma{}^{(n,M)}\sum_\sigma\int d\Omega\;\mathfrak{D}_{Jm,\Sigma\sigma}{}^{(n,M)*}(\Omega)U(\Omega)|p_4,\eta\epsilon\Sigma\sigma\rangle\\&=N_\Sigma{}^{(n,M)}\sum_{\sigma r\rho t\tau}\int d\Omega\;\mathfrak{D}_{Jm,\Sigma\sigma}{}^{(n,M)*}(\Omega)C_{\eta r}C_{\epsilon t}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho,\tau,\sigma)U(\Omega)|p_4,(r\rho),(t\tau)\rangle\\&=N_\Sigma{}^{(n,M)}\sum_{r\rho t\tau\rho'\tau'}\int d\Omega\;\mathfrak{D}_{Jm,\Sigma(\rho+\tau)}{}^{(n,M)*}(\Omega)D_{\rho'\rho}{}^{\frac{1}{2}}(\varphi,\theta,0)D_{\tau'\tau}{}^{\frac{1}{2}}(\varphi,\theta,0)\\&\qquad\times e^{-i(r\rho+t\tau)\psi}C_{\eta r}C_{\epsilon t}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho,\tau)|p(\Omega),(r\rho'),(t\tau')\rangle.\end{aligned} \tag{3.14}$$

From this equation we define the $O(4)$ spinor spherical harmonics as (to simplify notation let $\Upsilon=\{n,M,J,m,\eta,\epsilon,\Sigma\}$)

$$\begin{aligned}Z^\Upsilon(\Omega,\alpha,\beta)&\equiv Z_{Jm\eta\epsilon\Sigma}{}^{(n,M)}(\Omega,(r\rho),(t\tau))\equiv\langle p(\Omega),(r\rho),(t\tau)|nMJm\eta\epsilon\Sigma\rangle\\&=N_\Sigma{}^{(n,M)}\sum_{\rho'\tau'}\mathfrak{D}_{Jm,\Sigma(\rho'+\tau')}{}^{(n,M)*}(\Omega)D_{\rho\rho'}{}^{\frac{1}{2}}(\varphi,\theta,0)D_{\tau\tau'}{}^{\frac{1}{2}}(\varphi,\theta,0)e^{-i(r\rho'+t\tau')\psi}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho',\tau')C_{\eta r}C_{\epsilon t}.\end{aligned} \tag{3.15}$$

With the normalization chosen as

$$N_\Sigma{}^{(n,M)}=\frac{1}{\pi}\left(\frac{(n+M+1)(n-M+1)}{2(2\Sigma+1)}\right)^{1/2}, \tag{3.16}$$

Z is orthonormal and complete in the following sense:

$$\sum_{\alpha\beta}\int d\Omega\,Z^{\Upsilon*}(\Omega,\alpha,\beta)Z^{\Upsilon'}(\Omega,\alpha,\beta)=\delta_{\Upsilon\Upsilon'}\equiv\delta_{nn'}\delta_{MM'}\delta_{JJ'}\delta_{mm'}\delta_{\eta\eta'}\delta_{\epsilon\epsilon'}\delta_{\Sigma\Sigma'} \tag{3.17a}$$

and

$$\sum_\Upsilon Z^\Upsilon(\Omega,\alpha,\beta)Z^{\Upsilon*}(\Omega',\alpha',\beta')=\delta^3(\Omega-\Omega')\delta_{\alpha\alpha'}\delta_{\beta\beta'}, \tag{3.17b}$$

where the sum over Υ has the following range

$$\begin{aligned}&n=0,1,2,\cdots, && M=0,\pm1,\\&J=|M|,|M|+1,\cdots,n, && m=-J,\cdots,J,\\&\eta=\pm1, && \epsilon=\pm1,\\&\Sigma=0,1\quad\text{for}\quad M=0\\&\quad=1\quad\text{for}\quad|M|=1.\end{aligned} \tag{3.18}$$

Before proceeding with the $O(4)$ projections, it is worthwhile to note the invariants of the theory under discrete transformations.

D. Discrete Transformations

First we consider parity; from Appendix A we have $\mathcal{P}|p,\eta\rho,\epsilon\tau\rangle=\eta\epsilon|Pp,\eta\rho,\epsilon\tau\rangle$. Using this and the symmetry properties of the "boost" matrices,[7] we find

$$\mathcal{P}|nMJm\eta\epsilon\Sigma\rangle=(-)^{J-\Sigma}\eta\epsilon|n(-M)Jm\eta\epsilon\Sigma\rangle. \tag{3.19}$$

Next we consider particle exchange. Nucleon-antinucleon scattering is invariant under charge conjugation times particle exchange. But since we have already applied charge conjugation to one particle, the amplitude $R_{\alpha'\beta',\alpha\beta}(p',p,t)$ is invariant under particle exchange $\mathcal{J}$, defined by $\mathcal{J}|p,\alpha,\beta\rangle=|-p,\beta,\alpha\rangle$.

Actually it will be convenient to consider $\mathcal{J}$ as the product of two operators, one which interchanges the spinorial indices and one which reverses the relative momentum.

$$\mathcal{J}=\mathcal{J}_p\times\mathcal{J}_s, \tag{3.20a}$$

TABLE I. Solution to the nucleon-antinucleon Bethe-Salpeter equation at $t=0$.

Type	Amplitude[a]	State[b] $n-J$ even	State[b] $n-J$ odd	Comments
I	$R_{1+-}^{(n,0)}$, $R_{0-+}^{(n,0)}$, $R_{0--}^{(n,0)}$	1,(2)	...	
II	$R_{0+-}^{(n,0)}$, $R_{1-+}^{(n,0)}$, $R_{1--}^{(n,0)}$	u	0	
IIIa	$R_{1++}^{(n,1)}$, $R_{1-+}^{(n,1)}$, $R_{1--}^{(n,1)}$ }	0,2,(1)	u }	degenerate at $t=0$
IIIb	$R_{1++}^{(n,-1)}$, $R_{1-+}^{(n,-1)}$, $R_{1--}^{(n,-1)}$ }			
IV	$R_{1++}^{(n,0)}$	...	(1) }	degenerate at $t=0$ } Residue in partial-wave amplitude has a zero at $t=0$
V	$R_{0++}^{(n,0)}$	(0)	... }	
VIa	$R_{1+-}^{(n,1)}$ }	(u)	(2) }	degenerate at $t=0$
VIb	$R_{1+-}^{(n,-1)}$ }			

[a] The final-state quantum numbers have been suppressed. They must be equal to one of the initial states' quantum numbers in that type.
[b] $O(3)$ parity-conserving helicity state (see Table II) to which these amplitudes contribute. For those states listed in parenthesis the contribution is from the amplitude with $\omega=(+)$ and vanishes at $t=0$.

$$\mathscr{I}_p|p,\alpha,\beta\rangle=|-p,\alpha,\beta\rangle, \tag{3.20b}$$

$$\mathscr{I}_s|p,\alpha,\beta\rangle=|p,\beta,\alpha\rangle. \tag{3.20c}$$

From Eqs. (3.13) and (3.8) we find that

$$\mathscr{I}_s|nMJm\eta\epsilon\Sigma\rangle=(-)^{1-\Sigma}|nMJm\eta\epsilon\Sigma\rangle, \tag{3.21}$$

and taking $-p(\varphi,\theta,\psi)=p(\varphi,\theta,\psi-\pi)$ and using the symmetry properties of the boost representations, we get

$$\mathscr{I}|nMJm\eta\epsilon\Sigma\rangle=(-)^{n+M+1-\Sigma}|nMJm,-\epsilon,-\eta,\Sigma\rangle. \tag{3.22}$$

To make the final BS equation, after doing the $O(4)$ projection, as nearly diagonal as possible it is worth another change of basis to construct Z's diagonal in $\mathscr{I}$. Therefore, we let ω be the total intrinsic parity, $\omega=\eta\epsilon$ and define this new basis by

$$|nMJm\Sigma\omega\kappa\rangle=\frac{1}{\sqrt{2}}[\,|nMJm,\eta=+,\epsilon=\omega,\Sigma\rangle$$
$$+\kappa|nMJm,\eta=-,\epsilon=-\omega,\Sigma\rangle],$$
$$\omega=\pm,\quad\kappa=\pm. \tag{3.23}$$

Suppressing the common $(nMJm)$, this is

$$|\Sigma++\rangle=\frac{1}{\sqrt{2}}(|++\Sigma\rangle+|--\Sigma\rangle),$$

$$|\Sigma-+\rangle=\frac{1}{\sqrt{2}}(|+-\Sigma\rangle+|-+\Sigma\rangle),$$

$$|\Sigma+-\rangle=\frac{1}{\sqrt{2}}(|++\Sigma\rangle-|--\Sigma\rangle),$$

$$|\Sigma--\rangle=\frac{1}{\sqrt{2}}(|+-\Sigma\rangle-|-+\Sigma\rangle).$$

Then the symmetries become

$$\mathscr{P}|nMJm\Sigma\omega\kappa\rangle=(-)^{J-\Sigma}\omega|n,-M,Jm\Sigma\omega\kappa\rangle, \tag{3.24}$$

$$\mathscr{I}_p|nMJm\Sigma\omega\kappa\rangle=(-)^{n+M}\kappa|nMJm\Sigma\omega\kappa\rangle, \tag{3.25}$$

$$\mathscr{I}|nMJm\Sigma\omega\kappa\rangle=(-)^{n+M+1-\Sigma}(1-2\delta_{\omega+}\delta_{\kappa-})\times|nMJm\Sigma\omega\kappa\rangle. \tag{3.26}$$

Since we are using NN formalism to describe $N\bar{N}$ scattering, the equations for $\mathscr{P}$ and $\mathscr{I}$ must be used with care. We will find that an $O(4)$ state with a certain parity, as given by Eq. (3.24), will only contribute to $N\bar{N}$ partial-wave helicity amplitudes of the opposite parity, presented in Table I. Also $O(4)$ states of an $\mathscr{I}$ eigenvalue, Eq. (3.26), will contribute to a partial-wave amplitude with the opposite value of C_n, given in Table I (cf. Table II). The Z in the new basis is

$$Z_{Jm\Sigma\omega\kappa}^{(n,M)}(\Omega,\alpha,\beta)$$
$$=\frac{1}{\sqrt{2}}(Z_{Jm,+,\omega,\Sigma}^{(n,M)}+\kappa Z_{Jm,-,-\omega,\kappa}^{(n,M)})$$
$$=N_\Sigma^{(n,M)}\sum_{\rho'\tau'}\mathfrak{D}_{Jm,\Sigma(\rho'+\tau')}^{(n,M)*}(\Omega)$$
$$\times D_{\rho\rho'}^{\frac{1}{2}}(\varphi,\theta,0)D_{\tau\tau'}^{\frac{1}{2}}(\varphi,\theta,0)$$
$$\times e^{-i(r\rho'+t\tau')\psi}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho',\tau')\delta_{\kappa,rt}C_{\omega,t}. \tag{3.27}$$

IV. $O(4)$ PROJECTION OF THE BETHE-SALPETER EQUATION

Finally we are ready to take the projections

$$R_{\alpha'\beta',\alpha\beta}(p',p,t)=\sum_{\Upsilon'\Upsilon}Z^{\Upsilon'}(\Omega',\alpha',\beta')R^{\Upsilon'\Upsilon}(P',P,t)Z^{\Upsilon*}(\Omega,\alpha,\beta), \tag{4.1}$$

$$B_{\alpha'\beta',\alpha\beta}(p',p,t)=\sum_{\Upsilon'\Upsilon}Z^{\Upsilon'}(\Omega',\alpha',\beta')B^{\Upsilon'\Upsilon}(P',P,t)Z^{\Upsilon*}(\Omega,\alpha,\beta). \tag{4.2}$$

Here P and Ω are the magnitude and direction of p and likewise for the primed quantities. The projected BS equation is

$$\sum_{\Upsilon''}R^{\Upsilon'\Upsilon''}(P',P,t)\langle\Upsilon''|S^{-1}(\tfrac{1}{2}k+p)\otimes S^{-1}(\tfrac{1}{2}k-p)|\Upsilon\rangle$$
$$=-B^{\Upsilon'\Upsilon}(P',P,t)$$
$$+\frac{1}{(2\pi)^4}\int Q^3dQ\sum_{\Upsilon''}R^{\Upsilon'\Upsilon''}(P',Q,t)B^{\Upsilon''\Upsilon}(Q,P). \tag{4.3}$$

TABLE II. Parity-conserving helicity states for the $N\bar{N}$ system.

Name	State in $\lvert Jm\lambda_1\lambda_2\rangle$ basis[a]	LS description	C_n[b]	P[b]	Particles (trajectories)
$\lvert 0\rangle$	$\lvert\frac{1}{2},\frac{1}{2}\rangle-\lvert-\frac{1}{2},-\frac{1}{2}\rangle$	Singlet	$(-)^J$	$(-)^{J+1}$	π,η,B
$\lvert u\rangle$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle-\lvert-\frac{1}{2},\frac{1}{2}\rangle$	Uncoupled triplet	$(-)^{J+1}$	$(-)^{J+1}$	A_1
$\lvert 1\rangle$	$\lvert\frac{1}{2},\frac{1}{2}\rangle+\lvert-\frac{1}{2},-\frac{1}{2}\rangle$	First coupled triplet	$(-)^J$	$(-)^J$ }	P,P',ρ,ω,φ
$\lvert 2\rangle$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle+\lvert-\frac{1}{2},\frac{1}{2}\rangle$	Second coupled triplet	$(-)^J$	$(-)^J$ }	$A_2,(\sigma),(\pi_c)$

[a] The quantum numbers J and m are suppressed.
[b] See the warning after Eq. (3.26).

(Note that the homogeneous part is independent of the final-state quantum numbers so we will often suppress them.)

After finding expressions for the matrix elements of the inverse propagators and for the kernels in terms of the quantum numbers, this equation is continued to the complex J,n, and n' planes keeping $n-J=\mathcal{K}$ and $n'-J=\mathcal{K}'$ integer. The Lorentz poles (at $t=0$) are at those values of n for which the homogeneous equation has a solution (i.e., where the Fredholm determinant is zero).

Because we have the $O(4)$ symmetry at $t=0$ the BS equation is much simpler there, so we will investigate first its properties at $t=0$.

