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Sums of Direct-Channel Regge-Pole Contributions and Crossing Symmetry*

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Starting with a sequence of parallel rising trajectories in the s channel, we give examples of residue functions for which the Regge-Mandelstam pole contributions can be explicitly summed. The sum has Regge behavior in the s channel as well as the t channel, and satisfies fixed-t dispersion relations and finite-energy sum rules. The residue functions we start with do satisfy the usual analyticity properties and threshold properties, have the Mandelstam symmetry factor, and show the expected exponential behavior for large $|s|$. These results are achieved without fixing either the slope or the intercept of the leading trajectory and without specifying Imα in detail in the low- and intermediate-energy region. We use these examples to clarify some of the problems related to the use of finite-energy sum rules, both as phenomenological relations and as dynamical equations. The way a finite but large number of s-channel resonances can be summed to give Regge behavior in s is explicitly demonstrated. We also indicate how the observation by Schmid on the relation of the ρ-exchange contribution in pion-nucleon charge-exchange scattering to direct-channel resonances can be understood in terms of a direct-channel Regge-Mandelstam analysis. Finally, we point out a general method for generating more examples of other residue functions with the desired properties.

I. INTRODUCTION

FINITE-ENERGY sum rules[1] give a potentially useful relation between Regge residues in the crossed channel and the parameters of low- and intermediate-energy resonances in the direct channel. In a crossing-symmetric problem the resonances also lie on Regge trajectories, and in that case the infinite-energy sum rules, in the resonance approximation, give a set of integral equations for the residue functions. Mandelstam[2] proposed to use such a set of integral equations as the starting point of a dynamical bootstrap based on rising trajectories.

There are several problems that arise in the use of finite-energy sum rules both as kinematic restrictions on the relation between couplings in different channels, or as dynamical equations. It has been suggested that as dynamical equations they have no solution unless we have an infinite number of trajectories that arbitrarily approach each other as $s\rightarrow\infty$.[3] Another problem that comes up is the question of double counting. Namely, can a Regge term which behaves like $s^{\alpha(t)}$ for large s be actually built up by summing Regge resonances in the s channel? Or are the resonances and the Regge term independent to the extent that one could write the amplitude as a sum of $s^{\alpha(t)}$ plus resonance terms in some domain? With rising trajectories the answer to this question has not been clear. Schmid[4] has made a very interesting observation in this connection. He considered the pion-nucleon charge-exchange amplitude which for s larger than 2 BeV^2 is approximated by ρ exchange. He made a partial-wave projection of the simple ρ-exchange term in the s channel and plotted the resulting partial-wave amplitudes in an Argand diagram. The plots described circles, and Schmid identified the tops of these circles with resonances in the (πp) channel. In a plot of l versus resonance energy, Schmid obtained straight lines which approximated the known N^* trajectories.

In this paper we construct explicit examples which clarify some of the questions mentioned above. We start by showing how one can directly sum an infinite number of Regge terms in the s channel in such a way as to get a function that behaves like $s^{\alpha(t)}$ for large s. The set of trajectories we sum over are parallel and they have finite spacing even as $s\rightarrow\infty$. We perform this sum under the restriction that first our residue functions must satisfy the usual analyticity, reality, threshold properties, and must vanish at the Mandelstam symmetry points. Furthermore, we insist that the sum have the correct analyticity properties in s so that the functions we get automatically satisfy finite-energy sum rules and dispersion relations in s.

In a crossing-symmetric case we find that we can get the same leading trajectory in the s and t channels without any restriction on the slope of the rising trajectory. Furthermore, we find a finite region in t in which the residue function of one channel is numerically equal to the value that one would have obtained if he had used the expression for the residue of the direct channel. Since our sums satisfy finite-energy sum rules, then in that region of t in which we have approximate crossing symmetry in β, we achieve a solution of the finite-energy sum rule dynamical equations.

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[1] K. Igi, Phys. Rev. Letters **9**, 76 (1962); R. Dolen, D. Horn, and C. Schmid, *ibid.* **19**, 402 (1967); A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967).

[2] S. Mandelstam, Phys. Rev. **166**, 1539 (1968).

[3] J. E. Mandula and R. C. Slansky, Phys. Rev. Letters **20**, 1402 (1968).

[4] C. Schmid, Phys. Rev. Letters **20**, 689 (1968).

The method in which we carry out the summation is fairly general. It is easy to see how many more examples can be constructed in addition to the ones we study in detail. This leads one to the conclusion that, far from having no solutions, the finite-energy sum rules considered as dynamical equations have many solutions and are not a very restrictive starting point for a dynamical calculation. Without a serious inclusion of unitarity and other conditions, such a dynamical scheme has little information in it.

Although we sum over an infinite number of Regge trajectories, below any finite energy s, we have only a finite number of resonances. In fact, the resonances broaden out and disappear in our model beyond a certain energy, $s>M_\alpha^2$. The examples we have provide a very instructive ansatz to study the use of the narrow-resonance approximation in the finite-energy sum rules. We also explicitly show how in a certain high-energy region both resonances and Regge behavior can coexist with each other. Namely, the same function can be written in two equivalent ways, either as a sum of N resonances where N is proportional to s, or as a Breit-Wigner term times a factor $(s_N)^{\alpha(t)}$.

In Sec. II is shown how one can sum an infinite number of Mandelstam-Regge terms to get the desired properties mentioned above. In Sec. III we discuss crossing symmetry for the residues and the trajectories of both channels. We show how this gives us a bootstrap scheme essentially equivalent to that of the finite-energy sum rule. A solution of this bootstrap is given for a certain range of t values. We use our example in Sec. IV to study the resonance approximation used in the application of the finite-energy sum rules. We also show how both Regge behavior and resonance behavior can coexist at some large energies. We close this section by making a few remarks about imposing unitarity bounds in our model.

