Currents as Dynamical Variables*

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A dynamical theory of currents is presented. For each current there are a complete set of field equations and a complete set of equal-time commutators. These are related by expressing the energy-momentum tensor solely in terms of the currents and not of derivatives of the currents. The requirements of Lorentz covariance place severe restrictions on the commutators. Several models with and without internal symmetry are discussed.

I. INTRODUCTION

In recent years the algebra of current and charge densities has become a major tool in elementary-particle physics. Although many of the results which have been obtained are independent of any specific dynamical model for the currents, it is clear that further dynamical assumptions are required before one can say that one has a complete dynamical theory, a theory capable, in principle, of predicting everything. For example, to complete the picture one might provide a construction of the currents in terms of a set of canonical fields.[1] This is old-fashioned Lagrangian field theory and suffers (apart from calculational difficulties) from the fact that we do not know *a priori* which are the fundamental fields and which couplings are to be included in the Lagrangian.

An alternative which we should like to pursue in detail in this paper is to consider the currents themselves as fundamental (though not necessarily canonical) fields.[2] If in addition to the set of equal-time commutators between each pair of currents (which set constitutes the current algebra) we also specify a first-order differential equation of motion in time for each current, we obtain, in principle, a complete dynamical theory.

In the course of constructing such a theory we might be forced to introduce new operators which have no direct physical interpretation in terms of currents. In order to proceed, it is clear that we will have to treat these new operators on the same footing as the others.

The equations of motion and commutation relations are not independent of each other, but are connected via the Hamiltonian P^0 through the Heisenberg equation

$$\partial\Omega(\mathbf{x},t)/\partial t=-i[\Omega(\mathbf{x},t),P^0(t)].$$

If we construct the Hamiltonian operator at a given time as a function of current operators at that time, then, given the Hamiltonian, the equations of motion are determined by the equal-time commutators. If we make the assumption that the Hamiltonian is the spatial integral of a local energy-density operator, and make the corresponding assumption for the momentum operator P^k, we can write

$$P^\mu(t)=\int d^3x\, T^{0\mu}(\mathbf{x},t).$$

Part of our task, then, will be to specify the energy-momentum tensor $T^{\mu\nu}$ in terms of the currents.[3]

The equations of motion, the commutator algebra, and $T^{\mu\nu}$ must be consistent not only with each other but also with the requirements of Lorentz invariance and, to start with, parity, time-reversal, and charge-conjugation invariance, as well.

At this point the reader is likely to object. He willingly would admit that if only bosons are to be considered he could understand that $T^{\mu\nu}$ could be written in terms of currents. After all, boson fields are closely related to currents, and in some models, such as the algebra of fields, can be identified with them. But he reminds us that for fermions the kinetic part of the energy-momentum tensor is normally given as

$$T^{0k}=-\tfrac{1}{4}i([\bar\psi,\gamma^0\partial^k\psi]-[\partial^k\bar\psi,\gamma^0\psi]),$$

which is apparently not expressible in terms of the usual fermion currents. To allay this objection, at least temporarily, we point out that, in the world of one space and one time dimension, a massless free fermion has a momentum density identical to that above.[4] However, close examination reveals that all matrix elements of this theory involving the spatial derivative $\partial_1\psi$ are unchanged if this spatial derivative is replaced by $\frac{1}{2}i\pi\{(j^1+\gamma^5 j^0),\psi\}$, where $\gamma^5=\gamma^0\gamma^1$. The current j^μ is defined as the formal limit, as the space coordinates come together, of the bilocal form $\frac{1}{2}[\bar\psi,\gamma^\mu\psi]$, with the time coordinates set equal. We may therefore take

$$\partial_1\psi=\tfrac{1}{2}i\pi\{(j^1+\gamma^5 j^0),\psi\} \tag{1}$$

as an operator equation which, for some obscure reason, is true in this simple model. We are thereby enabled to

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[1] The inverse to this approach has been explored by R. F. Dashen and D. H. Sharp, Phys. Rev. **156**, 1857 (1968), and D. H. Sharp, *ibid.* **165**, 1867 (1968), who express canonical field theories solely in terms of currents.

[2] Suggestions to this effect have been made by M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); and in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1966).