V. PROJECTED BETHE-SALPETER EQUATION AT $t=0$

A. Propagators

At $t=0$ the inverse propagator is $S^{-1}(p)=(i\not{p}+m)$. The $O(4)$ matrix element of the progapators is

$$\langle\Upsilon'|S^{-1}(p)\otimes S^{-1}(-p)|\Upsilon\rangle=\sum_{\substack{\alpha\beta\\ \alpha'\beta'}}\int d\Omega_p \times Z^{\Upsilon'*}(\Omega',\alpha',\beta')S^{-1}{}_{\alpha'\alpha}(p)S^{-1}{}_{\beta'\beta}(-p)Z^{\Upsilon}(\Omega,\alpha,\beta).$$

From Eq. (A2) for $\not{p}$ we see that it can be expressed as

$$\not{p}_{\alpha'\alpha}\equiv\not{p}_{(r'\rho'),(r\rho)} =-P\delta_{r',-r}\sum_\nu D_{\rho'\nu}{}^{\frac{1}{2}}(\varphi,\theta,0)e^{-2i\nu r\psi}D_{\nu\rho}{}^{\frac{1}{2}}(0,-\theta,-\varphi).$$

It is easier to work out this matrix in the $\eta\epsilon\Sigma$ basis first. (We use the notation $A\otimes B$ for an operator in the Hilbert space of the two 4-component Dirac fields. A is the nucleon operator, B is the operator in the space of our C-transformed antinucleon, and I is the identity operator.) We find

$$\langle\Upsilon'|\not{p}\otimes I|\Upsilon\rangle=-P\eta\delta_{\Upsilon'\Upsilon}, \tag{5.1a}$$

$$\langle\Upsilon'|I\otimes\not{p}|\Upsilon\rangle=-P\epsilon\delta_{\Upsilon'\Upsilon}, \tag{5.1b}$$

so that

$$\langle\Upsilon'|S^{-1}(p)\otimes S^{-1}(-p)|\Upsilon\rangle=(P^2\eta\epsilon-iPm(\eta-\epsilon)+m^2)\delta_{\Upsilon'\Upsilon}.$$

In the $\Sigma\omega\kappa$ basis this is

$$[(P^2\omega+m^2)\delta_{\kappa'\kappa}-2iPm\delta_{\omega,-}\delta_{\kappa',-\kappa}] \times\delta_{n'n}\delta_{M'M}\delta_{J'J}\delta_{m'm}\delta_{\Sigma'\Sigma}\delta_{\omega'\omega}. \tag{5.2}$$

B. Expanding the Kernel in $O(4)$ Basis Functions

We need to calculate the projected Born terms to find the structure of the coupled integral equations. Of course, the symmetries limit the number of independent Born terms. At $t=0$ $B^{\Upsilon'\Upsilon}$ is diagonal in n and M and independent of J and m. For some simple exchanges without vector coupling B is independent of t and so is always diagonal in n and M and independent of J and m.

$$B^{\Upsilon'\Upsilon}(P',P,t=0)=B_{\Sigma'\omega'\kappa',\Sigma\omega\kappa}{}^{(n,M)}(P',P). \tag{5.3}$$

In this section we consider only the exchange of scalar or pseudoscalar particles, and later we will show how a more general interaction may be treated. For a scalar interaction

$$B_{\alpha'\beta',\alpha\beta}(p',p)=\frac{\eta_c g^2\delta_{\alpha\alpha'}\delta_{\beta\beta'}}{(p-p')^2+\mu^2} \tag{5.4}$$

and for a pseudoscalar exchange

$$B_{\alpha'\beta',\alpha\beta}(p',p)=\frac{\eta_c g^2\gamma_{\alpha'\alpha}{}^5\gamma_{\beta'\beta}{}^5}{(p-p')^2+\mu^2} =\frac{\eta_c g^2 rt\delta_{rr'}\delta_{tt'}\delta_{\rho\rho'}\delta_{\tau\tau'}}{(p-p')^2+\mu^2}. \tag{5.5}$$

We note that these simple interactions are invariant under (B commutes with) $\mathcal{J}_p$ as well as $\mathcal{P}$ and $\mathcal{J}$. From this and Eq. (3.24–6) the projected Born terms have the following properties:

$$\kappa'=\kappa,\quad(-)^{\Sigma'}\omega'=(-)^{\Sigma}\omega \text{ if } M=0,\quad \Sigma'=\Sigma \text{ if } \kappa=+,$$
$$\text{and } \omega'=\omega \text{ if both } \Sigma'=\Sigma \text{ and } \kappa=-.$$

Also, from conservation of parity we have

$$B_{\Sigma'\omega'\kappa',\Sigma\omega\kappa}{}^{(n,-M)}=(-)^{\Sigma'-\Sigma}\omega'\omega B_{\Sigma'\omega'\kappa',\Sigma\omega\kappa}{}^{(n,M)}$$

so that the $M=-1$ kernels are determined from the $M=1$ kernels.

For either scalar or pseudoscalar exchange B commutes with $I\otimes\gamma^5$ (identity for one nucleon, γ^5 for the other). We find

$$(I\otimes\gamma^5)|nMJm\Sigma\omega\kappa\rangle=|nMJm\Sigma,-\omega,\kappa\rangle,\quad (\gamma^5\otimes I)|nMJm\Sigma\omega\kappa\rangle=\kappa|nMJm\Sigma,-\omega,\kappa\rangle. \tag{5.6}$$

Since B commutes with $I\otimes\gamma^5$ (or $\gamma^5\otimes I$) it depends only upon the product $\omega\omega'$ and not upon both ω and ω' individually.

To take advantage of all the above symmetries we write

$$B_{\Sigma'\omega'\kappa',\Sigma\omega\kappa}{}^{(n,M)}(P',P)=\delta_{\kappa'\kappa}B_{\Sigma'\Sigma,\zeta\kappa}{}^{(n,M)}(P',P),\quad \zeta=\omega\omega'=\omega'/\omega \tag{5.7}$$

and for a given n the independent nonzero $B_{\Sigma'\Sigma,\zeta\kappa}{}^{(n,M)}(P,P')$ are $B_{11,++}{}^{(n,1)}$, $B_{11,+-}{}^{(n,1)}$, $B_{11,-+}{}^{(n,1)}$, $B_{00,++}{}^{(n,0)}$, $B_{00,+-}{}^{(n,0)}$, $B_{11,++}{}^{(n,0)}$, $B_{11,+-}{}^{(n,0)}$, and $B_{10,--}{}^{(n,0)}$. The above properties of γ^5 show that $B_{\Sigma'\Sigma,\zeta\kappa}{}^{(n,M)}$ for pseudoscalar exchange is just κ times the corresponding kernel for scalar exchange. Thus, we need only find the above eight kernels for scalar exchange.

Starting with

$$B_{\alpha'\beta',\alpha\beta}(p',p)=\frac{\eta_c g^2\delta_{\alpha'\alpha}\delta_{\beta'\beta}}{(p-p')^2+\mu^2}=\sum_{\mathrm{T}'\mathrm{T}}Z^{\mathrm{T}'}(\Omega',\alpha',\beta')B^{\mathrm{T}'\mathrm{T}}(P',P)Z^{\mathrm{T}*}(\Omega,\alpha,\beta)$$

and putting p' in the 4th direction and using

$$Z_{Jm\Sigma\omega\kappa}{}^{(n,M)}(\psi=0,\alpha,\beta)=N_\Sigma{}^{(n,M)}\delta_{J\Sigma}\delta_{m,(\rho+\tau)}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho,\tau)\delta_{\kappa,rt}C_{\omega,t}$$

we get

$$\sum_{\alpha'\beta'}\int d\Omega\, Z^{\mathrm{T}'*}(\Omega',\alpha',\beta')\frac{\eta_c g^2\delta_{\alpha'\alpha}\delta_{\beta'\beta}}{P^2+P'^2-2PP'\cos\psi'+\mu^2}=B^{\mathrm{T}'\mathrm{T}}N_\Sigma{}^{(n,M)}\delta_{J\Sigma}\delta_{m,(\rho+\tau)}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho,\tau)\delta_{\kappa,rt}C_{\omega t}$$

or

$$N_\Sigma{}^{(n,M)}B_{\Sigma'\Sigma,\zeta\kappa}{}^{(n,M)}(P',P)=\sum_{r\rho t\tau}\int d\Omega\, Z_{\Sigma m\Sigma'\zeta\kappa}{}^{(n,M)}(\Omega,(r\rho),(t\tau))\frac{\eta_c g^2}{2PP'(z-\cos\psi)}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho,\tau),$$

where

$$z=(P^2+P'^2+\mu^2)/2PP'.$$

The integration over φ and θ gives

$$B_{\Sigma'\Sigma\zeta\kappa}{}^{(n,M)}(P',P)=\frac{2\pi\eta_c g^2}{\{(2\Sigma+1)(2\Sigma'+1)\}^{1/2}PP'}\sum_{r,t=\pm}\sum_{\rho,\tau=\pm\frac{1}{2}}\int\sin^2\psi\, d\psi\, d_{\Sigma\Sigma'(\rho+\tau)}{}^{(n,M)}(\psi)\times\frac{e^{i(r\rho+t\tau)\psi}}{z-\cos\psi}C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\rho,\tau)C(\tfrac{1}{2},\tfrac{1}{2},\Sigma';\rho,\tau)\delta_{\kappa,rt}C_{\zeta t}. \tag{5.8}$$

The kernels are calculated from this using Ref. 7 (App.), the Gegenbauer polynomial identities, and the following integral[15]:

$$\int_{-1}^{1}\frac{x^m(1-x^2)^{1/2}}{(z-x)}C_n{}^1(x)dx=\pi z^m[z+(z^2-1)^{1/2}]^{-n-1}.$$

We find the eight independent kernels for scalar exchange are

$$\begin{aligned}
&B_{00,++}{}^{(n,0)}=\frac{1}{n+1}F_n, && B_{11,+-}{}^{(n,0)}=\frac{1}{n(n+1)(n+2)}[(n^2+2n+2)z+2(n+1)(z^2-1)^{1/2}]F_n,\\
&B_{00,+-}{}^{(n,0)}=\frac{1}{n+1}zF_n, && B_{01,--}{}^{(n,0)}=\frac{-1}{(n+1)[n(n+2)]^{1/2}}[z+(n+1)(z^2-1)^{1/2}]F_n,\\
&B_{11,++}{}^{(n,0)}=\frac{1}{n+1}F_n, && B_{11,++}{}^{(n,1)}=\frac{1}{n(n+2)}[(n+1)z+(z^2-1)^{1/2}]F_n,\\
&B_{11,+-}{}^{(n,1)}=\frac{1}{n+1}F_n, && B_{11,-+}{}^{(n,1)}=\frac{1}{n(n+2)}[z+(n+1)(z^2-1)^{1/2}]F_n,
\end{aligned} \tag{5.9}$$

where

$$F_n\equiv F_n(P',P)=2\pi^2\eta_c g^2\frac{[z+(z^2-1)^{1/2}]^{-n-1}}{PP'}\quad\text{and}\quad z=\frac{P^2+P'^2+\mu^2}{2PP'}.$$

[15] *Bateman Manuscript Project* (*Integral Transforms*), edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Eq. (16.3.5).

Some of these kernels have a pole at $n=0$, but the residue of this pole is independent of the mass of the exchanged particle.

Even for scalar (or pseudoscalar) exchange the BS equation for nucleon-nucleon scattering is not of Fredholm type. To avoid this we insert a cutoff in the propagator of the exchanged particle. In some approximate way the cutoff represents the structure of the exchanged particle. We replace $[(p-p')^2+\mu^2]^{-1}$ by

$$\frac{1}{(p-p')^2+\mu^2}-\frac{1}{(p-p')^2+\lambda^2}. \qquad (5.10)$$

To obtain the $O(4)$ kernels with this cutoff we just use Eq. (5.9) and subtract a similar term with μ^2 replaced by λ^2. This not only makes the equation of Fredholm type but it also eliminates the pole at $n=0$.[16] The pole at $n=0$ in the kernel seems to prevent Regge poles from crossing $J=0$ at finite t.

C. Six Solutions at $t=0$

The Bethe-Salpeter equation is now (suppressing final-state quantum numbers on R)

$$\sum_{\kappa'} R_{\Sigma\omega\kappa'}{}^{(n,M)}(P)[(\omega P^2+m^2)\delta_{\kappa\kappa'}-2iPm\delta_{\omega,-}\delta_{\kappa',-\kappa}]$$
$$=-\delta_{n_f n}\delta_{M_f M}\delta_{\kappa_f\kappa}B_{\Sigma_f\Sigma,(\omega/\omega_f)\kappa}{}^{(n,M)}(P_f,P)$$
$$+\frac{1}{(2\pi)^4}\sum_{\Sigma'\omega'}\int Q^3dQ$$
$$\times R_{\Sigma'\omega'\kappa}{}^{(n,M)}(Q)B_{\Sigma'\Sigma,(\omega'/\omega)\kappa}{}^{(n,M)}(Q,P). \qquad (5.11)$$

This equation is continued to the complex n-plane. The value of n for which the homogeneous equation has a solution is the position of a Lorentz pole. Of course, the equation is uncoupled to a large degree due to the $O(4)$ symmetry at this special value of t. The amplitudes $R_{0++}{}^{(n,0)}$, $R_{1++}{}^{(n,0)}$, $R_{1+-}{}^{(n,1)}$, and $R_{1+-}{}^{(n,-1)}$, each obey an uncoupled integral equation. The first two of these amplitudes satisfy identical equations, as do the last two. So the Lorentz poles of $R_{0++}{}^{(n,0)}$ and $R_{1++}{}^{(n,0)}$ are degenerate at $t=0$ in the n plane. And $R_{1+-}{}^{(n,1)}$ and $R_{1+-}{}^{(n,-1)}$ are another degenerate pair. This last degeneracy is understood as a parity doubling, since P changes the sign of M and we have over-all conservation of parity. We know of no reason (i.e., symmetry) that the first two should be degenerate. We will show later that for an R amplitude to contribute to any $O(3)$ helicity amplitude at $t=0$ it must have $\omega=(-)$. Thus the Lorentz poles in the above four amplitudes produce Regge poles in $O(3)$ amplitudes with a residue which has a zero at $t=0$. These then are the evasive solutions to the conspiracy condition.[17] They will be discussed in Sec. VII D after we determine which $O(4)$ amplitudes contribute to which $O(3)$ amplitudes for $t\neq 0$.

The amplitudes $R_{1+-}{}^{(n,0)}$, $R_{0-+}{}^{(n,0)}$, and $R_{0--}{}^{(n,0)}$ obey a set of coupled integral equations. Any Lorentz pole in one will appear in all three. This pole will be what Freedman and Wang[2] call a type-I Lorentz pole. Similarly, the three amplitudes $R_{+-}{}^{(n,0)}$, $R_{-+}{}^{(n,0)}$, and $R_{--}{}^{(n,0)}$ are coupled and will yield a Freedman and Wang type-II trajectory. The two sets, $M=\pm 1$, of three amplitudes $R_{1++}{}^{(n,M)}$, $R_{1-+}{}^{(n,M)}$, and $R_{1--}{}^{(n,M)}$ satisfy identical coupled integral equations and produce the parity-doublet type-III Lorentz poles. These results (and certain results from later sections) are summarized in Table I where we have continued Freedman and Wang's numbering to the evasive Lorentz poles.

VI. REGGE POLES IN HELICITY AMPLITUDES

Now we come to the important question of how a Lorentz pole contributes to a particular (parity-conserving) helicity amplitude. We consider this question for general t.