In Sec. V we discuss the problem of including signature factors in both each terms of the summation and in the result of the summation. The results of Schmid on charge-exchange scattering are also discussed, and it is shown explicitly how the ρ-exchange term can be built up of s-channel and u-channel resonances. Finally, in Sec. VI the general nature of the summation procedure used in this paper is discussed and it is shown how more examples could be made up.

II. SUMMING DIRECT-CHANNEL REGGE CONTRIBUTIONS

In this section we sum a sequence of direct-channel Regge-pole contributions to construct a class of functions, $F(s,t)$, with the following properties: Each function has Regge asymptotic behavior in both s and t. Each function is analytic in the cut s plane and satisfies a dispersion relation for fixed t. Since the functions are built up by summing s-channel Regge-pole terms, they have s-channel resonances at low and intermediate energies. The functions satisfy finite-energy sum rules and give us an ideal tool for studying some of the problems related to the sum rules mentioned in the Introduction.

We start by considering an infinite set of parallel Regge trajectories, $\alpha_n(s)$, given by

$$\alpha_n(s)\equiv\alpha_0(s)-n,\quad n=0,1,2,\cdots. \tag{1}$$

We take the leading trajectory, $\alpha_0(s)$, to be essentially a linear trajectory satisfying the following dispersion relation[5]:

$$\alpha_0(s)=as+b+\frac{s}{\pi}\int_4^\infty\frac{\mathrm{Im}\alpha_0(s')}{s'(s'-s)}ds', \tag{2}$$

with

$$\mathrm{Im}\alpha_0(s')\geqq 0, \tag{3}$$

and

$$a>0. \tag{4}$$

These last two conditions insure that $\alpha_0(s)$ is a Herglotz function; b is real. We shall later need to impose the condition that for large s, $\mathrm{Im}\alpha(s)\sim(\ln s)$ as $s\to\infty$. However, we can do this without changing the approximate relation $\alpha_0(s)\approx as+b$ at low energies by choosing $\mathrm{Im}\alpha_0$ small at low energies. The asymptotic behavior of $\alpha(s)$ will of course remain linear as $|s|\to\infty$,

$$\alpha_0(s)\to as+O(\ln^2 s). \tag{5}$$

Explicit examples of $\mathrm{Im}\alpha_0$ will be given later.

The residue functions, $\beta_n(s)$, will be chosen so that they satisfy the correct analyticity and threshold properties. They will also turn out to automatically have the Mandelstam symmetry factor.

The first problem we face is what to use for a Regge-pole contribution. Clearly, we cannot start with a sum of the form

$$\sum_n\beta_n(s)P_{\alpha_n}(-z)[2\alpha_n+1][\sin\pi\alpha_n]^{-1}.$$

For large enough n, $\mathrm{Re}\alpha_n<-\frac{1}{2}$, and the contributions of these trajectories will behave like $z^{-\alpha_n-1}$ for large z. There are several ways to write a Regge contribution in such a way that it will have the asymptotic behavior z^{α_n} regardless of whether $\mathrm{Re}\alpha_n<-\frac{1}{2}$ or $\mathrm{Re}\alpha_n>-\frac{1}{2}$. The most natural and most convenient for our present purpose is the way proposed by Mandelstam.[6] In the Mandelstam-Regge representation a pole contribution is proportional to $Q_{-\alpha_n-1}(-z)$, where Q_ν is a Legendre function of the second kind.

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[5] Equations (1) and (2) should be interpreted as an idealization of a situation where

$$\alpha_n(s)=as+b-n+\frac{s}{\pi}\int_4^{-\infty}\mathrm{Im}\alpha_n(s')[s'(s'-s)]^{-1}ds'.$$

This way the stack of resonances of different l which we get at a certain energy will not all have the same total width. Here $\mathrm{Im}\alpha_n$ depends on n and the trajectories are only approximately parallel to the extent that $\mathrm{Im}\alpha_n$ is small at low energies for all n.

[6] S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1962).

We define a function, $F(s,t)$, by the following series:

$$F(s,t)\equiv\sum_{n=0}^{\infty}\frac{\beta_n(s)[2\alpha_n(s)+1]Q_{-\alpha_n(s)-1}(-z)}{\cos\pi\alpha_n(s)}. \quad (6)$$

At this point we recall one property of the Mandelstam-Regge form which is crucial to our discussion. Namely, although we sum in (6) over an infinite number of Regge poles, below any finite energy s, we have only a finite number of resonances. For suppose $\mathrm{Re}\alpha_0(s_N)=N$, N an integer, and $\mathrm{Im}\alpha(s_N)$ is small. Then only the first $(N+1)$ terms in (6) give a resonance. The terms with $n\geqq N+1$ have no resonance poles. Using the identity, for the first $N+1$ terms,

$$-\frac{Q_{-\alpha_n-1}(z)}{\pi\cos\pi\alpha_n}=\frac{P_{\alpha_n}(z)}{\sin\pi\alpha_n}-\frac{Q_{\alpha_n}(z)}{\pi\cos\pi\alpha_n}, \quad (7)$$

we see that at $s\approx s_N$ we have $N+1$ resonances with angular momentum N, $N-1,\cdots,0$.

The series we have to sum is of the form

$$F(s,t)=\sum_{n=0}^{\infty}\gamma_n(s)Q_{n-\alpha_0-1}(+z). \quad (8)$$

We have collected all the extraneous factors in γ_n and set

$$\gamma_n(s)=[2\alpha_n(s)+1]\beta_n(s)e^{i\pi\alpha_0(s)}/\cos\pi\alpha_0(s). \quad (9)$$

To perform the sum in (7) it turns out to be convenient to use the following integral representation of the Q functions[7]:

$$Q_\nu(z)=(\tfrac{1}{2}\pi)^{1/2}\int_0^\infty e^{-yz}I_{\nu+1/2}(y)y^{-1/2}dy,$$
$$\mathrm{Re}z>1,\ \mathrm{Re}\nu>-1. \quad (10)$$

We carry out our summation in the region $t<0$ and $s<4$, where we can always find a domain such that

$$z=1+2t/(s-4)>1,$$

and

$$\mathrm{Re}\alpha_0(s)<-\tfrac{1}{2}. \quad (11)$$

Later we shall analytically continue the final answer in s to all values of s in the cut plane.