[3] The program of this work, though developed completely independently, is essentially identical to that of H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[4] C. M. Sommerfield, Ann. Phys. (N. Y.) **26**, 1 (1964).

write

$$T^{01}=\tfrac{1}{2}\pi\{j^0,j^1\}\,, \qquad (2)$$

since by computing

$$\partial_1\psi(x,t)=i\left[\psi(x,t),\int dx'T^{01}(x',t)\right]$$

$$=\tfrac{1}{2}i\pi\left[\psi(x,t),\int dx'\{j^0(x',t),j^1(x',t)\}\right]$$

and making use of the equal-time commutators

$$[\psi(x,t),j^0(x',t)]=\psi(x,t)\delta(x-x')\,,$$
$$[\psi(x,t),j^1(x',t)]=\gamma^5\psi(x,t)\delta(x-x')\,,$$

we obtain Eq. (1).

These considerations continue to apply even when a current-current coupling is introduced for the fermions, although the definition of j^μ must be changed slightly, and a new numerical factor depending on the coupling must take the place of π in T^{01}.

The relativistic generalization of Eq. (2) which also makes sense in these models is[5]

$$T^{\mu\nu}=\tfrac{1}{2}\pi(j^\mu j^\nu+j^\mu j^\nu-g^{\mu\nu}j^\lambda j_\lambda)\,.$$

It is perhaps asking too much for fermion fields, as normally conceived in four-dimensional space-time, to satisfy equations like Eq. (1). For this reason we will suppress the problem of what to do with fermions until after we have learned how to handle bosons.

In Sec. II we consider four-dimensional theories with no internal symmetry, and discover that the procedure proposed here leads to theories which are formally consistent. However, we discover severe restrictions on the commutator algebra.

In Sec. III we construct models which formally satisfy all of the requirements, and which are applicable to problems with $SU(2)$, $SU(2)\times SU(2)$, and broken $SU(2)\times SU(2)$ internal symmetry.

In Sec. IV we examine three specific models in detail. Of the first two, which involve no external symmetry, we find that one is directly given in canonical form, while the other can simply be put in such a form. The third model, with broken chiral symmetry, is similar to the nonlinear σ model, but is not simply expressible in canonical form.

II. THEORIES WITH NO INTERNAL SYMMETRY

For the sake of definiteness we introduce a set of Hermitian operators, called "currents," with well-defined transformation properties corresponding to those of the 16 bilinear covariants for a spin-$\frac{1}{2}$ field. The correspondence is as follows[6]:

$$S\leftrightarrow\tfrac{1}{2}[\bar\psi,\psi]\,,\quad V^\mu\leftrightarrow\tfrac{1}{2}[\bar\psi,\gamma^\mu\psi]\,,\quad M^{\mu\nu}\leftrightarrow\tfrac{1}{2}[\bar\psi,\sigma^{\mu\nu}\psi]\,,$$
$$A^\mu\leftrightarrow\tfrac{1}{2}[\bar\psi,\gamma^\mu\gamma^5\psi]\,,\quad P\leftrightarrow\tfrac{1}{2}[\bar\psi,i\gamma^5\psi]\,.$$

For each component of the currents, we must have a time-derivative equation. The demand of Lorentz invariance that time derivatives be accompanied by appropriate space derivatives can be satisfied by insisting that the form of the equation of motion be the specification of the four-gradient for scalars, and the four-divergence and four-curl for vectors and tensors. The results of the two-dimensional world suggest that we try a $T^{\mu\nu}$ which is quadratic in the covariants. The only formally covariant possibility is[7]

$$T^{\mu\nu}=\lambda_V(V^\mu V^\nu-\tfrac{1}{2}\alpha_V g^{\mu\nu}V^\lambda V_\lambda)+\lambda_A(A^\mu A^\nu-\tfrac{1}{2}\alpha_A g^{\mu\nu}A^\lambda A_\lambda)$$
$$+\lambda_M(M^{\mu\lambda}M^\nu{}_\lambda-\tfrac{1}{4}\alpha_M g^{\mu\nu}M^{\lambda\sigma}M_{\lambda\sigma})$$
$$-\tfrac{1}{2}\lambda_S g^{\mu\nu}S^2-\lambda_B g^{\mu\nu}S-\tfrac{1}{2}\lambda_P g^{\mu\nu}P^2\,.$$