A. Contribution of an $O(4)$ Amplitude to a Helicity Amplitude

First we continue $R_{\alpha'\beta',\alpha\beta}(p',p,t)$ to the mass shell. From

$$(\tfrac{1}{2}k+p)^2=-m^2=(\tfrac{1}{2}k-p)^2$$

and $k=(0,0,0,-i\sqrt{t})$, it follows that for $t\neq 0$, p_4 must be zero; hence, $\psi=\frac{1}{2}\pi$. (Although p_4 is not required by the above equations to be zero at $t=0$, it is required by continuation.) From $\mathbf{p}=\hat{p}P\sin\psi$ we learn that on the mass shell $P=-i(m^2-\frac{1}{4}t)^{1/2}\equiv Q(t)$.

Next, we sandwich the M amplitude between spinors to obtain the scattering amplitude. Then combining Eqs. (2.1), (2.3), and (4.1), we have

$$T_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(p',p,t)=\sum_{\Upsilon'\Upsilon}\bar{u}_{\alpha'}(\tfrac{1}{2}k+p',\lambda_1')u_{\beta'}(\tfrac{1}{2}k-p',\lambda_2')Z^{\Upsilon'}(\varphi',\theta',\tfrac{1}{2}\pi,\alpha',\beta')R^{\Upsilon'\Upsilon}(Q(t),Q(t),t)Z^{\Upsilon*}(\varphi,\theta,\tfrac{1}{2}\pi,\gamma,\delta)$$
$$\times S^{-1}{}_{\gamma\alpha}(\tfrac{1}{2}k+p)S^{-1}{}_{\delta\beta}(\tfrac{1}{2}k-p)u_\alpha(\tfrac{1}{2}k+p,\lambda_1)u_\beta(\tfrac{1}{2}k-p,\lambda_2)$$
$$=\sum_{\Upsilon'\Upsilon}(\bar{u}\bar{u}Z^{\Upsilon'})\lim_{P,P'\to Q(t)}[\sum_{\Upsilon''}R^{\Upsilon'\Upsilon''}(P',P,t)F^{\Upsilon''\Upsilon}(P,t)](Z^{\Upsilon*}uu), \qquad (6.1)$$

[16] We are indebted to Professor David Y. Wong for pointing out this fortunate property.

[17] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)]; M. Gell-Mann and E. Leader, in *Thirteenth International Conference on High-Energy Physics at Berkeley, 1966* (University of California Press, Berkeley, 1967).

where

$$F^{\Upsilon''\Upsilon}=\langle\Upsilon''|S^{-1}(\tfrac{1}{2}k+p)\otimes S^{-1}(\tfrac{1}{2}k-p)|\Upsilon\rangle.$$

Of course F will have a pole on the mass shell and R will have a zero but their product will be regular (except at a Regge pole where we expect a pole). Since the sum over RF is just the left-hand side of the projected BS equation, $RF=-B+\int RB$, we could substitute the right-hand side for it.

The quantity represented by $(Z^{\Upsilon}uu)$ is

$$(Z^{\Upsilon*}uu)\equiv\sum_{\alpha\beta} Z^{\Upsilon*}(\varphi,\theta,\tfrac{1}{2}\pi,\alpha,\beta)u_\alpha(\tfrac{1}{2}k+p,\lambda_1)u_\beta(\tfrac{1}{2}k-p,\lambda_2)$$

and we show later that it can be factored:

$$(Z^{\Upsilon*}uu)=D_{m\lambda}{}^J(\varphi,\theta,0)\tilde{X}^{\Upsilon}(t,\lambda_1,\lambda_2),\quad \lambda=\lambda_1-\lambda_2. \quad (6.2)$$

The quantity $(\bar{u}\bar{u}Z^{\Upsilon})$ when expressed in terms of the $O(4)$ angles (φ,θ,ψ) is just the complex conjugate of $(Z^{\Upsilon*}uu)$. We get this from using Eq. (A4) for $\bar{u}_\alpha(q,\lambda)$ and find that when it is expressed in terms of the $O(4)$ angles of q it is given by

$$\bar{u}_\alpha(\Omega_q,\lambda)=u_\alpha{}^*(\Omega_q,\lambda).$$

We can understand this by observing that it is consistent with the following facts:

$$\sum_\alpha \bar{u}_\alpha(q,\lambda)u_\alpha(q,\lambda)=\delta_{\lambda\lambda'},$$

$$u_\alpha(q,\lambda)=(1/\sqrt{2})\mathfrak{D}_{\rho\lambda}(\varphi_q,\theta_q,r\psi_q,0,0,0),\quad \alpha=(r,\rho),$$

and the fact that this $O(4)$ $\mathfrak{D}$ is unitary.

We have then

$$(\bar{u}\bar{u}Z^{\Upsilon'})=D_{m\mu}{}^{J*}(\varphi',\theta',0)\tilde{X}^{\Upsilon'*}(t,\lambda_1',\lambda_2'),\quad \mu=\lambda_1'-\lambda_2'.$$

Taking as the definition of a partial-wave amplitude

$$T_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(p',p,t)=\sum_{Jm}\frac{2J+1}{4\pi}D_{m\mu}{}^{J*}(\varphi',\theta',0) \times\langle\lambda_1'\lambda_2'|f^J(t)|\lambda_1\lambda_2\rangle D_{m\lambda}{}^J(\varphi,\theta,0), \quad (6.3)$$

we see that the partial-wave amplitude is given by

$$\langle\lambda_1'\lambda_2'|f^J(t)|\lambda_1\lambda_2\rangle =\frac{4\pi}{2J+1}\sum_{J,m\text{ fixed}}\sum_{\Upsilon\Upsilon'}\tilde{X}^{\Upsilon'*}(t,\lambda_1',\lambda_2') \times\lim_{P,P'\to Q(t)}\Big[\sum_{\Upsilon''}R^{\Upsilon'\Upsilon''}(P',P,t)F^{\Upsilon''\Upsilon}(P,t)\Big] \times\tilde{X}^{\Upsilon}(t,\lambda_1,\lambda_2). \quad (6.4)$$

From the above equation we observe that $\tilde{X}^{\Upsilon}(t,\lambda_1,\lambda_2)$ is really the projection of an $O(4)$ basis function on an $O(3)$ state. In bra-ket notation it is written as

$$\tilde{X}^{\Upsilon}(t,\lambda_1,\lambda_2)=\langle nMJm\Sigma\omega\kappa|tJm\lambda_1\lambda_2\rangle.$$

Equation (4.3) along with Eq. (6.4), the $O(4)$ projected BS equation, are the basic equations of this paper. As they are now written they give the partial-wave amplitude for integer J. Included in the sum over Υ and Υ' are

$$\sum_{n=J}^{\infty}\sum_{n'=J}^{\infty}.$$

We wish to continue this equation to noninteger J and have the partial-wave amplitude satisfy Carlson's theorem. If each term in the sums over n and n' satisfies Carlson's theorem and the sums converge uniformly in J, then the amplitude will satisfy Carlson's theorem. This happens if we let $n=J+\mathcal{K}$ and $n'=J+\mathcal{K}'$ and keep $\mathcal{K}$ and $\mathcal{K}'$ fixed as we continue J. Then

$$\sum_{n=J}^{\infty}\sum_{n'=J}^{\infty}f(J,n,n')\to\sum_{\mathcal{K}=0}^{\infty}\sum_{\mathcal{K}'=0}^{\infty}f(J,J+\mathcal{K},J+\mathcal{K}').$$

At the same time we continue the projected BS equation to the complex J,n and n' planes keeping $\mathcal{K}$ and $\mathcal{K}'$ fixed at integers. We shall show that this is the correct prescription of continuation for general t later, but now we note that at $t=0$ there is only one sum, $f(J,n,n')=\delta_{nn'}f(n)$, and the arguments of Ref. 2, Appendix B, show that this continuation is correct.

The final step is to form parity-conserving helicity amplitudes. For the nucleon-antinucleon system there are four helicity states. For convenience we name the four parity-conserving combinations by a single symbol according to Table II. Then the parity conserving helicity amplitudes of GGMW[18] are

$$\begin{aligned} &f_0{}^J=\langle 0|f^J|0\rangle,\quad f_{12}{}^J=\langle 1|f^J|2\rangle,\\ &f_1{}^J=\langle u|f^J|u\rangle,\quad f_{22}{}^J=\langle 2|f^J|2\rangle,\\ &f_{11}{}^J=\langle 1|f^J|1\rangle. \end{aligned}\quad (6.5)$$

As coefficients to the parity-conserving helicity states we define

$$X^{\Upsilon}(t,i)=(1/\sqrt{2})[\tilde{X}^{\Upsilon}(t,\lambda_1,\lambda_2)\pm\tilde{X}^{\Upsilon}(t,-\lambda_1,-\lambda_2)],\quad i=0,u,1,\text{ or }2, \quad (6.6)$$

where λ_1,λ_2 and the choice of sign are determined from Table II.

B. Special Results at $t=0$

At $t=0$ the calculation of $\tilde{X}$ is quite simple. From Eq. (A3) we have

$$u_\alpha(p,\lambda_1)=(1/\sqrt{2})\mathfrak{D}_{\rho\lambda_1}{}^{\frac{1}{2}0}(\varphi,\theta,r\psi,0,0,0),\quad \alpha=(r,\rho).$$

Using this and taking $-p(\varphi,\theta,\psi)=p(\varphi,\theta,\psi-\pi)$, we have for the second "nucleon"

$$u_\beta(-p,\lambda_2)=(1/\sqrt{2})\mathfrak{D}_{\tau,-\lambda_2}{}^{\frac{1}{2}0}(\varphi,\theta,t(\psi-\pi),0,0,0),\quad \beta=(t,\tau).$$

Then these equations and the definition of Z^{Υ}, Eq. (3.27), used in Eq. (6.2) give

$$\tilde{X}^{\Upsilon}(t=0,\lambda_1,\lambda_2)=(i^{-2\lambda_2}/\sqrt{2})N_{\Sigma}{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\tfrac{1}{2}\pi) \times C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\lambda_1,-\lambda_2,\lambda)\delta_{\omega,-}. \quad (6.7)$$

[18] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

Next, this equation substituted into Eq. (6.4) gives the same result as Freedman and Wang.[12]

We now see why $O(4)$ types IV, V, and VI do not contribute to partial-wave amplitudes at $t=0$; they have $\omega=(+)$. It might seem odd that only $\omega=(-)$ amplitudes contribute at $t=0$ when we have used an NN formalism. But when we treat unequal-mass scattering, Appendix C, we find that only $\omega=(+)$ amplitudes contribute at threshold and only $\omega=(-)$ contribute at pseudothreshold.

We can now observe the well-known conspiracy of Regge trajectories produced by a type-III Lorentz pole.[2] Table V b gives the behavior of $N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\frac{1}{2}\pi)$ near $J=0$. From this table we see that a type-III Lorentz pole, $|M|=1$, contributes to helicity states of both parities. There are then two Regge poles degenerate at $t=0$. When we solve the BS equation for $t\neq0$ we will find that there are two different sets of equations which reduce to the type-III equations at $t=0$, so these two Regge poles are in general degenerate only at $t=0$. We further observe from Table V b that if the pion had zero mass and had $|M|=1$ it would choose nonsense at $J=0$. That is, in the $\lambda=\mu=0$ (sense-sense at $J=0$) partial-wave amplitude the residue would vanish like α, and the $\lambda=\mu=1$ amplitude would be regular at $\alpha=0$.

VII. THE BETHE-SALPETER EQUATION FOR NONZERO t

At $t\neq0$ the Bethe-Salpeter equation, and therefore R, will not be invariant under simultaneous $O(4)$ rotations of p and p'.[19] (It is invariant to simultaneous rotations of p, p' and k but this is of no use in the Bethe-Salpeter equation where k is fixed.) Even though R is not invariant under the rotations of p and p' it is still useful to expand R in basis functions of the group $O(4)$. Then R is no longer diagonal in n and M, but since R still is invariant to $O(3)$ rotations of p and p' it is still diagonal in J and m and independent of m. The additional complexity at $t\neq0$ in the Bethe-Salpeter equation occurs both in the inverse propagators and in the kernels. For instance, there will be $\mathbf{k}$ in the propagators which will connect two different sets of (n,M) quantum numbers. We first consider this effect in the inverse propagators.

A. Matrix Elements of the Inverse Propagators

From

$$S^{-1}(\tfrac{1}{2}k+p)\otimes S^{-1}(\tfrac{1}{2}k-p)=[\tfrac{1}{2}(\sqrt{t})\gamma^4+ip\!\!\!/+m]\otimes[\tfrac{1}{2}(\sqrt{t})\gamma^4-ip\!\!\!/+m] \quad (7.1)$$

we get

$$\begin{aligned}\langle\Upsilon'|S^{-1}(\tfrac{1}{2}k+p)\otimes S^{-1}(\tfrac{1}{2}k-p)|\Upsilon\rangle &=\langle\Upsilon'|(ip\!\!\!/+m)\otimes(-ip\!\!\!/+m)|\Upsilon\rangle\\ &\quad-\tfrac{1}{2}i(\sqrt{t})P\langle\Upsilon'|\gamma^4\otimes\hat{p}\!\!\!/-\hat{p}\!\!\!/\otimes\gamma^4|\Upsilon\rangle\\ &\quad+\tfrac{1}{2}(\sqrt{t})m\langle\Upsilon'|\gamma^4\otimes I+I\otimes\gamma^4|\Upsilon\rangle\\ &\quad+\tfrac{1}{4}t\langle\Upsilon'|\gamma^4\otimes\gamma^4|\Upsilon\rangle. \quad (7.2)\end{aligned}$$

[19] See Appendix B for a more detailed explanation of this.

Since we already have the matrix elements of $p\!\!\!/\otimes I$ and $I\otimes p\!\!\!/$ we only need calculate the matrix elements of $\gamma^4\otimes I$ and $I\otimes\gamma^4$ to be able to calculate the matrix element of the above.

$$\gamma^4=-\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

and

$$\begin{aligned}\langle\Upsilon'|I\otimes\gamma^4|\Upsilon\rangle &=-\sum_{r\rho t\tau}\int d\Omega\, Z^{\Upsilon'*}(\Omega,r\rho,t\tau)Z^{\Upsilon}(\Omega,r\rho,-t\tau)\\ &=\delta_{J'J}\delta_{m'm}\delta_{\kappa',-\kappa}\frac{4\pi}{2J+1}N_{\Sigma'}{}^{(n',M')}N_\Sigma{}^{(n,M)}\\ &\quad\times\sum_\lambda\int\sin^2\psi d\psi E_{\Sigma\Sigma'\lambda}{}^{\omega\omega'}(\psi)\\ &\quad\times d_{J\Sigma'\lambda}{}^{(n',M')}(\psi)d_{J\Sigma\lambda}{}^{(n,M)}(\psi),\end{aligned}$$

where

$$\begin{aligned}E_{\Sigma\Sigma'\lambda}{}^{\omega\omega'}(\psi)&=\sum_\sigma C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\sigma,\lambda-\sigma)C(\tfrac{1}{2},\tfrac{1}{2},\Sigma';\sigma,\lambda-\sigma)\\ &\quad\times[\omega\delta_{\omega\omega'}\cos(2(\lambda-\sigma)\psi)-i\omega\delta_{\omega,-\omega'}\sin(2(\lambda-\sigma)\psi)].\end{aligned}$$

For the various values of Σ and Σ' the function E is:

Σ	Σ'	λ	$E_{\Sigma\Sigma'\lambda}{}^{\omega\omega'}(\psi)$
0	0	0	$\omega\delta_{\omega\omega'}\cos\psi$
0	1	0	$i\omega\delta_{\omega,-\omega'}\sin\psi$
1	0	0	$i\omega\delta_{\omega,-\omega'}\sin\psi$
1	1	$-1, 0, 1$	$\omega(\delta_{\omega\omega'}\cos\psi-i\lambda\delta_{\omega,-\omega'}\sin\psi)$

Using Eqs. (B15–B21) these remaining integrals are calculated and presented in part a of Table III. Part b of Table III for $\gamma^4\otimes I$ results from similar steps. These two parts are used to obtain parts c and d of Table III which are more convenient.