From (11) it follows that $\mathrm{Re}[n-\alpha_0(s)-1]>-\frac{1}{2}$ for $n=0,1,2,\cdots$. Hence we can use the representation (10) and get

$$F(s,t)=(\tfrac{1}{2}\pi)^{1/2}\int_0^\infty e^{-yz}y^{-1/2}H(s,y)dy. \quad (12)$$

The function $H(s,y)$ is given by the series

$$H(s,y)=\sum_{n=0}^{\infty}\gamma_n(s)I_{n-\alpha_0-1/2}(y). \quad (13)$$

[7] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 2, p. 56.

This series is a special case of a Neumann series of Bessel functions. The simplest example of such a series is the expansion of an exponential in terms of Bessel functions,

$$y^\nu e^{\eta y}=2^\nu\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)C_n{}^\nu(\eta)I_{\nu+n}(y), \quad (14)$$

where $C_n{}^\nu(\eta)$ are Gegenbauer polynomials in η. We set $\nu=-\alpha_0-\frac{1}{2}$ and choose $\gamma_n(s)$ as

$$\gamma_n(s)=[2^{-\alpha_0-1/2}\Gamma(-\alpha_0-\tfrac{1}{2})$$
$$\times(n-\alpha_0-\tfrac{1}{2})C_n{}^{-\alpha_0-1/2}(\eta)]f(s). \quad (15)$$

The two functions $f(s)$ and $\eta(s)$ will be specified below. This last equation, however, determines the dependence of the residues $\beta_n(s)$ on n. With (15) we obtain

$$H(s,y)=y^{-\alpha_0-1/2}e^{\eta y}f(s). \quad (16)$$

After substituting this result in (12) and carrying out the integration we find

$$F(s,t)=f(s)\Gamma(-\alpha_0)[z-\eta(s)]^{\alpha_0(s)}(\tfrac{1}{2}\pi)^{1/2}. \quad (17)$$

We note that the Γ function will have all the resonance poles on the second sheet in the s plane. It is also relevant to remark that the result we have in (17) looks like a Regge-type contribution of a simple Regge-type pole coming from expanding an amplitude $A(s,z)$ in power series of $(z-\eta)$, i.e., $A=\sum_\nu a_\nu(s)(z-\eta)^\nu$. If $a_\nu(s)$ has a pole at $\nu=\alpha_0(s)$, the contribution of that single pole in the ν plane to the Watson-Sommerfeld formula would be similar to (17).

At this stage we introduce the following transformation defining a new function $\lambda(s)$:

$$\eta(s)\equiv1+2/r\lambda(s)(s-4)-2c/(s-4), \quad (18)$$

where r and c are real constants and $r>0$. Written in terms of $\lambda(s)$, $F(s,t)$ is given by

$$F(s,t)=(-1)^{\alpha_0(s)}[\tfrac{1}{2}r(s-4)\lambda(s)]^{-\alpha_0}\Gamma(-\alpha_0)$$
$$\times(\tfrac{1}{2}\pi)^{1/2}f(s)[1-r(t+c)\lambda(s)]^{\alpha_0}. \quad (19)$$

Both $f(s)$ and $\lambda(s)$ are still to be specified. To simplify (19) we write

$$f(s)\equiv g(s)\left(\frac{2}{\pi}\right)^{1/2}(-1)^{\alpha_0(s)}[\tfrac{1}{2}r(s-4)\lambda(s)]^{\alpha_0}. \quad (20)$$

Finally, we get

$$F(s,t)=g(s)\Gamma(-\alpha_0(s))[1-r(t+c)\lambda(s)]^{\alpha_0(s)}. \quad (21)$$

We want now to impose sufficient conditions on $\lambda(s)$ and $g(s)$ to insure that $F(s,t)$ will be analytic in the cut s plane and will for large s have Regge asymptotic behavior. For large t the Regge behavior is obvious in (21).

The following conditions turn out to be sufficient: The function $\lambda(s)$ should have the representation,

$$\lambda(s)=\frac{1}{\pi}\int_0^\infty\frac{\sigma(s')}{s'-s}ds',\quad \sigma(s')\geqq0. \quad (22)$$

In other words, we choose $\lambda(s)$ to be an analytic function in the cut s plane, and furthermore the positivity of $\sigma(s')$ guarantees that $\lambda(s)$ is a Herglotz function. The weight function $\sigma(s')$ is otherwise arbitrary except for its asymptotic behavior, which we take such that

$$\lambda(s) \underset{|s|\to\infty}{\longrightarrow} \frac{\ln(4-s)}{(4-s)}+O(1/s^2). \qquad (23)$$

The two properties (22) and (23) are all we need for $\lambda(s)$. From (22) we see that the bracket in (21) cannot vanish for any complex s and real t and therefore we have no complex branch points in s. Moreover, for values of t such that

$$t<\frac{1}{r\lambda(4)}-c, \qquad (24)$$

the bracket in (21) will not vanish in the region $-\infty<s<4$. In this region $\lambda(s)$ is real, positive, and decreases monotonically as s approaches $-\infty$. Except for $g(s)$, which is still to be specified, $F(s,t)$ then is analytic in the cut s plane with a right-hand cut only, for all real t satisfying the inequality (24).