Since the coefficients of the δ functions in the equal-time commutators of the usual spin-$\frac{1}{2}$ realization are linear in the currents, we expect that the appropriate gradients, divergences, and curls of the currents will be given by expressions which are quadratic in the currents. We thus anticipate equations of the following forms:

$$\partial_\mu V^\mu=R_1\,, \qquad (3a)$$
$$\partial_\mu V_\nu-\partial_\nu V_\mu=R_{2\mu\nu}\,, \qquad (3b)$$
$$\partial_\nu M^{\mu\nu}=R_3{}^\mu\,, \qquad (3c)$$
$$\partial_\nu\bar M^{\mu\nu}=R_4{}^\mu\,,\quad (\bar M^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}M_{\lambda\sigma}) \qquad (3d)$$
$$\partial_\mu A^\mu=R_5\,, \qquad (3e)$$
$$\partial_\mu A_\nu-\partial_\nu A_\mu=R_{6\mu\nu}\,, \qquad (3f)$$
$$\partial_\mu S=R_{7\mu}\,, \qquad (3g)$$
$$\partial_\mu P=R_{8\mu}\,, \qquad (3h)$$

where R_1, R_2, $\cdots$, R_8 are quadratic expressions in the currents which do not involve any derivatives.

In deriving Eq. (3a) from

$$\partial_0V^0(x)=-i\left[V^0(x),\int d^3x'T^{00}(x')\right]_{\rm ET}$$

we must ask that the right-hand side of Eq. (3a) be equal to[8] $-\partial_kV^k+R_1$. In order to obtain ∂_kV^k, the equal-time commutator of V^0 with at least one of the currents

[5] This way of writing the energy-momentum tensor is also discussed by C. G. Callan, R. F. Dashen, and D. H. Sharp, Phys. Rev. **165**, 1883 (1968).

[6] Our covariant notation is such that we distinguish between upper and lower indices, and sum from 0 to 4 over repeated Greek indices, and from 1 to 3 over repeated Latin indices (which represent spatial components). The metric tensor is $g_{11}=g_{22}=g_{33}=-g_{00}=1$. The matrices γ^0 and $\gamma^5=-i\gamma^0\gamma^1\gamma^2\gamma^3$ are Hermitian, while γ^k are anti-Hermitian. Furthermore, $\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}$, $\sigma^{\mu\nu}=\frac{1}{2}i[\gamma^\mu,\gamma^\nu]$ and $\epsilon_{\mu\nu\lambda\sigma}$ is a completely antisymmetric tensor. We take $\epsilon_{0klm}=\epsilon_{klm}$, where $\epsilon_{123}=1$.

[7] In this and in succeeding equations involving operators at the same space-time point, we intend (without explicit indication) the product to be understood as appropriately symmetrized.

[8] The subscript ET signifies an equal-time commutator.

must contain a "Schwinger term" involving the gradient of a spatial δ function.[9] Now T^{00} can also be written

$$T^{00}=\tfrac{1}{2}\sum_{\alpha}\lambda^{(\alpha)}(j^{(\alpha)})^2,$$

where the 16 currents are represented by $j^{(\alpha)}$. Thus, let there be a particular β for which

$$[V^0(x),j^{(\beta)}(x')]_{\mathrm{ET}}=\partial_k[G^{(\beta)k}(x)\delta(\mathbf{x}-\mathbf{x}')]+Q^{(\beta)}(x)\delta(\mathbf{x}-\mathbf{x}'),$$

where we allow, at most, single derivatives of δ functions. Then

$$\partial_0V^0=i\sum_{\beta}\lambda^{(\beta)}\{\partial_k[G^{(\beta)k}j^{(\beta)}]+Q^{(\beta)}j^{(\beta)}\},$$

so that

$$\sum_{\beta}\lambda^{(\beta)}\partial_k[G^{(\beta)k}j^{(\beta)}]=-i\partial_kV^k.$$

From this we conclude that $G^{(\beta)k}=-i/(\lambda_V\alpha_V)$ for $X^{(\beta)}=V^k$, and $G^{(\beta)k}=0$ otherwise. A similar consideration of Eq. (3b) as derived from

$$\partial_0V^k(x)=-\partial_kV^0(x)+R_{20k}(x)=-i\left[V^k(x),\int d^3x'T^{00}(x')\right]_{\mathrm{ET}},$$

using the commutators

$$[V^k(x),j^{(\beta)}(x')]_{\mathrm{ET}}=\partial_l[H^{(\beta)lk}(x)\delta(\mathbf{x}-\mathbf{x}')]+W^{(\beta)k}(x)\delta(\mathbf{x}-\mathbf{x}'),$$

tells us that $H^{(\beta)lk}=-ig^{lk}/[\lambda_V(2-\alpha_V)]$ for $X^{(\beta)}=V^0$, and $H^{(\beta)lk}=0$ otherwise. From comparison of the two expressions for the equal-time commutator of V^0 with V^k we learn that $\alpha_V=1$.