B. General Born Terms

A general class of kernels can be represented by $f(k\cdot p,\mathbf{k},\cdots)/[(p-p')^2+\mu^2]$ with a possible integral over μ^2. The numerator is a polynomial in the possible invariants which can be formed from k,p,p', and the γ matrices. It is assumed to commute with $\mathcal{J}$ and $\mathcal{P}$. For such a kernel the projection can be found by considering it as

$$f(k\cdot p,\mathbf{k},\cdots)\times[(p-p')^2+\mu^2]^{-1}$$

and putting a complete set of ZZ^* between the two factors. The projection of the last factor has already been done so we only need to find the projection of such factors as $p\cdot k$, $\mathbf{k}$, $k^\mu p^\nu\sigma_{\mu\nu}$, $k\!\!\!/\gamma^5$, etc. But these connect the same pair of amplitudes with the same powers of t and J as the inverse propagator. One way to see this is that k^μ is an $O(4)$ $(\frac{1}{2},\frac{1}{2})$ operator (i.e., $j_1=j_2=\frac{1}{2}$), thus it will produce nonzero matrix elements with

TABLE III. $O(4)$-symmetry-breaking matrix elements, where $A_n{}^J = [(n-J+1)(n+J+2)/4(n+1)(n+2)]^{1/2}$, $B_n = [n(n+3)/(n+1)(n+2)]^{1/2}$, $C_n{}^J = [1/(n+1)][J(J+1)/2n(n+2)]^{1/2}$ and $G_n{}^J = [1/(n+1)]\{\frac{1}{2}[J(J+1)]\}^{1/2}$. To get matrix elements connecting states with $M=-1$ note that the parity operator, $\mathcal{P}$, commutes with the operators γ^4 and p and use Eq. (3.24). Part c of Table III may be conveniently used to get $\langle\Upsilon'|\gamma^4\otimes\gamma^4|\Upsilon\rangle$ from the equation $(I\otimes\gamma^4+\gamma^4\otimes I)(I\otimes\gamma^4+\gamma^4\otimes I)=2(I\otimes I+\gamma^4\otimes\gamma^4)$.

(a) $\langle\Upsilon'|I\otimes\gamma^4|\Upsilon\rangle$ $\qquad$ $\langle\Upsilon'|I\otimes\gamma^4|\Upsilon\rangle = \delta_{J'J}\delta_{m'm}\delta_{\kappa',-\kappa}\langle n'M'Jm\Sigma'\omega'-\kappa|I\otimes\gamma^4|nMJm\Sigma\omega\kappa\rangle$

M'	Σ'	n'	$M=0,\ \Sigma=0$	$M=0,\ \Sigma=1$	$M=1,\ \Sigma=1$
0	0	$n-1$	$-\omega\delta_{\omega'\omega}A_{n-1}{}^J$	$-\omega\delta_{\omega',-\omega}[(n+2)/n]^{1/2}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$\omega\delta_{\omega',-\omega}G_n{}^J$
		$n+1$	$-\omega\delta_{\omega'\omega}A_n{}^J$	$\omega\delta_{\omega',-\omega}[n/(n+2)]^{1/2}A_n{}^J$	$\cdots$
0	1	$n-1$	$-\omega\delta_{\omega',-\omega}[(n-1)/(n+1)]^{1/2}A_{n-1}{}^J$	$-\omega\delta_{\omega'\omega}B_{n-1}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$\omega[-\delta_{\omega'\omega}+(n+1)\delta_{\omega',-\omega}]C_n{}^J$
		$n+1$	$\omega\delta_{\omega',-\omega}[(n+3)/(n+1)]^{1/2}A_n{}^J$	$-\omega\delta_{\omega'\omega}B_nA_n{}^J$	$\cdots$
1	1	$n-1$	$\cdots$	$\cdots$	$\omega(-\delta_{\omega'\omega}+\delta_{\omega',-\omega})A_{n-1}{}^J$
		n	$-\omega\delta_{\omega',-\omega}G_n{}^J$	$-\omega[\delta_{\omega'\omega}+(n+1)\delta_{\omega',-\omega}]C_n{}^J$	$\cdots$
		$n+1$	$\cdots$	$\cdots$	$-\omega(\delta_{\omega'\omega}+\delta_{\omega',-\omega})A_n{}^J$

(b) $\langle\Upsilon'|\gamma^4\otimes I|\Upsilon\rangle$ $\qquad$ $\langle\Upsilon'|\gamma^4\otimes I|\Upsilon\rangle = \delta_{J'J}\delta_{m'm}\delta_{\kappa',-\kappa}\langle n'M'Jm\Sigma'\omega'-\kappa|\gamma^4\otimes I|nMJm\Sigma\omega\kappa\rangle$

M'	Σ'	n'	$M=0,\ \Sigma=0$	$M=0,\ \Sigma=1$	$M=1,\ \Sigma=1$
0	0	$n-1$	$-\delta_{\omega'\omega}A_n{}^J$	$-\kappa\delta_{\omega',-\omega}[(n+2)/n]^{1/2}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$\kappa\delta_{\omega',-\omega}G_n{}^J$
		$n+1$	$-\delta_{\omega'\omega}A_n{}^J$	$\kappa\delta_{\omega',-\omega}[n/(n+2)]^{1/2}A_n{}^J$	$\cdots$
0	1	$n-1$	$-\kappa\delta_{\omega',-\omega}[(n-1)/(n+1)]^{1/2}A_{n-1}{}^J$	$-\delta_{\omega'\omega}B_{n-1}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$[-\delta_{\omega'\omega}-\kappa(n+1)\delta_{\omega',-\omega}]C_n{}^J$
		$n+1$	$\kappa\delta_{\omega',-\omega}[(n+3)/(n+1)]^{1/2}A_n{}^J$	$-\delta_{\omega'\omega}B_nA_n{}^J$	$\cdots$
1	1	$n-1$	$\cdots$	$\cdots$	$-(\delta_{\omega'\omega}+\kappa\delta_{\omega',-\omega})A_{n-1}{}^J$
		n	$-\kappa\delta_{\omega',-\omega}G_n{}^J$	$[-\delta_{\omega'\omega}+\kappa(n+1)\delta_{\omega',-\omega}]C_n{}^J$	$\cdots$
		$n+1$	$\cdots$	$\cdots$	$(-\delta_{\omega'\omega}+\kappa\delta_{\omega',-\omega})A_n{}^J$

(c) $\langle\Upsilon'|I\otimes\gamma^4+\gamma^4\otimes I|\Upsilon\rangle$ $\qquad$ $\langle\Upsilon'|I\otimes\gamma^4+\gamma^4\otimes I|\Upsilon\rangle = \delta_{J'J}\delta_{m'm}\delta_{\kappa',-\kappa}\langle n'M'Jm\Sigma'\omega'-\kappa|I\otimes\gamma^4+\gamma^4\otimes I|nMJm\Sigma\omega\kappa\rangle$

M'	Σ'	n'	$M=0,\ \Sigma=0$	$M=0,\ \Sigma=1$	$M=1,\ \Sigma=1$
0	0	$n-1$	$-(1+\omega)\delta_{\omega'\omega}A_{n-1}{}^J$	$-(\omega+\kappa)\delta_{\omega',-\omega}[(n+2)/n]^{1/2}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$(\omega+\kappa)\delta_{\omega',-\omega}G_n{}^J$
		$n+1$	$-(1+\omega)\delta_{\omega'\omega}A_n{}^J$	$(\omega+\kappa)\delta_{\omega',-\omega}[n/(n+2)]^{1/2}A_n{}^J$	$\cdots$
0	1	$n-1$	$-(\omega+\kappa)\delta_{\omega',-\omega}[(n-1)/(n+1)]^{1/2}A_{n-1}{}^J$	$-(1+\omega)\delta_{\omega'\omega}B_{n-1}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$[-(1+\omega)\delta_{\omega'\omega}+(\omega-\kappa)(n+1)\delta_{\omega',-\omega}]C_n{}^J$
		$n+1$	$(\omega+\kappa)\delta_{\omega',-\omega}[(n+3)/(n+1)]^{1/2}A_n{}^J$	$-(1+\omega)\delta_{\omega'\omega}B_nA_n{}^J$	$\cdots$
1	1	$n-1$	$\cdots$	$\cdots$	$[-(1+\omega)\delta_{\omega'\omega}+(\omega-\kappa)\delta_{\omega',-\omega}]A_{n-1}{}^J$
		n	$-(\omega+\kappa)\delta_{\omega',-\omega}G_n{}^J$	$[-(1+\omega)\delta_{\omega'\omega}-(\omega-\kappa)(n+1)\delta_{\omega',-\omega}]C_n{}^J$	$\cdots$
		$n+1$	$\cdots$	$\cdots$	$[-(1+\omega)\delta_{\omega'\omega}-(\omega-\kappa)\delta_{\omega',-\omega}]A_n{}^J$

(d) $\langle\Upsilon'|\gamma^4\otimes\hat{p}-\hat{p}\otimes\gamma^4|\Upsilon\rangle$ $\qquad$ $\langle\Upsilon'|\gamma^4\otimes p-p\otimes\gamma^4|\Upsilon\rangle = \delta_{J'J}\delta_{m'm}\delta_{\kappa'\kappa}P\langle n'M'Jm\Sigma'\omega'\kappa|\gamma^4\otimes\hat{p}-\hat{p}\otimes\gamma^4|nMJm\Sigma\omega\kappa\rangle$

M'	Σ'	n'	$M=0,\ \Sigma=0$	$M=0,\ \Sigma=1$	$M=1,\ \Sigma=1$
0	0	$n-1$	$\cdots$	$-\omega(1+\kappa)\delta_{\omega',-\omega}[(n+2)/n]^{1/2}A_{n-1}{}^J$	$\cdots$
		n	$\cdots$	$\cdots$	$\omega(1+\kappa)\delta_{\omega',-\omega}G_n{}^J$
		$n+1$	$\cdots$	$\omega(1+\kappa)\delta_{\omega',-\omega}[n/(n+2)]^{1/2}A_n{}^J$	$\cdots$
0	1	$n-1$	$-\omega(1+\kappa)\delta_{\omega',-\omega}[(n-1)/(n+1)]^{1/2}A_{n-1}{}^J$	$\cdots$	$\cdots$
		n	$\cdots$	$\cdots$	$\omega(1-\kappa)\delta_{\omega',-\omega}(n+1)C_n{}^J$
		$n+1$	$\omega(1+\kappa)\delta_{\omega',-\omega}[(n+3)/(n+1)]^{1/2}A_n{}^J$	$\cdots$	$\cdots$
1	1	$n-1$	$\cdots$	$\cdots$	$\omega(1-\kappa)\delta_{\omega',-\omega}A_{n-1}{}^J$
		n	$-\omega(1+\kappa)\delta_{\omega',-\omega}G_n{}^J$	$-\omega(1-\kappa)\delta_{\omega',-\omega}(n+1)C_n{}^J$	$\cdots$
		$n+1$	$\cdots$	$\cdots$	$-\omega(1-\kappa)\delta_{\omega',-\omega}A_n{}^J$

$n=n'$, $M=M'\pm1$, or $n=n'\pm1$, $M=M'$. Therefore the basic structure of coupling is not changed by generalizing the kernels. We will henceforth consider only scalar exchange kernels.

C. Perturbation Expansion

Now because of the symmetry breaking the various types of $O(4)$ amplitudes found at $t=0$ will be coupled together. This includes the parent amplitudes, $n=n'=J$, and an infinite number of daughters, $n'=J+\mathcal{K}'$, $n=J+\mathcal{K}$; $\mathcal{K}$, $\mathcal{K}'=1, 2, 3, \cdots$. But since the coupling coefficients are of order $\sqrt{t}$ or t, a perturbation expansion is possible. Only the equation for R^{nn} has a inhomogeneous term, so from the coupling between amplitudes we see that $R^{n'n}\propto(t/4m)^{|n-n'|/2}\times R^{n'n}$. This shows us two things. First, to have a parent amplitude R^{nn}, $n=J$, accurate to order $(t/4m)^{\mathcal{K}}$ and only $2\mathcal{K}$ nondiagonal amplitudes need be included, the amplitudes $R^{nn'}$, $n'=J+1,\cdots,J+2\mathcal{K}$. Second, in the expression for a partial-wave amplitude as a sum of $O(4)$ amplitudes, Eq. (6.4), the sum over $\mathcal{K}$ (after replacing n by $J+\mathcal{K}$) converges uniformly in J and n' for $t<4m^2$.

There is also coupling from the $M=1$ to the $M=-1$ amplitudes. Actually, we expect this since neither is an eigenstate of parity. Parity transforms a state of M into one of $-M$. When we were defining states in Sec. III we could have made eigenstates of parity from the $M=\pm1$ states but at that point it seemed that we had already made more than enough changes of basis. So now we take the equivalent step by adding and subtracting the integral equations to obtain equations for $(R^{M=1}\pm R^{M=-1})$. (The equations are unchanged at $t=0$ since there $R^{M=1}$ obeys the same equation as $R^{M=-1}$.) Because the two different combinations have different parity they will couple to different amplitudes; thus the two Regge poles, degenerate at $t=0$, will split as t increases from zero.

Parity and particle interchange diagonalize the BS equation so there will be four separate sets of coupled equations. Since the eigenvalue of $\mathcal{I}$ depends on $(n-J)$ the parent and even daughter amplitudes will be in a different set than the odd daughters. In terms of the $O(4)$ types found at $t=0$, Table I, the four sets are ("e" is for $\mathcal{K}$ and $\mathcal{K}'$ even including the parent, and "o" for $\mathcal{K}$ and $\mathcal{K}'$ odd):

(a) II(e), III(o), V(o), and VI(e) which will contribute to the parity-conserving partial-wave NN state $|u\rangle$,

(b) II(o), III(e), V(e), and VI(o) which will contribute to $|0\rangle$,

(c) I(o), III(o), IV(e), and VI(e) which will contribute to no mass-shell amplitude, and

(d) I(e), III(e), IV(o), and VI(o) which will contribute to $|1\rangle$ and $|2\rangle$.