For fixed real s and for $-\infty<s<4$, $F(s,t)$ is analytic in the cut t plane with a cut starting at $t=[r\lambda(s)]^{-1}-c$. If we take $[r\lambda(4)]^{-1}>c$, then this branch point always occurs on the positive t axis. For large values of $(-s)$ this branch point moves out farther and farther on the positive t axis. This unphysical nature of the analyticity in t of our sums is a demonstration of the fact pointed out by Mandelstam in Ref. 6, that one cannot completely ignore the background term in the Mandelstam-Regge representation even if the background integral is pushed back to a line $\mathrm{Re}\,l\to-\infty$.

As for $g(s)$, we define it by an expression which exactly cancels the asymptotic behavior of $\Gamma(-\alpha_0(s))$ in any direction in the s plane for which $|\arg(-\alpha_0)|<\pi$. We write

$$g(s)=A/(2\pi)^{1/2}\exp(-\alpha_0)\exp[(\alpha_0+\tfrac{1}{2})\ln(-\alpha_0)]. \qquad (25)$$

We choose the phase of (α_0) such that $\arg(-\alpha_0)=0$ for $s<0$ and $\arg(-\alpha_0)\to-\pi$ as $s\to+\infty$ above the cut and $\arg(-\alpha_0)\to+\pi$ below the cut.

To simplify the discussion we limit ourselves at this stage to the case where there are no bound states or ghosts and take $\alpha_0(4)<0$. In that case $\alpha_0(s)$ does not have any zeros on the physical sheet in the s plane. This makes $g(s)$ analytic in the cut s plane with the cut running from $s=4$ to $s=\infty$. The branch point of $\ln(-\alpha_0)$ when $\alpha_0=0$ is on the second sheet of the s plane.

It is easy to verify now that with (23) and (25) the asymptotic behavior of $F(s,t)$ is given by

$$F(s,t)=A(4-s)^{art+arc}[1+O(\ln^2 s/s)], \qquad \epsilon<\arg s<2\pi-\epsilon. \qquad (26)$$

The product $\Gamma(-\alpha_0(s))g(s)$ approaches the constant A by construction and the Regge behavior of the bracket term in (21) follows from (23). The asymptotic behavior along the cut needs more careful handling. However, if $\mathrm{Im}\alpha(s)$ grows as $s\to+\infty$ above the cut, then we have the same answer as in (26). One only has to rewrite $g(s)\Gamma(-\alpha_0)$ as

$$g(s)\Gamma(-\alpha_0(s))=g(s)\pi/-\sin\pi\alpha_0\,\Gamma(1+\alpha_0(s)). \qquad (27)$$

As $s\to+\infty$, $\sin\pi\alpha_0\sim(i/2)\exp[-i\pi\alpha_0(s)]$, since the other exponential in the sine function is damped by a factor $\exp[-\pi\,\mathrm{Im}\alpha_0(s)]$. If $\mathrm{Im}\alpha_0$ grows asymptotically like $\ln s$, then $g(s)\Gamma(-\alpha_0)\sim A(1+O(1/s))$. Therefore, with this condition on $\mathrm{Im}\alpha$, (26) holds in all directions of the physical sheet of the s plane.

The function $F(s,t)$ has all the desired properties we are looking for. We are left with the task of checking whether the form we have picked for the residue functions also satisfies all the analyticity and threshold properties. On substituting (15) into (9) we obtain

$$\beta_n(s)=g(s)[\tfrac{1}{4}r(s-4)\lambda(s)]^{\alpha_0(s)} \times\frac{C_n^{-\alpha_0-1/2}(\eta(s))}{\Gamma(\alpha_0(s)+\frac{3}{2})}(\tfrac{1}{2}\pi^{1/2}). \qquad (28)$$

Here g is given by (25), and $\eta(s)$ is related to $\lambda(s)$ by (18). The Gegenbauer polynomial is a polynomial of degree n in $(\frac{1}{2}-\frac{1}{2}\eta)$. As $(s-4)\to0$, $C_n^{\nu}(\eta(s))\sim(s-4)^{-n}$ and it is easy to check that near threshold

$$\beta_n(s)\sim[(s-4)/2]^{\alpha_n(s)}, \qquad (29)$$

which is the correct threshold behavior. Furthermore, it is clear from (28) that the reduced residue functions $\beta_n(s)/[\frac{1}{2}(s-4)]^{\alpha_0(s)}$ are real analytic functions in s with only a right-hand cut. The factor $\Gamma(\alpha_0(s)+\frac{3}{2})$ gives zeros in $\beta_n(s)$ at negative half-integer values of $\alpha_0(s)$ as required by the Mandelstam symmetry.

As expected, the residue functions have exponential behavior for large s; however, the behavior is different from that suggested by Teplitz and Jones.[8] The leading trajectory has a residue function $\beta_0(s)$,

$$\beta_0(s)=g(s)[\tfrac{1}{4}r(s-4)\lambda(s)]^{\alpha_0(s)} \times[1/\Gamma(\alpha_0(s)+\tfrac{3}{2})]\tfrac{1}{2}\pi^{1/2}. \qquad (30)$$

For large positive s this grows like $A\exp[as\log\log s]$. We shall see later that this does not lead to a contradiction with unitarity as long as $\mathrm{Im}\alpha_0(s)$ starts to grow for large s. We also show in Sec. III how our example can be slightly modified so that the elastic widths of the resonances decrease with s.

In closing, we point out that many other examples can be generated from (21) by differentiation with respect to the parameter r. The function F is a function of s,t and the parameters r and c. For example, we can define

$$F^{(1)}(s,t)\equiv(-1)/r\lambda(s)d/dr[F(s,t;r,c)]. \qquad (31)$$

[8] C. E. Jones and V. L. Teplitz, Phys. Rev. Letters **19**, 135 (1967).