Similar arguments can be applied to each of the other time-derivative equations in the set (3). We learn that $\alpha_A=\alpha_M=1$. We are also able to conclude that a necessary and sufficient condition for the structure of the left-hand sides of (3) to be properly reproduced upon commutation with T^{00} of the operator whose time derivative is being computed, is that all equal-time commutators have no Schwinger terms except for the following, which must have the c-number Schwinger terms as indicated[10]:

$$[V^0(x),V^k(x')]_{\mathrm{ET,ST}}=-i\lambda_V^{-1}\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$
$$[A^0(x),A^k(x')]_{\mathrm{ET,ST}}=-i\lambda_A^{-1}\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$
$$[M^{0l}(x),M^{kl}(x')]_{\mathrm{ET,ST}}=-i\lambda_M^{-1}\partial_k\delta(\mathbf{x}-\mathbf{x}'),\quad(\text{no sum on } l).$$

It will be noted that these Schwinger terms are precisely the ones which an analysis of the Källén-Lehmann representation for the two-point functions of the currents shows to have nonvanishing expectation values.[11] Furthermore, the same Schwinger terms properly account for all of the derivatives when we consider ∂_kV_l, ∂_kA_l, ∂_kM^{0k}, and $\partial_k\bar{M}^{0k}$.

Introducing the notation

$$[j^{(\alpha)}(x),T^{0\mu}(x')]_{\mathrm{ET}}=(j^{(\alpha)},T^{0\mu})\delta(\mathbf{x}-\mathbf{x}')+\text{Schwinger terms},$$

which exhibits an equal-time commutator as a sum of an "ordinary" part and a Schwinger term, we may rewrite Eqs. (3) as

$$\begin{aligned}\partial_\mu V^\mu&=-i(V^0,T^{00}),\\ \partial_\mu V_\nu-\partial_\nu V_\mu&=i(V_\nu,T^0{}_\mu),\\ \partial_\nu M^{k\nu}&=i(M^{k\nu},T^0{}_\nu),\\ \partial_kM^{0k}&=\tfrac{1}{3}i(M^{0k},T^0{}_k),\\ \partial_\mu S&=i(S,T^0{}_\mu),\end{aligned}\qquad(4)$$

together with relations obtained from these with the replacements $V_\mu\to A_\mu$, $M_{\mu\nu}\to\bar{M}_{\mu\nu}$, and $S\to P$.

We must also try to derive Eq. (3a) by considering

$$\partial_kV^k(x)=i\left[V^k(x),\int d^3x'T^0{}_k(x')\right]_{\mathrm{ET}}=\partial_kV^k(x)+i(V^k,T^0{}_k).$$

We are thereby led to the constraint

$$(V^k,T^{0k})=0.$$

By starting with the spatial derivatives in each of Eqs. (3), we obtain the constraints

$$\begin{aligned}(V^k,T^{0l})&=-(V^l,T^{0k}),\\ (V^0,T^{0k})&=0,\\ (M^{0k},T^{0l})&=\tfrac{1}{3}g^{kl}(M^{0m},T^0{}_m),\end{aligned}$$

together with their counterparts, obtained by replacing V_μ and $M_{\mu\nu}$ by A_μ and $\bar{M}_{\mu\nu}$, respectively.

We have assumed that the ordinary part of the equal-time commutator between any two currents is linear in the currents. Thus, if we introduce a 17th $j^{(\alpha)}$ which is just a constant c number, we find that the commutators satisfy a Lie algebra

$$(j^{(\alpha)},j^{(\beta)})=i\sum_{\gamma}C^{\alpha\beta\gamma}j^{(\gamma)}\qquad(5)$$

with real structure constants $C^{\alpha\beta\gamma}$. Since each current $j^{(\alpha)}$ has well-defined properties under space inversion, time reversal, and charge conjugation, it is not hard to show that there is at most one nonvanishing term in the sum over γ in Eq. (5), and that all of the nonvanishing commutators must be proportional to those obtained from the spin-$\frac{1}{2}$ realization of the currents. The only exception is if $C^{\alpha\beta\gamma}\neq0$, where $j^{(\gamma)}=S$, then $C^{\alpha\beta(17)}$ is

[9] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[10] The subscript ST indicates the Schwinger-term part of the commutator.