In types III and VI, the combination of $M=1$ and $M=-1$ is chosen which has the same parity as the other $O(4)$ states in the set. Since we are considering the pion trajectory we only display the equations with the pion's parity and charge conjugation. The equations are of the form

$$R^{\Upsilon'\Upsilon''}F^{\Upsilon''\Upsilon}=-B^{\Upsilon'\Upsilon}+\int R^{\Upsilon'\Upsilon''}B^{\Upsilon''\Upsilon}.$$

We write this as

$$R^{\Upsilon'\Upsilon''}H^{\Upsilon''\Upsilon}=-B^{\Upsilon'\Upsilon},$$

where H is the operator $(F-\int B)$. Table IV is the part of H containing all the parent and first-daughter amplitudes which contributes to the $\langle 0|f|0\rangle$ partial-wave amplitude (i.e., the pion).

We will not explicitly solve these equations but rather will use the t and J dependence of the nondiagonal terms to infer the t and J dependence of the residue of the pion trajectory.

D. X's for $t\neq0$

Finally we need to calculate X^{Υ} for $t\neq0$ to determine how these amplitudes contribute to partial-wave amplitudes. As for the case of $t=0$ we use Eq. (A3) for u,

$$u_\alpha(\tfrac{1}{2}k+p,\lambda_1)=(1/\sqrt{2})\mathfrak{D}_{\rho\lambda_1}{}^{\frac{1}{2}0}(\varphi,\theta,r\psi_1,0,0,0)\,,\quad \alpha=(r,\rho)$$

TABLE IV. $R^{\Upsilon''}H^{\Upsilon''\Upsilon}$ from the equation $R^{\Upsilon'\Upsilon''}H^{\Upsilon''\Upsilon}=B^{\Upsilon'\Upsilon}$. Beneath each matrix element the behavior of that element near $t=0$ and $J=0$ is given in brackets. Here $F_1{}^J=\frac{1}{4}(J+1)/(J+2)$, $F_2{}^J=\frac{1}{4}(J-1)/(J+1)$, $F_3{}^J=\frac{1}{4}J/(J+3)$, $F_4{}^J=\frac{1}{4}(J-3)/(J+3)$, $F_5{}^J=\frac{1}{2}(H_J)^{-1}(A_{J+1}{}^J)^2-\frac{1}{2}H_J(A_J{}^J)^2$, $F_6{}^J=\frac{1}{2}(H_J)^2(A_J{}^J)^2+\frac{1}{2}(H_J)^{-2}(A_{J+1}{}^J)^2-\frac{1}{4}$, $H_J=[(J+3)/(J+1)]^{1/2}$, and $\int\equiv(2\pi)^{-4}\int_0^\infty Q^3dQ$. The remaining symbols are defined in Table III.

$(R_{1++}{}^{(J,1)}-R_{1++}{}^{(J,-1)})/\sqrt{2}$	$P^2+m^2-tF_1{}^J$ $-\int B_{11++}{}^{(J,1)}$ [1]	$-\frac{1}{4}t(J+2)^{-1}$ $-\int B_{11-+}{}^{(J,1)}$ [1]			$-mt^{1/2}A_J{}^J$ [$t^{1/2}$]		$-(t/\sqrt{2})(J+2)$ $\times C_{J+1}{}^JA_J{}^J$ [$J^{1/2}t$]	
$(R_{1-+}{}^{(J,1)}+R_{1-+}{}^{(J,-1)})/\sqrt{2}$	$-\frac{1}{4}t(J+2)^{-1}$ $-\int B_{11-+}{}^{(J,1)}$ [1]	$m^2-P^2-tF_1{}^J$ $-\int B_{11++}{}^{(J,1)}$ [1]	$-2iPm$ [1]	$iP(2t)^{1/2}G_J{}^J$ [$(Jt)^{1/2}$]	$mt^{1/2}A_J{}^J$ [$t^{1/2}$]		$(t/\sqrt{2})(J+2)$ $\times C_{J+1}{}^JA_J{}^J$ [$J^{1/2}t$]	
$(R_{1--}{}^{(J,1)}+R_{1--}{}^{(J,-1)})/\sqrt{2}$		$-2iPm$ [1]	$m^2-P^2+tF_2{}^J$ $-\int B_{11+-}{}^{(J,1)}$ [1]	$-m(2t)^{1/2}G_J{}^J$ [$(Jt)^{1/2}$]	$-iPt^{1/2}A_J{}^J$ [$t^{1/2}$]	$(t/\sqrt{2})G_J{}^JA_J{}^J$ [$J^{1/2}t$]		$-(t/\sqrt{2})$ $\times H_JG_J{}^JA_J{}^J$ [$J^{1/2}t$]
$R_{0++}{}^{(J,0)}$		$iP(2t)^{1/2}G_J{}^J$ [$(Jt)^{1/2}$]	$-m(2t)^{1/2}G_J{}^J$ [$(Jt)^{1/2}$]	$m^2+P^2+\frac{1}{4}t$ $-\int B_{00++}{}^{(J,0)}$ [1]		$-mt^{1/2}A_J{}^J$ [$t^{1/2}$]	$-iPt^{1/2}H_JA_J{}^J$ [$t^{1/2}$]	$mt^{1/2}H_JA_J{}^J$ [$t^{1/2}$]
$(R_{1+-}{}^{(J+1,1)}+R_{1+-}{}^{(J+1,-1)})/\sqrt{2}$	$-mt^{1/2}A_J{}^J$ [$t^{1/2}$]	$mt^{1/2}A_J{}^J$ [$t^{1/2}$]	$-iPt^{1/2}A_J{}^J$ [$t^{1/2}$]		$m^2+P^2+\frac{1}{4}t$ $-\int B_{11+-}{}^{(J+1,1)}$ [1]		$m(2t)^{1/2}(J+2)$ $\times C_{J+1}{}^J$ [$(Jt)^{1/2}$]	$-iP(2t)^{1/2}(J+2)$ $\times C_{J+1}{}^J$ [$(Jt)^{1/2}$]
$R_{0+-}{}^{(J+1,0)}$			$(t/\sqrt{2})G_J{}^JA_J{}^J$ [$J^{1/2}t$]	$-mt^{1/2}A_J{}^J$ [$t^{1/2}$]		$m^2+P^2-tF_3{}^J$ $-\int B_{00+-}{}^{(J+1,0)}$ [1]		$tF_5{}^J$ $-\int B_{01--}{}^{(J+1,0)}$ [1]
$R_{1-+}{}^{(J+1,0)}$	$-(t/\sqrt{2})(J+2)$ $\times C_{J+1}{}^JA_J{}^J$ [$J^{1/2}t$]	$(t/\sqrt{2})(J+2)$ $\times C_{J+1}{}^JA_J{}^J$ [$J^{1/2}t$]		$-iPt^{1/2}H_JA_J{}^J$ [$t^{1/2}$]	$m(2t)^{1/2}(J+2)$ $\times C_{J+1}{}^J$ [$(Jt)^{1/2}$]		$m^2-P^2+tF_4{}^J$ $-\int B_{11++}{}^{(J+1,0)}$ [1]	$-2iPm$ [1]
$R_{1--}{}^{(J+1,0)}$			$-(t/\sqrt{2})$ $\times H_JG_J{}^JA_J{}^J$ [$J^{1/2}t$]	$mt^{1/2}H_JA_J{}^J$ [$t^{1/2}$]	$-iP(2t)^{1/2}(J+2)$ $\times C_{J+1}{}^J$ [$(Jt)^{1/2}$]	$tF_5{}^J$ $-\int B_{10--}{}^{(J+1,0)}$	$-2iPm$	$m^2-P^2+tF_6$ $-\int B_{11+-}{}^{(J+1,0)}$

but here ψ_1 is not the boost angle of p but rather the boost angle of $\frac{1}{2}k+p$. For the second "nucleon" we have

$$u_\beta(\tfrac{1}{2}k-p,\lambda_2)=(1/\sqrt{2})\mathfrak{D}_{p-\lambda_2}{}^{\frac{1}{2}0}(\varphi,\theta,i\psi_2,0,0,0)\,,\quad \beta=(t,\tau)$$

where ψ_2 is the boost angle of $\frac{1}{2}k-p$. Putting this all together in (Zuu) we get

$$Z^*uu=D_{m\lambda}{}^J(\varphi,\theta,0)\tilde{X}^\Upsilon(t,\lambda_1,\lambda_2)\,,\quad \lambda=\lambda_1-\lambda_2\,,$$

where

$$\tilde{X}^\Upsilon(t,\lambda_1,\lambda_2)=\tfrac{1}{2}N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\psi)C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\lambda_1,-\lambda_2)\times\sum_{\tau t}e^{-i[\tau\lambda_1(\psi_1-\psi)-t\lambda_2(\psi_2-\psi)]}\delta_{\kappa,\tau t}C_{\omega,t}\,. \quad (7.3)$$

The angle ψ is still the boost angle of p and for equal-mass scattering in the center-of-mass system must be $\frac{1}{2}\pi$. The angle ψ_1 is defined by

$$\cos\psi_1=(k_4+p_4)/|k+p|$$

and, evaluated on the mass shell for equal-mass particles in the center-of-mass system, this is $t^{1/2}/2m$. The cosine of ψ_2 has the same value. As $t\to 0$ we know that $\psi_1\to\frac{1}{2}\pi$ and $\psi_2\to\frac{1}{2}\pi$; this gives

$$\cos\psi_1=\cos\psi_2=t^{1/2}/2m\,,$$
$$\sin\psi_1=-\sin\psi_2=(4m^2-t)^{1/2}/2m\,.$$

Using these, we find that

$$\tilde{X}^\Upsilon(t,\lambda_1,\lambda_2)=(1/\sqrt{2})N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\tfrac{1}{2}\pi)C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\lambda_1,-\lambda_2)\times\left[-i^{2\lambda_2}\delta_{\omega-}\left(\delta_{\kappa\lambda_1,\lambda_2}+\delta_{\kappa\lambda_1,-\lambda_2}\frac{(4m^2-t)^{1/2}}{2m}\right)+\delta_{\omega+}\delta_{\kappa\lambda_1,\lambda_2}\frac{t^{1/2}}{2m}\right]. \quad (7.4)$$

For all the d's used in these equations

$$d_{J\Sigma\lambda}{}^{(n,M)}(\tfrac{1}{2}\pi)=(-)^{n-J+M+\Sigma+\lambda}d_{J\Sigma\lambda}{}^{(n,M)}(\tfrac{1}{2}\pi)$$

[see Ref. 7 (App.)]; thus to calculate X we can consider two separate cases depending on the sign of $(-)^{n-J+M}$. These are presented in Table V. Using this table we can determine which $O(4)$ amplitudes contribute to which $O(3)$ parity-conserving helicity amplitudes. This important information is included in Table I.

From these tables we see that types IV, V, and VI $O(4)$ amplitudes can produce evasive Regge trajectories, i.e., Regge trajectories whose residues are zero at $t=0$. Parent evasive Regge trajectories of this type can occur only in the singlet, $|0\rangle$ and uncoupled triplet, $|u\rangle$, states. Dynamical evasion, where the residue of a Lorentz pole in an $O(4)$ amplitude is zero at $t=0$, is possible with type I, II, and III but we do not consider it as likely.

E. σ and ν Designation of $O(4)$ States

We are interested in the behavior of the pion trajectory near $J=0$ and of course near $t=0$. Therefore, we have written the highest power of J and t of the matrix elements in Table IV under each of the elements. We did the same in part b of Table V for the X's, and from these two tables we can see a very important regularity of the $J^{1/2}$'s. When n and J are integer, Z vanishes unless $n\geq J\geq|M|$ and $n\geq\Sigma\geq|M|$. We therefore label $O(4)$ states "σ" or "ν" depending on whether the above conditions are satisfied or not satisfied, respectively, at $J=0$. σ and ν are then the four-dimensional analogs of "sense" and "nonsense" in $O(3)$. Then we see (Table IV) that the matrix element between σ and ν states is proportional to $J^{1/2}$, while the matrix elements between σ and σ or between ν and ν are regular at $J=0$. The X's, which are the projection of an $O(4)$ state, Υ, on an $O(3)$ helicity state, $|J\lambda_1\lambda_2\rangle$, have the following regularity near $J=0$: An X between an $O(4)$ ν state and an $O(3)$ nonsense state or between a ν state and a sense state vanishes like $\sqrt{J}$, while an X between σ and sense or between ν and nonsense is regular at $J=0$. From this it is easy to see how the pion chooses nonsense if perturbation theory holds and if the pion is $|M|=1$ at $t=0$ (or any other ν state for that matter). It is convenient to make use of the fact that the residues of a pole in the $O(4)$ projected M-functions factor, i.e.,

$$M^{\Upsilon'\Upsilon}(t)\equiv\lim_{P,P'\to Q(t)}\Big[\sum_{\Upsilon''}R^{\Upsilon'\Upsilon''}(P',P,t)F^{\Upsilon''\Upsilon}(P,t)\Big]=\frac{\Gamma_{\Upsilon'}(t)\Gamma_\Upsilon(t)}{J-\alpha(t)}\,. \quad (7.5)$$

This allows the residue of a Regge pole in the parity-conserving partial-wave amplitude to factor

$$\frac{2J+1}{4\pi}\langle i|f^J(t)|j\rangle=\frac{\gamma_i(t)\gamma_j(t)}{J-\alpha(t)}\,,\quad i,j=0,u,1,2$$

and

$$\gamma_i(t)=\sum_\Upsilon\Gamma_\Upsilon(t)X^\Upsilon(t,i)\,. \quad (7.6)$$

If perturbation theory holds

$$\Gamma_\Upsilon(t)\propto\langle\Upsilon_0|S^{-1}\otimes S^{-1}|\Upsilon\rangle\,,$$

where Υ_0 is the $O(4)$ state which exhibits the pole at $t=0$. Assuming the pion trajectory is $|M|=1$ at $t=0$ means Υ_0 is a ν state so that $\Gamma_\nu(t)\propto t^\beta$ and $\Gamma_\sigma(t)\propto J^{1/2}t^\beta$. Breaking the sum over Υ in Eq. (7.6) into two sums, one over $O(4)$ ν states and one over σ states we get

$$\gamma_i(t)\propto(tJ)^{1/2}\,,\quad i=\text{sense}$$
$$\propto 1\,,\quad i=\text{nonsense}.$$

Thus the pion chooses nonsense, and since the pion's conspirator also has $|M|=1$ at $t=0$, it too will choose nonsense.

TABLE V. X^T's for general t.