From (21) this gives

$$F^{(1)}(s,t)=(\alpha_0(s)/a)g(s)\Gamma(-\alpha_0(s))(t+c)\times[1-r(t+c)\lambda(s)]^{\alpha_0(s)-1}. \quad (32)$$

The asymptotic behavior is now given by

$$F^{(1)}(s,t)=A(t+c)(-s)^{art+arc+1}[1+O(\ln^2 s/s)]. \quad (33)$$

The residue functions corresponding to $F^{(1)}$ are calculated by differentiating (28) with respect to r. Given any polynomial in t, $P(t)$, we can, by taking a linear combination of different $F^{(n)}(s,t;r,c)$ for different values of c, construct a function $A(s,t)$ which has the asymptotic behavior

$$A(s,t)=AP(t)(-s)^{at+b}[1+O(\ln^2 s/s)]. \quad (34)$$

The functions $F^{(n)}(s,t;r,c)$ are defined by

$$F^{(n)}(s,t;r,c)=(-1/ra\lambda(s)(d/dr))^n F(s,t;c,r). \quad (35)$$

III. CROSSING-SYMMETRIC TRAJECTORIES AND RESIDUES AND ALMOST EXACT SOLUTIONS TO THE FINITE-ENERGY SUM RULES

The function $F(s,t)$ given in (21) has Regge asymptotic behavior in both the s and the t channels. In a crossing-symmetric problem one would require that at least the leading s-channel and t-channel trajectories are the same. In our simple example this can be achieved by setting

$$r=1 \quad \text{and} \quad ac=b. \quad (36)$$

With this choice the asymptotic behavior of F is

$$F(s,t)\cong A(-s)^{at+b}. \quad (37)$$

As long as $\mathrm{Im}\,\alpha_0(s)$ is small for low and intermediate values of s, we effectively have the same leading trajectory in both channels, $\alpha_0(s)\approx as+b$ and $\alpha_0(t)\simeq at+b$.

The more difficult problem is to have the same residues in both channels. We want to compare the form of the residue of the t-channel Regge pole that we can infer from (37) with the form of the s-channel residue function. From (37) and the Regge asymptotic formula we get

$$\beta_0(t)=-\frac{A}{\pi}\frac{\sin\pi\alpha_0(t)}{[2\alpha_0(t)+1]}\frac{\Gamma(1+\alpha_0(t))}{\Gamma(\frac{1}{2}+\alpha_0(t))}\times\left(\frac{t-4}{2}\right)^{\alpha_0(t)}\frac{\pi^{1/2}}{2^{\alpha_0(t)}}. \quad (38)$$

This result is obtained by comparing (37) with the usual Regge asymptotic formula. Note that so far we are ignoring signature factors and working with a pure two-channel problem. In the recent literature the Γ functions in (38) are usually absorbed into the definition of β_0. We prefer to exhibit them. Our β_n is the actual residue of the partial-wave amplitude at $l=\alpha_n(s)$. The fact that the usual factor $[\sin\pi\alpha(t)]^{-1}$ does not appear on the right-hand side of (37) makes $\beta_0(t)$ proportional to $\sin\pi\alpha_0(t)$. However, since we are mainly interested in the region $t<0$, this does not lead to serious problems. The function $F(s,t)$ does not have t-channel resonances.

To have the residues in both channels have the same mathematical form would require $\beta_0(t)$ to be given by (30),

$$\beta_0(t)=g(t)[\tfrac{1}{4}(t-4)\lambda(t)]^{\alpha_0(t)}\times 1/\Gamma(\alpha_0(t)+\tfrac{3}{2})\times\tfrac{1}{2}\pi^{1/2}. \quad (30')$$

An exact crossing-symmetric bootstrap condition would require (38) and (30′) to be equal. Equating (30′) and (38), we get the bootstrap condition

$$A/\Gamma[-\alpha_0(t)]=g(t)[\lambda(t)]^{\alpha_0(t)}. \quad (39)$$

It is immediately apparent that (39) *cannot* be satisfied for all t. For large t, $\Gamma(-\alpha_0(t))g(t)\to A$, but $\lambda(t)\sim[\ln(-t)]/(-t)$. However, we can always seek numerical and approximate solutions of (39) for some finite range of t. Only the asymptotic behavior of $\lambda(t)$ has been specified so far and we can fix its value in a finite t domain to approximately satisfy (39).

To give an example of a solution of (39) let us concentrate on the region $t<0$. More specifically, let us look at the region in t for which $\alpha_0(t)\lesssim -1$, i.e.,

$$-t_0<t<(-1-b)/a, \quad (40)$$

where $t_0\ll M_\lambda^2$ and M_λ^2 is the mass value as $\lambda(t)$ becomes asymptotic to $\ln(4-t)/(-t)$. In the domain given in (40) $[-\alpha_0(t)]\geq 1$. One recalls the remarkable fact that $\Gamma(x)$ is extremely well approximated by the first two terms in the Stirling expansion,

$$\Gamma(x)\cong e^{-x}e^{(x-1/2)\ln x}(1+1/12x)(2\pi)^{1/2}. \quad (41)$$

The accuracy of this approximation is less than 1% even at $x=1$. Substituting (41) in (39) and using the definition of g, (25), we get

$$[\lambda(t)]^{-\alpha_0(t)}\cong(1-1/12\alpha_0(t)),\ -t_0<t<(-1-b)/a. \quad (42)$$

This gives the approximate numerical solution for $\lambda(t)$,

$$\lambda(t)\cong 1+1/12\alpha_0^2(t);\ -t_0<t<(-1-b)/a. \quad (43)$$

One can arrange for the validity of (43) to a very good accuracy without affecting the asymptotic behavior of $\lambda(t)$ for $|t|>M_\lambda^2$. The expression on the right-hand side of (43) is positive and decreases monotonically as $(-t)$ increases.

For $t<0$, $F(s,t)$ satisfies a finite-energy sum rule. As long as t lies in the domain given in (40) we have also a "solution" to the finite-energy sum rule, when it is considered as a dynamical equation for $\beta(t)$. The accuracy of this solution depends on how closely we match $\lambda(t)$ to the right-hand side of (43).