[11] L. S. Brown, Phys. Rev. **150**, 1338 (1966).

not necessarily 0. The most stringent requirement on these commutators is that the right-hand sides of Eqs. (4) must come out with the same covariance properties as the left-hand sides. A detailed but straightforward calculation shows that the following are the only possible nonvanishing (and more conveniently annotated) ordinary commutators:

$$\begin{aligned}(A^0,S)&=iC_1P\,,\\(A^0,P)&=-iC_2S-iC_2'\,,\\(A^0,M^{0k})&=-iC_3\bar{M}_{0k}\,,\\(A^0,\bar{M}_{0k})&=iC_4M^{0k}\,,\\(M^{0k},M^{0l})&=iC_5\epsilon_{klm}A^m\,,\\(\bar{M}_{0k},\bar{M}_{0l})&=iC_6\epsilon_{klm}A^m\,,\\(V^k,M^{0l})&=-iC_7g^{kl}S-iC_7'g^{kl}\,,\\(V^k,\bar{M}_{0l})&=iC_8g^{kl}P\,,\end{aligned}\tag{6}$$

and that the conditions

$$\lambda_MC_5=\lambda_AC_3\,,\quad \lambda_MC_6=\lambda_AC_4\tag{7}$$

must be satisfied as well. The signs have been chosen so that each unprimed C would be 2 for the spin-$\frac{1}{2}$ realization. The Jacobi identity requires the following additional relations:

$$C_4C_7=C_2C_8\,,\quad C_3C_8=C_1C_7\,,\quad C_4C_7'=C_3'C_8.\tag{8}$$

The equations of motion then read

$$\begin{aligned}&\partial_\mu V^\mu=0\,,\\&\partial_\mu V_\nu-\partial_\nu V_\mu=\lambda_MC_7SM_{\mu\nu}+\lambda_MC_8P\bar{M}_{\mu\nu}+\lambda_MC_7'M_{\mu\nu}\,,\\&\partial_\mu A^\mu=(\lambda_SC_1-\lambda_PC_2)SP+\tfrac{1}{4}\lambda_M(C_4-C_3)\bar{M}^{\mu\nu}M_{\mu\nu}\\&\qquad+(\lambda_BC_1-\lambda_PC_2')P\,,\\&\partial_\mu A_\nu-\partial_\nu A_\mu=0\,,\\&\partial_\mu S=\lambda_AC_1PA_\mu\,,\\&\partial_\mu P=-\lambda_AC_2SA_\mu-\lambda_AC_2'A_\mu\,,\\&\partial_\nu M^{\mu\nu}=-\lambda_VC_7SV^\mu+\lambda_AC_3\bar{M}^{\mu\nu}A_\nu-\lambda_VC_7'V^\mu\,,\\&\partial_\nu\bar{M}^{\mu\nu}=-\lambda_VC_8PV^\mu-\lambda_AC_4M^{\mu\nu}A_\nu.\end{aligned}\tag{9}$$

There are ten C's and six λ's, which seems to indicate, given the five constraint equations (7) and (8), that 11 parameters are needed to specify a theory of this type. However, most of these may be eliminated by dimensionless scale transformations on the currents, or by changes in λ_S, λ_P, λ_B. For example, any theory for which C_4, C_7, C_8, and the six λ's are assigned to be nonzero is equivalent to that for which $C_7=C_8=1$, $C_1=C_3=C_5$, $C_2=C_4=C_6=1$, $C_2'=C_7'$, $\lambda_V=\lambda_A=\lambda_M$, $\lambda_S=0$, and which contains only five arbitrary parameters, one of which determines the mass scale.

We take Eqs. (6)–(9) as the complete formal solution of the limited problem with no internal degrees of freedom. We defer discussion of specific choices of the parameters to Sec. IV.