State	LS description	λ	X^T for $(-)^{n-J+M}=1$	X^T for $(-)^{n-J+M}=-1$
(a) $X^T(t,i)$ in terms of $N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\frac{1}{2}\pi)$ (called Nd in this part of the table).				
0	Singlet	0	$\sqrt{2}Nd\delta_{\Sigma 0}\delta_{\omega+}\delta_{\kappa+}t^{1/2}/2m$	$-i\sqrt{2}Nd\delta_{\Sigma 1}\delta_{\omega-}\left[\delta_{\kappa-}+\delta_{\kappa+}\left(1-\frac{t}{4m^2}\right)^{1/2}\right]$
u	Uncoupled triplet	1	$iNd\delta_{\Sigma 1}\delta_{\omega-}\left[\delta_{\kappa+}+\delta_{\kappa-}\left(1-\frac{t}{4m^2}\right)^{1/2}\right]$	$Nd\delta_{\Sigma 1}\delta_{\omega+}\delta_{\kappa-}$
1	First coupled triplet	0	$-i\sqrt{2}Nd\delta_{\Sigma 0}\delta_{\omega-}\left[\delta_{\kappa-}+\delta_{\kappa+}\left(1-\frac{t}{4m^2}\right)^{1/2}\right]$	$\sqrt{2}Nd\delta_{\Sigma 1}\delta_{\omega+}\delta_{\kappa+}$
2	Second coupled triplet	1	$Nd\delta_{\Sigma 1}\delta_{\omega+}\delta_{\kappa-}t^{1/2}/2m$	$iNd\delta_{\Sigma 1}\delta_{\omega-}\left[\delta_{\kappa+}+\delta_{\kappa-}\left(1-\frac{t}{4m^2}\right)^{1/2}\right]$

(b) Values of $N_\Sigma{}^{(n,M)}d_{J\Sigma 2}{}^{(n,M)}(\frac{1}{2}\pi)$ in the vicinity of $J=0$

n	$\|M\|$	Σ	ν or σ	Sense at $J=0$ $N_\Sigma{}^{(n,M)}d_{J\Sigma 0}{}^{(n,M)}(\frac{1}{2}\pi)$	Nonsense at $J=0$ $N_\Sigma{}^{(n,M)}d_{J\Sigma 1}{}^{(n,M)}(\frac{1}{2}\pi)$
J	1	1	ν	$J^{1/2}$	c
J	0	1	ν	0	c
J	0	0	σ	c	0
$J+1$	1	1	ν	0	c
$J+1$	0	1	σ	c	0
$J+1$	0	0	σ	0	0
$J+2$	1	1	ν	$J^{1/2}$	c
$J+2$	0	1	σ	0	$J^{1/2}$
$J+2$	0	0	σ	c	0

VIII. CROSSING UNCOUPLED TRAJECTORIES

Although nondegenerate perturbation theory easily leads to a trajectory that is $M=1$ at $t=0$ choosing nonsense at $J=0$, it is not as easy to find what happens in a degenerate situation. Therefore it is instructive to consider a simplified model. We consider the following equations

$$\sum_{j=1,2} R^{ij}F^{jk}=\lambda B^{ik}+\lambda\sum_{j=1,2}\int R^{ij}B^{jk},\quad i,k=1,2,$$
$$F=\begin{pmatrix}1 & (bJt)^{1/2}\\ (bJt)^{1/2} & 1\end{pmatrix},\qquad (8.1)$$
$$B=\begin{pmatrix}B^1(J,t) & 0\\ 0 & B^2(J,t)\end{pmatrix}.$$

These equations are a model of the $n=J$, $M=1$ and $n=J$, $\Sigma=0$, $M=0$, parts of the equations which determine the pion trajectory. The superscripts one and two correspond to these two parts. We have kept the off-diagonal part of $F\equiv\langle\Upsilon'|S^{-1}\otimes S^{-1}|\Upsilon\rangle\propto(Jt)^{1/2}$, but we have made the following approximations: (a) We approximated the three $n=J$, $M=1$ equations by one equation, (b) we suppressed the P dependence of F, and (c) we set its diagonal elements equal to unity. In solving these equations it is easier to consider λ as a function of J and t and then once this is found to solve $\lambda(J,t)=$ constant for the trajectory $J=\alpha(t)$. Now consider the uncoupled equations (i.e., with $F^{12}=F^{12}=0$)

$$R^{ik}(P',P)=\lambda\delta_{ik}B^k(P',P)+\lambda\int Q^3dQR^{ik}(P',Q)B^k(Q,P),\quad i,k=1,2\qquad (8.2)$$

then, assuming the equations are of Fredholm type

$$B^k(P',P)=\sum_\mu\frac{v_\mu{}^k(P')v_\mu{}^k(P)}{\lambda_\mu{}^{(k)}},\qquad (8.3)$$

where $v_\mu{}^k$ and $\lambda_\mu{}^{(k)}$ are, respectively, the μth eigenfunction and eigenvalue of homogeneous equation and are functions of J and t. The $v_\mu{}^k$'s are orthonormal, i.e.,

$$\int P^3dPv_\mu{}^k(P)v_\nu{}^k(P)=\delta_{\mu\nu}.$$

Suppose that $\lambda_1{}^{(k)}(J,t)=\lambda_0$ produces two trajectories, $\alpha^k(t)$, $k=1,2$ which both have J slightly less than zero at $t=0$ and which cross near $t=0$. Also we assume all the trajectories determined from $\lambda_\mu{}^{(k)}(J,t)=\lambda_0$, $\mu>1$ are lower than either of these for all $t\leq m_\pi{}^2$. Then these other trajectories can be omitted now and inserted later using simple perturbation theory.

We expand the P dependence of $R^{ik}(P',P)$ in the eigenfunctions of $B^k(P',P)$,

$$R^{ik}(P',P)=\sum_\mu w_\mu{}^{ik}(P')v_\mu{}^k(P),\qquad (8.4)$$

where the $w_\mu{}^{ik}(P')$ are unknown functions which also depend on J and t. Putting Eqs. (8.3 and 8.4) into the coupled integral equations and integrating on the right by $\int v_1{}^k(P)P^3dP$ gives

$$w_1{}^{ik}(P)+(bJt)^{1/2}\sum_\mu\left\langle\begin{matrix}j\\ \mu\end{matrix}\middle|\begin{matrix}k\\ 1\end{matrix}\right\rangle w_\mu{}^{ij}(P)=\lambda\delta_{ik}\frac{v_1{}^k(P)}{\lambda_1{}^{(k)}}+\lambda\frac{w_1{}^{ik}(P)}{\lambda_1{}^{(k)}},\quad j\neq k,\qquad (8.5)$$

where

$$\left\langle {}^{j}_{\mu}\middle|{}^{k}_{\nu}\right\rangle=\int v_\mu{}^j(P)v_\nu{}^k(P)P^3dP$$

is a function of J and t. The assumption that the other uncoupled trajectories are far from those with $\mu=1$ and $t=0$ allows the sum over μ to contain just the $\mu=1$ term. Then dropping the 1 subscript and setting $i=1$ and dropping that superscript, the two coupled equations are (at $\lambda=\lambda_0$)

$$\begin{aligned}(\lambda^{(1)}-\lambda_0)w^{(1)}(P)&\\ +\lambda^{(1)}(bJt)^{1/2}\langle 2|1\rangle w^{(2)}(P)&=\lambda_0 v^{(1)}(P)\,,\\ (\lambda^{(2)}-\lambda_0)w^{(2)}(P)&\\ +\lambda^{(2)}(bJt)^{1/2}\langle 1|2\rangle w^{(1)}(P)&=0\,.\end{aligned}\tag{8.6}$$

Expanding $\lambda^{(k)}(J,t)-\lambda_0$ in a power series in J and t and keeping only first-order terms gives

$$(\lambda^{(k)}-\lambda_0)/\lambda_0=a_k[J-\alpha_k(t)]\,,\tag{8.7}$$

where

$$\alpha_k(t)=-c_k+d_kt\,.$$

In the coupling terms we approximate $\lambda^{(k)}$ by λ_0 and set $\langle 1|2\rangle^2 b=b'$ and then neglect its dependence on J and t so that we get

$$\begin{pmatrix}a_1[J-\alpha_1(t)] & (b'Jt)^{1/2}\\ (b'Jt)^{1/2} & a_2[J-\alpha_2(t)]\end{pmatrix}\begin{pmatrix}w^{(1)}\\ w^{(2)}\end{pmatrix}=\begin{pmatrix}v^{(1)}\\ 0\end{pmatrix}.\tag{8.8}$$

If the final state were state two instead of state one, the right-hand side would be

$$\begin{pmatrix}0\\ v^{(2)}\end{pmatrix}.$$

Even without the approximations made at the beginning of this section we would have two coupled equations, similar to the above, to solve.[20]

[20] To relax the approximations of this model we must use the three equations in Table IV as the uncoupled (from the $M=0$ amplitudes) $M=1$ equation. Then we have for the uncoupled equation

$$\sum_j R^{ij}(P',P)F^{jk}(P)=\lambda B^{ik}(P',P)$$
$$+\sum_j\int Q^3dQR^{ij}(P',Q)B^{jk}(Q,P),\quad i,j=1,2,3,$$

where the three values of the indices correspond to the three sets of ω and κ for the $M=1$ trajectory. Then $B(Q,P)$ can be written as

$$B^{ij}(Q,P)=[F^{1/2}(Q)]^{ik}\sum_\nu\frac{v_\mu{}^k(Q)u_\mu{}^l(Q)}{\lambda_\mu}[F^{1/2}(P)]^{lj},$$

where

$$\sum_i\int u_\mu{}^i(P)v_\nu{}^i(P)P^3dP=\delta_{\mu\nu}$$

and $[F^{1/2}]^{ik}$ is the i,kth element of the matrix $F^{1/2}$, and the eigenfunctions are

$$\sum_l u_\mu{}^l(P)[F^{1/2}(P)]^{lj}.$$

After coupling these equations to the other equations, we use

$$R^{ij}(P',P)=\sum_{l\mu}w_\mu{}^{ij}(P')u_\mu{}^l(P)[F^{1/2}(P)]^{lj}$$

as a solution and get two equations similar to the previous equation.

We see the nondiagonal parts of the matrix have the same J and t dependence, near $J=0$ and $t=0$, as the corresponding elements in the $\langle\Upsilon'|S^{-1}\otimes S^{-1}|\Upsilon\rangle$ matrix. Thus if we wanted to study the possibility that the uncoupled trajectory crossing the $M=1$ uncoupled trajectory is a $n=J+1$, $M=0$, $\Sigma=1$ we would have a similar 2×2 matrix but with $t(J)^{1/2}$ for the nondiagonal elements.[21] Since we are assuming that one of the two states in the model is the $M=1$ state which is a ν state the other must be an σ state for the trajectory to have the possibility of choosing sense. Then the nondiagonal term is proportional to $(t^\beta J)^{1/2}$, $\beta=1,2,\cdots$.

The trajectories are determined by the solution, α, of the equation

$$\begin{aligned}\bar{D}\equiv a_\nu a_\sigma[\alpha-\alpha_\nu(t)][\alpha-\alpha_\sigma(t)]-(b')^2t^\beta J&=0\\ \alpha_i(t)=-c_i+d_it\,,\quad i=\nu,\sigma\,,&\end{aligned}\tag{8.9}$$

where we have replaced the state labels 1 and 2 by the more informative labels ν and σ. Since the equation is quadratic it yields two solutions. At $t=0$ and again at $J=0$ the two solutions are $\alpha=\alpha_\nu(t)$ and $\alpha=\alpha_\sigma(t)$. Without knowing the sign of $a_\nu a_\sigma/(b')^2$ and whether $\alpha_\nu(t)$ and $\alpha_\sigma(t)$ cross in the $J<0$, $t>0$ region, it is impossible to determine anything more about the trajectories. In Fig. (2) we show the various possible trajectories. We only include cases where $\alpha_\nu(0)>\alpha_\sigma(0)$ since the high-energy data seem to require an $M=1$ trajectory to be highest at $t=0$.

We find the residues of the trajectories by examining the solution of the 2×2 matrix equations. These are

$$\begin{aligned}R^{\sigma\sigma}&=a_\nu[J-\alpha_\nu(t)]v^\sigma(P)v^\sigma(P')/\bar{D}\,,\\ R^{\nu\sigma}&=(b'Jt)^{1/2}v^\nu(P)v^\sigma(P')/\bar{D}\,,\\ R^{\nu\nu}&=a_\sigma[J-\alpha_\sigma(t)]v^\nu(P)v^\nu(P')/\bar{D}\,,\\ R^{\sigma\nu}&=(b'Jt)^{1/2}v^\sigma(P)v^\nu(P')/\bar{D}\,.\end{aligned}\tag{8.10}$$

The residue of the poles obviously factor. If we had kept the correct P dependence in the inverse propagators we would find that M on the mass shell factors into equal factors for initial and final states

$$M^{ij}=\Gamma_i\Gamma_j/(J-\alpha(t))\,.\quad i,j=\nu,\sigma\,.\tag{8.11}$$

Examining these residues we see that the vertex functions for the trajectory which at $J=0$ crosses α_σ are $\Gamma_\nu\propto\alpha^{1/2}$ and $\Gamma_\sigma\propto 1$ and the vertex functions of the trajectory which crosses α_ν at $J=0$ are $\Gamma_\nu\propto 1$ and $\Gamma_\sigma\propto\alpha^{1/2}$. Combining this with the properties of the X's yields the following: The trajectory which equals (crosses) α_σ at $J=0$ chooses sense and the other one chooses nonsense. The pion's position in Fig. 1 is consistent with this.

The last drawing in Fig. 1 seems the most likely to us. It has the pion lying on the same trajectory which is $M=1$ at $t=0$. This would agree with the fact that the vertex function used in the high-energy fits extrapolated

[21] This particular choice of an $O(4)$ σ state was used to compare with PCAC in Ref. 9.

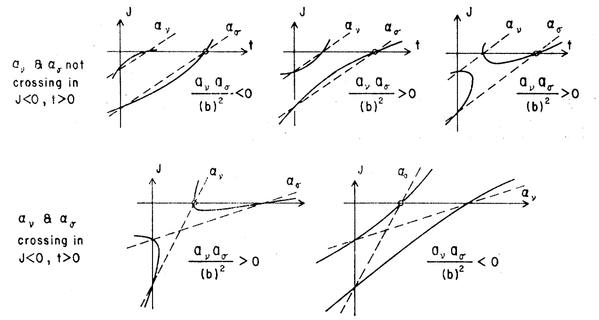

FIG. 1. Trajectories from the 2×2 matrix model. Dashed line, uncoupled trajectory; solid line, trajectory with coupling; circle, position of the pion.

to the pion-nucleon coupling constant at $t=m_\pi^2$. For this case $\Gamma_\sigma \propto t^{1/2}$ and $\Gamma_\nu \propto \alpha^{1/2}$. Then the $O(3)$ vertex function for the singlet partial-wave state is

$$\begin{aligned}\gamma_0 &= \Gamma_\sigma X^\sigma(t,0)+\Gamma_\nu X^\nu(t,0)\\ &= g_\sigma t+g_\nu \alpha .\end{aligned} \tag{8.12}$$

This causes the vertex function to have a first-order zero in the vicinity of $t=0$. This agrees with the photoproduction fit. But the charge-exchange scattering was fit with a first-order zero in the residue while we predict a second-order zero (from two vertex functions).

An obvious question to ask is, what is the other trajectory? We see that it chooses nonsense at $J=0$, and so produces no particle. Its $O(4)$ vertex functions are $\Gamma_\sigma \propto \alpha^{1/2}$ and $\Gamma_\nu \propto t^{1/2}$, thus its singlet partial-wave vertex function is

$$\gamma_0 = (g_\sigma + g_\nu)(t\alpha)^{1/2} \tag{8.13}$$

so that it makes no contribution to forward scattering in crossed channels. It does not have a parity-reversed partner trajectory to contribute to other partial waves as does the pion's trajectory above. So it could easily escape experimental verification.