The approximate relation (43) still does not completely determine $\lambda(t)$ for values of $t>4$. Furthermore, the slope a, and the intercept b, of the leading trajectory are still not fixed as is the value of $\text{Im}\alpha_0(s)$ at low energies. Even though we have only imposed crossing at negative values of t, we must conclude that the finite-energy sum rules are not very restrictive as a starting point for a dynamical scheme. Without a serious introduction of unitarity it is possible to write many functions which contain most of the crossing information in the finite-energy sum rules.

For $t>4$, we can, of course, try to match both sides of (39) numerically. However, the physical meaning of such a matching would be obscure since the function $F(s,t)$ does not have any t-channel resonances.

IV. RESONANCE APPROXIMATION TO THE FINITE-ENERGY SUM RULES

In this section we discuss some of the features of the class of examples constructed in Sec. II which clarify some of the problems related to the use of finite-energy sum rules. At the end of the section we indicate how at least the simplest unitarity bounds are not violated by our examples.

For large enough s_M, the function $F(s,t)$ we have constructed satisfies the sum rule

$$\int_4^{s_M} \text{Im}F(s',t)ds' \approx A \sin\pi\alpha_0(t)\frac{s_M^{\alpha_0(t)+1}}{[\alpha_0(t)+1]}. \quad (44)$$

We are considering only the region $t<0$ and treating for simplicity the case $\alpha_0(0)<0$.

There are two questions related to the application of a sum rule like (44). First, what determines what we mean by "large enough s_M"? Second, what happens when we replace $\text{Im}F$ on the left-hand side by a sequence of resonance contributions?

It is clear from the construction of our examples that we have two asymptotic regions intrinsic in the construction. One is the asymptotic region for $\text{Im}\alpha_0(s)$ and the other is the asymptotic region for $\lambda(s)$. In other words, there could be two independent scale factors M_α^2 and M_λ^2 which determine the asymptotic region for $\text{Im}\alpha_0$ and $\lambda(s)$, respectively. We have large squared masses M_α^2 and M_λ^2 such that

$$\begin{aligned}\text{Im}\alpha_0(s) &\approx \ln s+O(M_\alpha^2/s), & s>M_\alpha^2;\\ \lambda(s)&\approx \ln(4-s)/-s+O(M_\lambda^4/s^2), & s>M_\lambda^2.\end{aligned} \quad (45)$$

For example, we could think of a concrete ansatz for $\text{Im}\alpha_0(s)$, to facilitate the discussion, and write

$$\text{Im}\alpha_0(s)=\epsilon/p\ \ln[1+((s-4)/M_\alpha^2)^p],\ p\geq 1. \quad (46)$$

As long as $(s-4)<M_\alpha^2$, $\text{Im}\alpha_0(s)$ is small, and we have a set of narrow resonances whenever $\text{Re}\alpha(s_N)=N$ and $s_N\ll M_\alpha^2$. For $(s-4)>M_\alpha^2$, $\text{Im}\alpha_0(s)$ begins to grow like $\ln s$. The resonances become much broader than the spacing between them and effectively disappear. Thus the region $s<M_\alpha^2$ is the resonance region, while $s>M_\alpha^2$ is the no-resonance region. For large p in our ansatz the transition between these two regions occurs quite sharply and the end of the resonance region will be a well-defined energy. However, it is more likely that in the real world one moves more gradually from one regime to the other.

Regge behavior begins where s is larger than both M_α^2 and M_λ^2. The sum rule (44) is exactly satisfied as long as $s_M>M_\alpha^2$ and $s_M>M_\lambda^2$. In actual practice, however, one approximates the left-hand side of the sum rule by resonance contributions and usually the background is ignored. If $M_\lambda^2\gg M_\alpha^2$, even though both are large, then the resonance approximation is very bad. For if we take $s_M\approx M_\alpha^2$, then (44) is not valid because Regge behavior has not set in yet. On the other hand, if we set $s_M\approx M_\lambda^2$, then in the region $M_\alpha^2<s<M_\lambda^2$ we have no resonances and setting the integrand equal to zero in that region would be a disastrous approximation.

The other possibility, $M_\alpha^2\gtrsim M_\lambda^2$, is more interesting and does not lead to difficulties in the resonance approximation. In that case one must choose $s_M\approx M_\alpha^2$ in (44). For values of s in the interval $M_\lambda^2<s<M_\alpha^2$ we have mixed resonance behavior and Regge behavior. For example, suppose $\text{Re}\alpha_0(s_N)=N$, where N is a large integer, and $M_\lambda^2<s_N<M_\alpha^2$. Then for $s\approx s_N$ we can write,

$$\begin{aligned}F(s,t)&=[g(s_N)/\Gamma(1+N)][(-\pi)/\sin\pi\alpha_0(s_N)]\\&\qquad\times[1-(t+c)\lambda(s_N)]^N\\&\approx[g(s_N)/a\Gamma(1+N)]\{(-1)^N/[(s-s_N)+i(\epsilon_N/a)]\}\\&\qquad\times[1-(t+c)\lambda(s_N)]^N,\end{aligned} \quad (47)$$

where $\epsilon_N=\text{Im}\alpha_0(s_N)$ and is assumed to be small. The last bracket in (47) can be written in two ways. Either we can expand it in terms of $N+1$ Legendre polynomials $P_N(z)$, $P_{N-1}(z),\cdots,P_0(z)$, and recover the original resonances we started with. Or since s_N is large enough for $\lambda(s_N)$ to assume its asymptotic value, we get

$$[1-(t+c)\lambda(s_N)]^N\sim s_N^{at+b}, \quad (48)$$

where $N\approx as_N+b$. At this stage we see how this class of functions provide an explicit counterexample to some of the assertions of Ref. 3.[9] There is no difficulty in using the narrow resonance approximation in (44) as long as $s_M\approx M_\alpha^2$.