III. INTERNAL SYMMETRY

Our program is somewhat simplified when we consider systems having internal symmetry. Then, many of the commutation relations are merely statements of how we expect the various operators to behave under the transformations which make up the symmetry group.

A. $SU(2)$ Symmetry

If we introduce isospin invariance as an internal symmetry, we should include the isovector current $\mathbf{V}^\mu$ as a dynamical variable.[12] The significance of $\int d^3x\,\mathbf{V}^0(x)$ as the generator of isospin transformations means that any quantity $\mathbf{Z}$ which transforms like an isovector must satisfy the equal-time commutation relation

$$(\mathbf{a}\cdot\mathbf{V}^0,\mathbf{b}\cdot\mathbf{Z})=i\mathbf{a}\times\mathbf{b}\cdot\mathbf{Z}\,,$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary unit vectors in isospin space. All isoscalars commute with $\mathbf{V}^0$ for equal times, except for possible Schwinger terms.

An energy-momentum tensor incorporating $\mathbf{V}$ only would be

$$T^{\mu\nu}=\lambda(\mathbf{V}^\mu\cdot\mathbf{V}^\nu-\tfrac{1}{2}g^{\mu\nu}\mathbf{V}^\lambda\cdot\mathbf{V}_\lambda)\,.$$

Arguments similar to those in the preceding section show that we must have the Schwinger term

$$[\mathbf{a}\cdot\mathbf{V}^0(x),\mathbf{b}\cdot\mathbf{V}^k(x')]_{\mathrm{ET,ST}}=-i\mathbf{a}\cdot\mathbf{b}\lambda^{-1}\partial_k\delta(\mathbf{x}-\mathbf{x}')$$

as well as the vanishing of the commutators $(\mathbf{V}^k,\mathbf{V}^l)$. The resulting equations of motion are then

$$\begin{aligned}&\partial_\mu\mathbf{V}^\mu=0\,,\\&\partial_\mu\mathbf{V}_\nu-\partial_\nu\mathbf{V}_\mu=-\lambda\mathbf{V}_\mu\times\mathbf{V}_\nu.\end{aligned}$$

B. Chiral Symmetry

A simple extension of the above to chiral $SU(2)\times SU(2)$ internal symmetry requires the introduction of an axial isovector current $\mathbf{A}^\mu$, and the interpretation of $\frac{1}{2}\int d^3x(\mathbf{V}^0\pm i\mathbf{A}^0)$ as the generators of two independent $SU(2)$ transformations. The corresponding commutation relations are then

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{V}^0,\mathbf{b}\cdot\mathbf{V}^\mu)=(\mathbf{a}\cdot\mathbf{A}^0,\mathbf{b}\cdot\mathbf{A}^\mu)=i\mathbf{a}\times\mathbf{b}\cdot\mathbf{V}^\mu\,,\\(\mathbf{a}\cdot\mathbf{V}^0,\mathbf{b}\cdot\mathbf{A}^\mu)=(\mathbf{a}\cdot\mathbf{A}^0,\mathbf{b}\cdot\mathbf{V}^\mu)=i\mathbf{a}\times\mathbf{b}\cdot\mathbf{A}^\mu.\end{aligned}$$

The generalization of the energy-momentum tensor to this case is

$$T^{\mu\nu}=\lambda(\mathbf{V}^\mu\cdot\mathbf{V}^\nu-\tfrac{1}{2}g^{\mu\nu}\mathbf{V}^\lambda\cdot\mathbf{V}_\lambda)+\lambda(\mathbf{A}^\mu\cdot\mathbf{A}^\nu-\tfrac{1}{2}g^{\mu\nu}\mathbf{A}^\lambda\cdot\mathbf{A}_\lambda)\,.$$

Applying the analysis of Sec. II, we determine that the only Schwinger terms are

$$\begin{aligned}[\mathbf{a}\cdot\mathbf{V}^0(x),\mathbf{b}\cdot\mathbf{V}^k(x')]_{\mathrm{ET,ST}}&=[\mathbf{a}\cdot\mathbf{A}^0(x),\mathbf{b}\cdot\mathbf{A}^k(x')]_{\mathrm{ET,ST}}\\&=-i\lambda^{-1}\mathbf{a}\cdot\mathbf{b}\partial_k\delta(\mathbf{x}-\mathbf{x}')\,,\end{aligned}$$

and that all space-space commutators must vanish.