In the review paper we have shown that for certain choices of the $O(4)$ state whose uncoupled trajectory crosses the $M=1$ trajectory, we get no obvious contradiction of PCAC.

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APPENDIX A: DIRAC SPINORS AS REPRESENTATIONS OF THE LORENTZ GROUP

So that we will end in the normal Euclidean metric after the Wick rotation we choose the metric

$$g_{\mu\nu}=g^{\mu\nu}=\begin{bmatrix}-1 &&& \\ & 1 && \\ && 1 & \\ &&& 1\end{bmatrix}.$$

Contravariant 4-vectors are defined by

$$p^\mu \equiv (p^0,p^1,p^2,p^3)=(E,p_x,p_y,p_z)$$

and covariant by $p_\mu = g_{\mu\nu}p^\nu$. Then the mass-shell condition is

$$p^\mu p_\mu = |\mathbf{p}|^2 - E^2 = -m^2 .$$

Much of this Appendix is contained in, or follows from, Weinberg's "Feynman rules for any spin"[22] and is presented here only for completeness.

Using the transformation properties under the homogeneous Lorentz group as a definition of helicity (or spin) states of a particle we have

$$U[\Lambda]|p\lambda\rangle = \sum_{\lambda'} D_{\lambda'\lambda}{}^j[L^{-1}(\Lambda p)\Lambda L(p)]|\Lambda p\lambda'\rangle ,$$

where if $p^0=m\cosh\tilde{\psi}$, $p^3=m\sinh\tilde{\psi}\cos\theta$, $p^2=m\sinh\tilde{\psi}\times\sin\theta\sin\varphi$, and $p^1=m\sinh\tilde{\psi}\sin\theta\cos\varphi$. $L(p)=R_3(\varphi)\times R_2(\theta)B_3(\tilde{\psi})$ if λ's are helicities, or $L(p)=R_3(\varphi)R(\theta)\times B_3(\tilde{\psi})R_2(-\theta)R_3(-\varphi)$ if λ's are spins.

We use the normal four-component Dirac fields for the nucleon. We define the state $|p,\alpha\rangle$ as the state created by the α component of the adjoint field. We choose a representation for the fields which diagonalizes the Lorentz transformation matrices. To use this representation easily α is replaced by the set (r,ρ), $r=\pm$, $\rho=\pm\frac{1}{2}$. The $r=+$ components transform according to $\mathfrak{D}^{\frac{1}{2}0}$; $r=-$ according to $\mathfrak{D}^{0\frac{1}{2}}$. Thus we have

$$U[\Lambda]|p,(r\rho)\rangle = \sum_{r'\rho'} \mathfrak{D}_{(r'\rho'),(r\rho)}(\Lambda)|\Lambda p,(r'\rho')\rangle ,$$

where

$$\mathfrak{D}_{(r'\rho'),(r\rho)}(\Lambda)=\begin{pmatrix}\mathfrak{D}_{\rho'\rho}{}^{\frac{1}{2}0}(\Lambda) & 0\\ 0 & \mathfrak{D}_{\rho'\rho}{}^{0\frac{1}{2}}(\Lambda)\end{pmatrix}_{r'r}. \tag{A1}$$

Three important properties of the representations are

(1) $\mathfrak{D}^{\frac{1}{2}0\dagger}(\Lambda)=\mathfrak{D}^{0\frac{1}{2}}(\Lambda^{-1})$, where Λ is any Lorentz transformation,

(2) $\mathfrak{D}^{\frac{1}{2}0}(R)=\mathfrak{D}^{0\frac{1}{2}}(R)=D^{\frac{1}{2}}(R)$, where $D^{\frac{1}{2}}$ is a representation of $O(3)$ and R is any rotation, and

(3) $\mathfrak{D}^{\frac{1}{2}0}(\tilde{\psi})=\mathfrak{D}^{0\frac{1}{2}}(-\tilde{\psi})$, where $\tilde{\psi}$ is a pure boost.

If we express a general Lorentz transformation in terms of five rotations and one boost by

$$\Lambda(\varphi,\theta,\tilde{\psi},\alpha,\beta,\gamma)=R_3(\varphi)R_2(\theta)B_3(\tilde{\psi})R_3(\alpha)R_2(\beta)R_3(\gamma)$$

and use the last two properties above, we can reexpress $\mathfrak{D}$ as

$$\begin{aligned}\mathfrak{D}_{(r'\rho'),(r\rho)}(\Lambda)=\delta_{rr'}\sum_{\mu=\pm\frac{1}{2}} & D_{\rho'\mu}{}^{\frac{1}{2}}(\varphi,\theta,0)\\ & \times\mathfrak{D}_{\mu\mu}{}^{\frac{1}{2}0}(r\tilde{\psi})D_{\mu\rho}{}^{\frac{1}{2}}(\alpha,\beta,\gamma).\end{aligned} \tag{A2}$$

Since the Dirac spinors, $u_\alpha(p,\lambda)$ are the coefficients of the particle annihilation operators in the expression for

[22] S. Weinberg, Phys. Rev. 133, B1318 (1964).

the fields we have

$$|p\lambda\rangle=\sum_\alpha u_\alpha(p,\lambda)\,|p,\alpha\rangle\,,$$

$$u_\alpha(p\lambda)=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathfrak{D}_{\rho\lambda}{}^{\frac{1}{2}0}[L(p)]\\ \mathfrak{D}_{\rho\lambda}{}^{0\frac{1}{2}}[L(p)]\end{pmatrix}_r. \tag{A3}$$

Next $\bar{u}(p,\lambda)$ is determined by the equation

$$\sum_\alpha \bar{u}_\alpha(p,\lambda)u_\alpha(p,\lambda')=\delta_{\lambda\lambda'}\,.$$

Because the representations of the Lorentz group used to define $u(p,\lambda)$ are not unitary $\bar{u}_\alpha(p,\lambda)$ is not simply the complex conjugate of $u_\alpha(p,\lambda)$. Instead of unitarity we use the first of the three properties of the $\mathfrak{D}$'s mentioned previously to reach the definition

$$\bar{u}_\alpha(p,\lambda)=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathfrak{D}_{\rho\lambda}{}^{0\frac{1}{2}*}[L(p)]\\ \mathfrak{D}_{\rho\lambda}{}^{\frac{1}{2}0*}[L(p)]\end{pmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathfrak{D}_{\lambda\rho}{}^{\frac{1}{2}0}[L^{-1}(p)]\\ \mathfrak{D}_{\lambda\rho}{}^{0\frac{1}{2}}[L^{-1}(p)]\end{pmatrix}. \tag{A4}$$

We have fixed the representation of the Dirac fields by specifying their Lorentz transformation properties, and, as a result, the representation of the Dirac matrices is now determined. From the Lorentz transformation of the Dirac equation we know that

$$\mathfrak{D}(\Lambda)\gamma^\nu\mathfrak{D}(\Lambda^{-1})=\gamma^\mu\Lambda_\mu{}^\nu\,,$$

where $\mathfrak{D}(\Lambda)$ is given by Eq. (A1). We next show that for our choice of representation this is satisfied by

$$\not{p}\equiv\gamma^\mu p_\mu=\boldsymbol{\gamma}\cdot\mathbf{p}-\gamma^0p^0$$

$$=i\bar{P}\begin{bmatrix}0 & \mathfrak{D}^{\frac{1}{2}0}[L(p)]^2\\ \mathfrak{D}^{0\frac{1}{2}}[L(p)]^2 & 0\end{bmatrix}, \tag{A5}$$

where $L(p)=R_3(\varphi)R_2(\theta)B_3(\psi)R_2(-\theta)R_3(-\varphi)$ and $\bar{P}^2=-p^\mu p_\mu$.

Proof:

$$\mathfrak{D}(\Lambda)\gamma^\mu p_\mu\mathfrak{D}(\Lambda^{-1})=\gamma^\mu\Lambda_\mu{}^\nu p_\nu=\gamma^\mu(\Lambda p)_\mu\,,$$

$$\begin{pmatrix}\mathfrak{D}^{\frac{1}{2}0}(\Lambda) & 0\\ 0 & \mathfrak{D}^{0\frac{1}{2}}(\Lambda)\end{pmatrix}\begin{pmatrix}0 & \mathfrak{D}^{\frac{1}{2}0}[L(p)]^2\\ \mathfrak{D}^{0\frac{1}{2}}[L(p)]^2 & 0\end{pmatrix}\begin{pmatrix}\mathfrak{D}^{\frac{1}{2}0}(\Lambda^{-1}) & 0\\ 0 & \mathfrak{D}^{0\frac{1}{2}}(\Lambda^{-1})\end{pmatrix}$$

$$=\begin{pmatrix}0 & \mathfrak{D}^{\frac{1}{2}0}(\Lambda)\mathfrak{D}^{\frac{1}{2}0}[L(p)]^2\mathfrak{D}^{0\frac{1}{2}}(\Lambda^{-1})\\ \mathfrak{D}^{0\frac{1}{2}}(\Lambda)\mathfrak{D}^{0\frac{1}{2}}[L(p)]^2\mathfrak{D}^{\frac{1}{2}0}(\Lambda^{-1}) & 0\end{pmatrix}.$$

Now considering only the upper right element and using the fact that $L(p)$ is a pure boost so that $\mathfrak{D}^{\frac{1}{2}0}[L(p)]=\mathfrak{D}^{0\frac{1}{2}}[L^{-1}(p)]$ we have

$$\mathfrak{D}^{\frac{1}{2}0}(\Lambda)\mathfrak{D}^{\frac{1}{2}0}[L(p)]\mathfrak{D}^{0\frac{1}{2}}[L^{-1}(p)]\mathfrak{D}^{0\frac{1}{2}}(\Lambda^{-1})\,.$$

Now $L^{-1}(\Lambda p)\Lambda L(p)$ is a pure rotation, called R_w, so that our element is

$$\mathfrak{D}^{\frac{1}{2}0}[\Lambda L(p)]D^{\frac{1}{2}}(R_w{}^{-1})D^{\frac{1}{2}}(R_w)\mathfrak{D}^{0\frac{1}{2}}[L^{-1}(p)\Lambda^{-1}]$$
$$=\mathfrak{D}^{\frac{1}{2}0}[L(\Lambda p)]\mathfrak{D}^{0\frac{1}{2}}[L(\Lambda p)]$$
$$=\mathfrak{D}^{\frac{1}{2}0}[L(\Lambda p)]^2.\quad\text{Q.E.D.}$$

Of course we have not established that our choice of normalization is correct. To see that it is we consider the anticommutation relation

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}I\,.$$

From Eq. (A5) and taking p at rest we find that

$$\gamma^0=-i\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{A6}$$

which has correct normalization to satisfy the anticommutation relation.

Since γ^5 is invariant under proper Lorentz transformation, that is, it obeys

$$\mathfrak{D}(\Lambda)\gamma^5\mathfrak{D}(\Lambda^{-1})=\gamma^5$$

and is normalized by $\gamma^5\gamma^5=I$, and is different from the identity, it is (to within a sign)

$$\gamma^5=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{A7}$$

The Dirac equation for spinors is

$$(b\not{p}+m)u(p)=0\,,$$

where b is a constant determined by our choice of the metric and the normalization of γ^0. For a particle at rest

$$u=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \alpha\end{pmatrix},$$

Eq. (A3), where α is a 2-component spinor depending on the helicity. At rest

$$\not{p}=-p^0\gamma^0=im\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

and hence $b=i$. This gives for the inverse propagator

$$S^{-1}(p)=i\not{p}+m \tag{A8}$$

and for the propagator

$$S(p)=-i\not{p}+m/p^2+m^2\,,\quad\text{where } p^2=p^\mu p_\mu\,. \tag{A9}$$

We note also that since a particle at rest has

$$u=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \alpha\end{pmatrix},$$

and hence an antiparticle at rest must have

$$v=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix},$$

the intrinsic parity (or particle-antiparticle description) is not diagonal. In fact, since $\mathcal{P}u(p^0)=\gamma^0 u(p^0)$ we have

$$\mathcal{P}|p^0,(r\rho)\rangle=|p^0,(-r\rho)\rangle.$$

We can construct a new basis which is diagonal in the intrinsic parity, η, and use η to label this basis, thus

$$\mathcal{P}|p,\eta\rho\rangle=\eta|\mathcal{P}p,\eta\rho\rangle.$$

Then the transformation between the two bases is

$$|p,\eta\rho\rangle=\sum_r C_{\eta r}|p,(r\rho)\rangle,$$

$$C_{\eta r}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}_{\eta r}. \qquad \text{(A10)}$$

To see how to change an antiparticle line in a Feynman diagram into a particle line, consider the part of a 2-rung ladder diagram (box diagram) drawn in Fig. 2(a) and the following expression for it:

$$\bar{v}(p,\lambda)\Gamma\frac{(-\boldsymbol{p}-\boldsymbol{k}+im)}{(p+k)^2+m^2}\Delta v(q,\lambda').$$

Now since

$$v(q)=\mathcal{C}^{-1}\bar{u}^T(q),$$
$$u(q)=\mathcal{C}^{-1}\bar{v}^T(q),$$

and

$$\mathcal{C}\boldsymbol{k}\mathcal{C}^{-1}=-\boldsymbol{k},$$

where $\mathcal{C}$ is the charge-conjugation operator,[23] it is possible to define new vertex functions so that one may use the Feynman rules for a particle line. Let $\Gamma_C=\mathcal{C}\Gamma^T\mathcal{C}^{-1}$ and $\Delta_C=\mathcal{C}\Delta\mathcal{C}^{-1}$. Then the following expression for a particle line (Figure 2b) has the same value:

$$\bar{u}(q,\lambda')\Delta_c\frac{(\boldsymbol{p}+\boldsymbol{k}+im)}{(p+k)^2+m^2}\Gamma_c u(p,\lambda).$$

Since the vertex functions are invariant under the total (meaning all three particles) operator $\Gamma_C=\eta_C\Gamma$, where η_C is the eigenvalue of the exchange particle under $\mathcal{C}$. Thus we may use the Bethe-Salpeter equation for the

[23] In the (r,ρ) representation $\mathcal{C}_{(r'\rho'),(r\rho)}=\begin{pmatrix}C_{\rho'\rho}&0\\0&-C_{\rho'\rho}\end{pmatrix}_{r'r}$, where $C=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

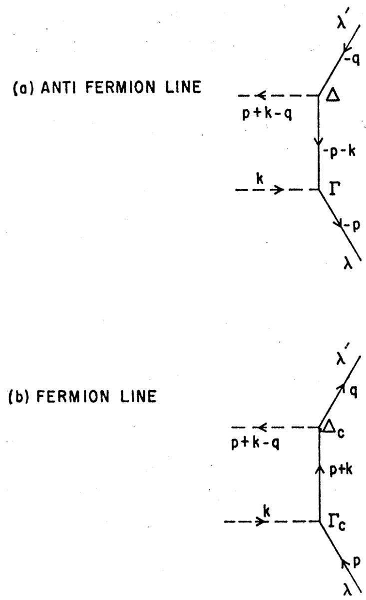

FIG. 2. Replacing a fermion line in a Feynman diagram by an antifermion line.

scattering of two nucleons if we change the sign of the Born terms and kernels of odd-C exchange particles. (If we included isospin, we would change the sign for exchange of odd-G particles.)