In summary, we can say that for the resonance approximation to the finite-energy sum rules to be valid it is necessary that first one knows at what energy Regge behavior sets in and that this energy be roughly the same as that at which resonance behavior

[9] Other counterarguments to some of the statements of Ref. 3 were given independently by C. Goebel [Phys. Rev. Letters 21, 383 (1968)].

begins to disappear. If in the real world resonances broaden out and disappear long before Regge behavior sets in, then the resonance approximation is obviously useless.

We close this section by making a few remarks about unitarity. For $s>M_\alpha^2$ and M_λ^2, $F(s,t)$ satisfies the unitarity bounds for $t<0$, provided the constant A is not large. For $s<M_\alpha^2$, we can also choose A such that at least near each resonance we do not violate unitarity. The constant A can always be chosen so that for all $s_N<M_\alpha^2$

$$|\beta_n(s_N)[2\alpha_n(s_N)+1]/\epsilon_N|<1\,, \tag{49}$$

where $\mathrm{Re}\alpha_0(s_N)=N$ and $\mathrm{Im}\alpha_0(s_N)=\epsilon_N$.

One feature of our functions $\beta_n(s)$ is rather unphysical. In the region $M_\lambda^2<s<M_\alpha^2$, $\beta_n(s)$ grows like $\exp[s\ln\ln s]$. This means that the elastic widths of our resonances grow with energy in this region. To correct this deficiency without changing any of the essential features of the model we can replace the constant A in (25) by a function $A(s)$ which tends to a constant as $|s|\to\infty$ and decreases for $s<M_\alpha^2$. For example, take

$$A(s)=A'(-\ln[M_\alpha^2-(s-4)]/\lambda(s)h(s))^{\alpha_0(s)}. \tag{50}$$

We take $h(s)$ to be a Herglotz function defined by a representation similar to (22) but with asymptotic behavior $h(s)\to s$, as $|s|\to\infty$. For large $|s|$, $|s|\gg M_\alpha^2$, the function $A(s)$ approaches a constant. The function F will now have two new branch points at $s\approx M_\alpha^2$, but nothing else is changed. With a proper choice of $h(s)$, the elastic widths will decrease in the region $M_\lambda^2<s<M_\alpha^2$. For $s\gg M_\alpha^2$, $\beta_n(s)$ will still grow like $\exp[as\ln\ln s]$ as $s\to+\infty$, but in this region the resonance picture is no longer applicable.

V. SIGNATURE AND THE CHARGE-EXCHANGE AMPLITUDE

So far we have been working with a purely two-channel set of examples and ignoring signature factors. There are two questions regarding signature to consider. The first is to include signature factors in the terms of the original series (6). The second is to get signature factors out on the right-hand side in (26).

As to the first problem, we take the set $\alpha_n(s)$ to have alternating signature in analogy to the daughter sequence.[10] We set the signature factor τ_n for each α_n to be

$$\tau_n=(-1)^n. \tag{51}$$

The leading trajectory we have taken to be even, but obviously a similar summation could be done with an odd trajectory. Instead of the series (6), we have

$$A(s,t)=\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{\beta_n(s)[2\alpha_n(s)+1]}{\cos\pi\alpha_n(s)}\times[Q_{-\alpha_n-1}(-z)+(-1)^nQ_{-\alpha_n-1}(z)]. \tag{52}$$

This reduces to the series

$$A(s,t)=\tfrac{1}{2}(1+e^{-i\pi\alpha_0})\sum_{n=0}^{\infty}\gamma_n(s)Q_{n-\alpha_0-1}(z). \tag{53}$$

The γ_n's here are the same as in (9). We carry out the summation in the same way and get

$$A(s,t)=G(s)[\Gamma(-\alpha_0(s))/\Gamma(-\tfrac{1}{2}\alpha_0-\tfrac{1}{2})]\times[1-r(t+c)\lambda(s)]^{\alpha_0(s)}, \tag{54}$$

where

$$G(s)=2A\exp[-\tfrac{1}{2}\alpha_0(s)(1-\ln 2)]\times\exp[(\tfrac{1}{2}\alpha_0-\tfrac{1}{2})\ln(-\alpha_0(s))]. \tag{55}$$

The function f is now different than in (20) and is given by

$$f(s)=(-1/\pi)G(s)(2/\pi)^{1/2}(-1)^{\alpha_0(s)}[\tfrac{1}{2}r(s-4)\lambda(s)]^{\alpha_0(s)}\times\exp[+\tfrac{1}{2}i\pi\alpha_0]\Gamma(\tfrac{1}{2}\alpha_0+\tfrac{3}{2}). \tag{56}$$

The residues are given by (15) and (9) with f as in (56). The new Γ function in (56) makes the residue functions have poles when $\alpha_0(s)=-(2j+1)$, $j=0,1,2,\cdots$. However, these are values for which the signature factor $[1+\exp(-i\pi\alpha_0)]$ vanishes and no poles appear in $A(s,t)$. The Mandelstam symmetry factor still remains in $\beta_n(s)$ as before, and the threshold behavior is unchanged. However, one feature is now different. The factor $\exp(+\frac{1}{2}i\pi\alpha_0)$ in (56) gives an extra phase factor in $\beta_n(s)$. The ratio $\beta_n(-i)^{\alpha_0}/[\frac{1}{2}(s-4)]^{\alpha_n}$ is now real for $s<4$ instead of the reduced residue, $\beta_n/[\frac{1}{2}(s-4)]^{\alpha_n}$. In view of the remarks in Sec. VI, it should be possible to construct other examples where this unphysical phase is not present.