[12] Isovectors will be indicated in boldface type and dot- and cross-product notation will be used.

The equations of motion read

$$\partial_\mu \mathbf{V}^\mu = 0\,, \tag{10}$$
$$\partial_\mu \mathbf{A}^\mu = 0\,,$$
$$\partial_\mu \mathbf{V}_\nu - \partial_\mu \mathbf{V}_\nu = -\lambda(\mathbf{V}_\mu \times \mathbf{V}_\nu + \mathbf{A}_\mu \times \mathbf{A}_\nu)\,, \tag{11}$$
$$\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu = -\lambda(\mathbf{A}_\mu \times \mathbf{V}_\nu + \mathbf{V}_\mu \times \mathbf{A}_\nu)\,. \tag{12}$$

The equality of the coefficients of the **A** and **V** parts of $T^{\mu\nu}$ is necessary in order that the right-hand side of this last equation be antisymmetric in μ and ν.[13]

C. Broken Chiral Symmetry

To break the chiral symmetry we should add a term to $T^{\mu\nu}$ which does not commute with $\mathbf{A}^0$. It must be an isoscalar to keep isospin symmetry, and hence must commute with $\mathbf{V}^0$. It if were a function of $\mathbf{V}^\mu$ and $\mathbf{A}^\mu$, new spatial derivatives would be introduced into the field equations because of the Schwinger terms, and it would be very difficult (if not impossible) to preserve the proper covariant structure of these equations. One possibility would be to introduce a term containing new vector, axial-vector, or tensor currents. But under simple assumptions for the commutation relations, these either do not do the job, or lead to very complicated equations. Another approach is to introduce a $g^{\mu\nu}$ term as follows[14]:

$$T^{\mu\nu} = \lambda(\mathbf{V}^\mu\cdot\mathbf{V}^\nu - \tfrac{1}{2}g^{\mu\nu}\mathbf{V}^\lambda\cdot\mathbf{V}_\lambda + \mathbf{A}^\mu\cdot\mathbf{V}^\nu - \tfrac{1}{2}g^{\mu\nu}\mathbf{A}^\lambda\cdot\mathbf{V}_\lambda) - g^{\mu\nu}S\,.$$

The nonvanishing equal-time commutator

$$(\mathbf{A}^0, S) = i\mathbf{P}$$

is taken to define a pseudoscalar isovector field **P**. We find that all other operators must commute with S. The theory is still not complete, for we do not know the commutation relations involving **P**. The isovector character of **P** requires that we take

$$(\mathbf{a}\cdot\mathbf{V}^0, \mathbf{b}\cdot\mathbf{P}) = i\mathbf{a}\times\mathbf{b}\cdot\mathbf{P}\,.$$

But what of its commutator with $\mathbf{A}^0$? We may write, in general,

$$(\mathbf{a}\cdot\mathbf{A}^0, \mathbf{b}\cdot\mathbf{P}) = i\mathbf{a}\cdot\boldsymbol{\Omega}\cdot\mathbf{b}$$

and distinguish three cases:

(i) The algebra closes in the sense of the σ model[15] and

$$\mathbf{a}\cdot\boldsymbol{\Omega}\cdot\mathbf{b} = -\mathbf{a}\cdot\mathbf{b}S\,;$$

(ii) the algebra closes, but involves isospin two terms, e.g.,

$$\mathbf{a}\cdot\boldsymbol{\Omega}\cdot\mathbf{b} = (\mathbf{a}\cdot\mathbf{P})(\mathbf{P}\cdot\mathbf{b})\,;$$

(iii) the algebra does not close, and $\boldsymbol{\Omega}$ must be taken to be an independent dyadic operator whose commutation relations must be specified. We shall here discuss only cases (i) and (ii).