APPENDIX B: $O(4)$ PROJECTION FOR NONZERO ENERGY

In this Appendix we present an alternative way of understanding our $O(4)$ projection for nonzero energy (t).

We start by presenting the invariance (actually covariance) of the scattering amplitude to a general $O(4)$ rotation, g. This follows from the Lorentz covariance of the amplitude before the Wick rotation.

$$T_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(p',p,k)=\sum_{\mu_1\mu_2\mu_1'\mu_2'}D_{\lambda_1'\mu_1'}{}^{\frac12}[R_w^{-1}(g,\tfrac12k+p')]$$
$$\times D_{\lambda_2'\mu_2'}{}^{\frac12}[R^{-1}(g,\tfrac12k-p')]T_{\mu_1'\mu_2',\mu_1\mu_2}(gp',gp,gk)$$
$$\times D_{\mu_1\lambda_1}{}^{\frac12}[R_w(g,\tfrac12k+p)]D_{\mu_2\lambda_2}{}^{\frac12}[R_w(g,\tfrac12k-p)], \qquad \text{(B1)}$$

where

$$R_w(g,p)=L^{-1}(gp)gL(p)$$

and $D^{\frac12}$ is the normal $J=\frac12$ representation of the $O(3)$ rotation [or $SU(2)$] group.

One way to consider this equation is in the Hilbert space of two particle (asymptotic) states. Then the operator T which transforms the state before scattering into the state after scattering commutes with the unitary operator of an $O(4)$ transformation.

$$U^{-1}(g)TU(g)=T.$$

Eq. (B1) is a matrix element of this equation.

The two-particle states, with k and p the total and relative momentum, transform under an $O(4)$ rotation by

$$U(g)|kp\lambda_1\lambda_2\rangle=\sum_{\mu_1\mu_2} D_{\mu_1\lambda_1}{}^{\frac{1}{2}}[R_w(g,\tfrac{1}{2}k+p)] \times D_{\mu_2\lambda_2}{}^{\frac{1}{2}}[R_w(g,\tfrac{1}{2}k-p)]|gk,gp,\mu_1,\mu_2\rangle. \quad \text{(B2)}$$

For comparison we write the $O(4)$ covariance of the M amplitude

$$M_{\alpha'\beta',\alpha\beta}(p',p,k)=\sum_{\gamma\gamma'\delta\delta'}\mathfrak{D}_{\alpha'\gamma'}(g^{-1})\mathfrak{D}_{\beta'\delta'}(g^{-1}) \times M_{\gamma'\delta',\gamma\delta}(gp',gp,gk)\mathfrak{D}_{\delta\alpha}(g)\mathfrak{D}_{\beta\delta}(g)\,, \quad \text{(B3)}$$

where $\mathfrak{D}$, the $O(4)$ representation to which the Dirac field belongs, is given by Eq. (3.4). M also can be considered an operator in a Hilbert space and this is a matrix element of the operator equation

$$U^{-1}(g)MU(g)=M\,.$$

But now the Hilbert space is different. We call this the spinorial space; it is actually the space of field components. A two-field ket in this space transforms according to

$$U(g)|kp\alpha\beta\rangle=\sum_{\alpha'\beta'}\mathfrak{D}_{\alpha'\alpha}(g)\mathfrak{D}_{\beta'\beta}(g)|gk,gp,\alpha',\beta'\rangle. \quad \text{(B4)}$$

We will show later that this transformation law is more convenient for our use than Eq. (B2). But first we need to determine the properties we want our $O(4)$ projection to have.

In the BS equation k is held constant while the integral over p covers all of four space. And in the partial-wave projection the final relative momentum p' varies while k is fixed. Thus, we need an $O(4)$ projection which keeps k fixed.

A particularly useful form of the partial-wave projection for us is the "transformation to the angular momentum basis" one would write if he did not know that the rotations commute with T. It is

$$\langle J'm'\lambda_1'\lambda_2'|T|Jm\lambda_1\lambda_2\rangle =\frac{1}{64\pi^4}\int dR_1 D_{\lambda'm'}{}^{J'}(R_1)\int dR_2 D_{\lambda m}{}^{J*}(R_2) \times\langle k_4,p',\lambda_1',\lambda_2'|U^{-1}(R_1)TU(R_2)|k_4,p,\lambda_1,\lambda_2\rangle\,,$$

where the integrals are over the $O(3)$ rotation group. (Since the rotation operators do commute with T we have

$$\langle J'm'\lambda_1'\lambda_2'|T|Jm\lambda_1\lambda_2\rangle=\delta_{JJ'}\delta_{mm'}f_{\lambda_1'\lambda_2',\lambda_1\lambda_2}{}^{J})\,.$$

We need a transformation to an $O(4)$ basis, like the above, but with the $O(4)$ rotation operators operating only on the relative momenta and on the spins, but not on k.

By comparing Eq. (B2) to Eq. (B4) we see the superiority of the spinorial Hilbert space; it can be considered the direct product of two Hilbert spaces, one containing k and the other p and the two field indices. Then

$$|k,p,\alpha,\beta\rangle=|k\rangle\otimes|p,\alpha,\beta\rangle\,,$$

and we let

$$U(g)=U_k(g)\otimes U_p(g)\,,$$

such that

$$U_k(g)|k\rangle=|gk\rangle$$

and

$$U_p(g)|p,\alpha,\beta\rangle=\sum_{\alpha'\beta'}\mathfrak{D}_{\alpha'\alpha'}(g)\mathfrak{D}_{\beta'\beta'}(g)|gp,\alpha',\beta'\rangle.$$

$I\otimes U_p(g)$ is then the operator we need for the $O(4)$ projection. Since it is not possible to make a similarly simple decomposition of the two-particle Hilbert space, M instead of T is the "natural" function for the $O(4)$ projection.

We can reformulate our $O(4)$ projection in the $|p,\alpha,\beta\rangle$ space by defining $M(k)\equiv\langle k|M|k\rangle$. This is an operator in the $|p,\alpha,\beta\rangle$ space which does not commute with $U_p(g)$. In fact

$$U_p^{-1}(g)M(k)U_p(g)=M(gk)\,.$$

Since it does not commute with U_p, the operator used in the $O(4)$ projections, $M(k)$, will have nondiagonal $O(4)$ matrix elements.

APPENDIX C: UNEQUAL-MASS FERMION-ANTIFERMION SCATTERING

In this Appendix we outline the generalization of the paper to unequal-mass scattering. We will call this baryon-antinucleon scattering to give different names to the two particles. In the summary paper[9] this scattering was used to investigate pion-nucleon s-wave scattering. If the baryon is assumed to be a pion and a nucleon in a relative s state the baryon-nucleon-pion vertex is the pion-nucleon s-wave scattering amplitude. In this particular case the baryon has the opposite "intrinsic" parity than the nucleon.

Because of the unequal masses, C_n no longer is a good quantum number in the neutral baryon-antinucleon system. Hence the singlet partial-wave state, $|0\rangle$, and the "uncoupled triplet," $|u\rangle$, are coupled and there is a new partial wave amplitude, $\langle 0|f|u\rangle$. This manifests itself in the BS equation (using baryon-nucleon formalism) by not having I invariance. So for $t\neq 0$ there are two instead of four disjoint classes of coupled integral equations. Even at $t=0$ there is coupling between the $O(4)$ types. Type I and IV couple, as do II and V and also III and VI.

The kernels of the $O(4)$-projected BS equation do not change in any important way.

TABLE VI. X's for unequal-mass $N\bar{N}$ states. Here $d_{J11\pm}{}^{(n,M)}(\psi)=d_{J11}{}^{(n,M)}(\psi)\pm d_{J1,-1}{}^{(n,M)}(\psi)$, $\tau(t)=(2m_1^2+2m_2^2-t)^{1/2}$, $P(t)=[t-(m_1-m_2)^2]^{1/2}$, and $N(t)=[(m_1+m_2)^2-t]^{1/2}$, $\Delta=m_1^2-m_2^2$, and $G=PN$.

State	$X(t,\lambda_1\lambda_2)\times 2(m_1m_2)^{1/2}$
(a) X's in terms of $N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\Psi)$	
0	$\sqrt{2}N_\Sigma{}^{(n,M)}d_{J\Sigma 0}{}^{(n,M)}(\psi)\left[\delta_{\Sigma 0}\delta_{\omega+}P(t)\left(\delta_{\kappa+}+\delta_{\kappa-}\frac{m_1-m_2}{\tau(t)}\right)-i\delta_{\Sigma 1}\delta_{\omega-}N(t)\left(\delta_{\kappa+}+\delta_{\kappa-}\frac{m_1+m_2}{\tau(t)}\right)\right]$
u	$\frac{1}{\sqrt{2}}N_1{}^{(n,M)}\delta_{\Sigma 1}\left[\delta_{\omega+}d_{J11-}{}^{(n,M)}(\psi)P(t)\left(\delta_{\kappa+}\frac{m_1-m_2}{\tau(t)}+\delta_{\kappa-}\right)+i\delta_{\omega-}d_{J11+}{}^{(n,M)}(\psi)N(t)\left(\delta_{\kappa+}\frac{m_1+m_2}{\tau(t)}+\delta_{\kappa-}\right)\right]$
1	$\sqrt{2}N_\Sigma{}^{(n,M)}d_{J\Sigma 0}{}^{(n,M)}(\psi)\left[\delta_{\Sigma 1}\delta_{\omega+}P(t)\left(\delta_{\kappa+}+\delta_{\kappa-}\frac{m_1-m_2}{\tau(t)}\right)-i\delta_{\Sigma 0}\delta_{\omega-}N(t)\left(\delta_{\kappa+}+\delta_{\kappa-}\frac{m_1+m_2}{\tau(t)}\right)\right]$
2	$\frac{1}{\sqrt{2}}N_1{}^{(n,M)}\delta_{\Sigma 1}\left[\delta_{\omega+}d_{J11+}{}^{(n,M)}(\psi)P(t)\left(\delta_{\kappa+}\frac{m_1-m_2}{\tau(t)}+\delta_{\kappa-}\right)+i\delta_{\omega-}d_{J11-}{}^{(n,M)}(\psi)N(t)\left(\delta_{\kappa+}\frac{m_1+m_2}{\tau(t)}+\delta_{\kappa-}\right)\right]$

(b) $N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda\pm}{}^{(n,M)}(\psi)$ in the vicinity of $J=0$

n	Υ $\lvert M\rvert$	Σ	ν or σ	$N_\Sigma{}^{(n,M)}d_{J\Sigma 0}{}^{(n,M)}(\psi)$	$N_\Sigma{}^{(n,M)}d_{J\Sigma 1+}{}^{(n,M)}(\psi)$	$N_\Sigma{}^{(n,M)}d_{J\Sigma 1-}{}^{(n,M)}(\psi)$
J	1	1	ν	$(Jt)^{1/2}\tau/G$	Δ/G	c
J	0	1	ν	$\Delta J^{1/2}/G$	$\tau t^{1/2}/G$	0
J	0	0	σ	c	0	0
$J+1$	1	1	ν	$\Delta J^{1/2}/G$	$(c\Delta^2-G^2)/(t)^{1/2}G\tau$	$\Delta/(t)^{1/2}\tau$
$J+1$	0	1	σ	$(cJ\Delta^2-G^2)/(t)^{1/2}G\tau$	$(J)^{1/2}\Delta/G$	0
$J+1$	0	0	σ	$\Delta/(t)^{1/2}\tau$	0	0
$J+2$	1	1	ν	$(J)^{1/2}(c\Delta^2-G^2)/(t)^{1/2}G\tau$	$(c\Delta^2-G^2)\Delta/tG\tau^2$	$(c\Delta^2-G^2)/t\tau^2$
$J+2$	0	1	σ	$\Delta[cJ(\Delta^2+kG^2)-G^2]/tG\tau^2$	$(J)^{1/2}(c\Delta^2-G^2)/(t)^{1/2}G\tau$	0
$J+2$	0	0	σ	$(c\Delta^2-G^2)/t\tau^2$	0	0

The inverse propagator is changed by the introduction of two new terms, both proportional to the mass difference. They are

$$\tfrac{1}{2}(m_2-m_1)\langle\Upsilon'|p\otimes I+I\otimes p|\Upsilon\rangle$$

and

$$\tfrac{1}{4}(\sqrt{t})(m_2-m_1)\langle\Upsilon'|\gamma^4\otimes I-I\otimes\gamma^4|\Upsilon\rangle.$$

These are the terms that provide the coupling between the different $O(4)$ types.

But the most important change produced by the generalization to unequal masses occurs in the calculation of the X's. Equation (7.3) for $\tilde{X}(t,\lambda_1,\lambda_2)$ is still valid but the angles in it have new values. The mass shell constraints in the center-of-mass system yield (see Ref. 7)

$$\cos\psi=\frac{m_1^2-m_2^2}{[t(2m_1^2+2m_2^2-t)]^{1/2}},\quad \sin\psi=\frac{NP}{[t(2m_1^2+2m_2^2-t)]^{1/2}},$$
$$\cos\psi_1=\frac{t+m_1^2-m_2^2}{2m_1\sqrt{t}},\quad \sin\psi_1=\frac{NP}{2m_1\sqrt{t}},\tag{C1}$$
$$\cos\psi_2=\frac{t-m_1^2+m_2^2}{2m_2\sqrt{t}},\quad \sin\psi_2=\frac{-NP}{2m_2\sqrt{t}},$$

where $N=[(m_1+m_2)^2-t]^{1/2}$ and $P=[t-(m_1-m_2)^2]^{1/2}$. Using these in Eq. (7.3) gives

$$\tilde{X}^\Upsilon(t,\lambda_1,\lambda_2)=(1/\sqrt{2})N_\Sigma{}^{(n,M)}d_{J\Sigma\lambda}{}^{(n,M)}(\psi)C(\tfrac{1}{2},\tfrac{1}{2},\Sigma;\lambda_1,-\lambda_2,\lambda)$$
$$\times\left[i^{-2\lambda_2}\delta_{\omega-}\frac{N}{2(m_1m_2)^{1/2}}\left(\delta_{\kappa\lambda_1,\lambda_2}+\delta_{\kappa\lambda_1,-\lambda_2}\frac{m_1+m_2}{(2m_1^2+2m_2^2-t)^{1/2}}\right)\right.$$
$$\left.+\delta_{\omega+}\frac{P}{2(m_1m_2)^{1/2}}\left(\delta_{\kappa\lambda_1,\lambda_2}+\delta_{\kappa\lambda_1,-\lambda_2}\frac{m_1-m_2}{(2m_1^2+2m_2^2-t)^{1/2}}\right)\right].\tag{C2}$$

From this we calculate the X's which are presented in Table VI.