The next question to consider is how to get signature factors in the resulting sum. We propose to discuss this in the context of the problem studied in Ref. 4, namely, charge-exchange scattering. We consider charge-exchange scattering of pions on some spin-zero, $I=\frac{1}{2}$ target. The standard t-channel Regge analysis for the charge-exchange amplitude gives us the contribution of the ρ trajectory for large s:

$$M^{\mathrm{CEX}}(s,t)\approx C\{(-s)^{\alpha_\rho(t)}-(-u)^{\alpha_\rho(t)}/\Gamma(\alpha_\rho(t))\sin\pi\alpha_\rho(t)\}. \tag{57}$$

This is essentially the same expression as that used by Schmid in Ref. 4. For pion-nucleon charge exchange it fits the data fairly well. In the region of t in which the fit is made, the factor $[\Gamma(\alpha(t))\sin\pi\alpha(t)]^{-1}$ can be closely approximated by a polynomial in t:

$$M^{\mathrm{CEX}}(s,t)\approx CP(t)[(-s)^{\alpha_\rho(t)}-(-u)^{\alpha_\rho(t)}]. \tag{58}$$

[10] D. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967).

The charge-exchange amplitude can be written

$$M^{\text{CEX}}(s,t)=\sqrt{2}[M(\pi^+)-M(\pi^-)], \tag{59}$$

where $M(\pi^\pm)$ is the amplitude for $\pi^\pm$ scattering on the target. If we write $M(\pi^+)$ as a function of s and t, $M(s,t)$, then $M(\pi^-)$ is given by the same function of u and t, $M(u,t)$. Clearly, if we approximate $M(\pi^+)$ by a sum of s-channel Regge contributions, $A(s,t)$, then $M(\pi^-)$ is given by a sum of u-channel Regge contributions, $A(u,t)$:

$$M^{\text{CEX}}(s,t)\approx\sqrt{2}[A(s,t)-A(u,t)], \tag{60}$$

where A is defined by a series like (52). An $I=1$ t-channel amplitude must have odd $s\leftrightarrow u$ symmetry.

As we pointed out in (34), it is possible to construct $A(s,t)$ such that asymptotically

$$A(s,t)\approx CP(t)(-s)^{at+b}. \tag{61}$$

Thus we obtain the same answer as in (59) by summing s- and u-channel Regge contributions and taking their difference.

A possible way to understand the results of Ref. 4 is as follows. One starts by carrying out a Regge-Mandelstam analysis for π^+p and π^-p scattering in the direct s channel. One writes a Regge-Mandelstam formula for each of these amplitudes and pushes the background integral to a line $\text{Re}l\rightarrow-\infty$. On taking the difference between the two amplitudes, the difference of the backgrounds, though not negligible for all s and t, is probably negligible in the region considered by Schmid. The sum over the Regge-Mandelstam pole contributions could then give for large s an expression almost identical to that obtained from ρ exchange.

In the present examples there does not seem to be any reason for the slopes of the ρ trajectory and direct-channel πN trajectories to be the same. If they are, it must be for some deeper physical reason unrelated to the mechanism we are discussing here.

VI. REMARKS ABOUT THE METHOD OF SUMMATION

One might raise two questions about the summation performed in Sec. II. First, one might question whether the particle spectrum is rich enough to accommodate an infinite set of almost parallel trajectories $\alpha_n(s)\approx\alpha_0(s)-n$. At present there is no good answer to this question. However, one can say that such a rich spectrum cannot be ruled out by the present experimental data. In this connection one should mention the recent work of Gell-Mann and Zweig based on the quark model, which proposes such a spectrum.[11] On the theoretical side, there is also a recent work of Toller which suggests that the daughter sequence is parallel to the parent trajectory.[12]

The second objection one can raise to the examples constructed in this paper is the fact that they seem to depend on choosing a specific n dependence for the residues, $\beta_n(s)$, namely, having β_n proportional to a Gegenbauer polynomial. If this was the only choice that would give Regge behavior, then our example could only be of mathematical interest as a counterexample, say, to the statements in Ref. 3, but of little positive physical significance. This is by far not the case.

The advantage of using the representation (10) is that by expressing Q_ν in terms of modified Bessel functions, $I_{\nu+1/2}$, we converted our summation to a standard series.

The series (13) is a Neumann series. Any function $h(y)$ which has the power series expansion

$$h(y)=\sum_{j=0}^{\infty} a_j y^j \tag{62}$$

can also be written as a series of Bessel functions of the form

$$h(y)=y^{-\nu}\sum_{n=0}^{\infty} b_n J_{n+\nu}(y). \tag{63}$$

The new coefficients b_n are related to a_j by

$$b_n=2^{\nu+n}(\nu+n)\sum_{l=0}^{\leqq 1/2n} 2^{-2l}\frac{\Gamma(\nu+n-l)}{l!}a_{n-2l}. \tag{64}$$

Thus one can construct more examples by choosing an $H(y,s)=h(y,s)y^{-\alpha_0-1/2}$ such that the Laplace transform, (12), of $H(y,s)y^{-1/2}$ is a function $F(s,t)$ which has Regge behavior for large s. The residue functions of this new example will be determined by the coefficients $a_j(s)$ of the power series expansion of $h(y,s)$ through (64). If these new residue functions do not have any unphysical properties, then we have another useful example.

Note added in proof. After the completion of this work we learned of an interesting model proposed by Veneziano for the $\pi\pi\rightarrow\pi\omega$ bootstrap which exhibits several of the features discussed in this paper. [See G. Veneziano, Nuovo Cimento **57**, 190 (1968).] However, the hope that this model does give a unique solution to an F. E. S. R. bootstrap has turned out to be over-optimistic.

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[11] M. Gell-Mann and G. Zweig (private communication) (to be published). See also H. R. Rubenstein, A. Schwimmer, G. Veneziano, and M. Virasoro, Phys. Rev. Letters **21**, 491 (1968).

[12] M. Toller, Report to the Goteborg Conference, 1968 (unpublished). We thank Dr. G. Zweig for this information. See also R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **164**, 1981 (1967).