The equations of motion are (10)–(12), together with

$$\partial_\mu \mathbf{A}^\mu = \mathbf{P}\,,$$
$$\partial_\mu S = \lambda \mathbf{A}_\mu \cdot \mathbf{P}\,, \tag{13}$$
$$\partial_\mu \mathbf{P} = -\lambda \mathbf{V}_\mu \times \mathbf{P} + \lambda \boldsymbol{\Omega}\cdot\mathbf{A}_\mu\,. \tag{14}$$

The operator $\boldsymbol{\Omega}$ is not entirely unrestricted. Our commutation relations must be consistent with the Jacobi identity or, equivalently, the equations of motion must be consistent with one another. Thus the four-curl of the right-hand sides of Eqs. (13) and (14) must each vanish. From (13), we learn that $\mathbf{A}_\mu\cdot\boldsymbol{\Omega}\cdot\mathbf{A}_\nu = \mathbf{A}_\nu\cdot\boldsymbol{\Omega}\cdot\mathbf{A}_\mu$, whereupon we infer that $\boldsymbol{\Omega}$ is a symmetric dyadic. From (14), we find

$$(\partial_\mu\boldsymbol{\Omega})\cdot\mathbf{A}_\nu + \lambda\mathbf{V}_\mu\times\boldsymbol{\Omega}\cdot\mathbf{A}_\nu - \lambda\boldsymbol{\Omega}\times\mathbf{V}_\mu\cdot\mathbf{A}_\nu - (\mu\leftrightarrow\nu) = \lambda\mathbf{P}\times(\mathbf{A}_\mu\times\mathbf{A}_\nu)\,. \tag{15}$$

For case (i), this is identically satisfied. For case (ii) we expect that if $\boldsymbol{\Omega}$ depends on $\mathbf{A}^\mu$ or $\mathbf{V}^\mu$, great complication will result from taking the gradient (since only curls and divergences are simple for these operators), and it is hard to see how Eq. (15) could follow. On the other hand, if $\boldsymbol{\Omega}$ depends on **P** and S, we can use the equations of motion themselves to take gradients of **P** and S. Thus we write

$$\mathbf{a}\cdot\boldsymbol{\Omega}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b}X(S,\mathbf{P}^2) + (\mathbf{a}\cdot\mathbf{P})(\mathbf{b}\cdot\mathbf{P})Y(S,\mathbf{P}^2)$$

and discover that Eq. (15) is equivalent to

$$[\partial/\partial S + 2(X+\mathbf{P}^2Y)\partial/\partial\mathbf{P}^2]X + 1 = XY\,.$$

On the other hand, multiplying Eq. (14) by **P** and comparing with Eq. (13), we learn that

$$\partial_\mu(\tfrac{1}{2}\mathbf{P}^2) = (X+\mathbf{P}^2Y)\partial_\mu S\,,$$

so that there is a differential constraint connecting S with **P**. Thus X and Y may be considered functions of S only, and we find that Eq. (15) now reads

$$(d/dS)X(S) + 1 = XY\,.$$

Introducing

$$S' = \exp\left[-\int^S dt/X(t)\right], \tag{16}$$

and $\mathbf{P}' = -i(\mathbf{A}^0, S')$, we learn that

$$(\mathbf{a}\cdot\mathbf{A}^0, \mathbf{b}\cdot\mathbf{P}') = -i\mathbf{a}\cdot\mathbf{b}S'\,.$$

Thus, if we solve Eq. (16) for S as a function of S' and substitute that in $T^{\mu\nu}$, we recover the commutation relations of case (i) in terms of S' and $\mathbf{P}'$. Therefore the choice of energy-momentum tensor

$$T^{\mu\nu} = \lambda(\mathbf{V}^\mu\cdot\mathbf{V}^\nu - \tfrac{1}{2}g^{\mu\nu}\mathbf{V}^\lambda\cdot\mathbf{V}_\lambda + \mathbf{A}^\mu\cdot\mathbf{A}^\nu - \tfrac{1}{2}g^{\mu\nu}\mathbf{A}^\lambda\cdot\mathbf{A}_\lambda) - g^{\mu\nu}f(S)\,, \tag{17}$$

[13] It has been shown that these equations can be obtained from those for a Yang-Mills field [C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954)] in the limit that the bare mass and charge vanish, but that $g_0^2/m_0^2 = \lambda$ remains finite. See K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968).

[14] This method of breaking the symmetry was also given by Bardakci, Frishman, and Halpern, Ref. 13.

[15] M. Gell-Mann and M Lévy, Nuovo Cimento **16**, 705 (1960).

where f is an arbitrary function, and the commutator

$$(\mathbf{a}\cdot\mathbf{A}^0,\mathbf{b}\cdot\mathbf{P})=-i\mathbf{a}\cdot\mathbf{b}S$$

covers both cases (i) and (ii).[16] The equations of motion are then (10)–(13) and

$$\partial_\mu\mathbf{A}^\mu=f'(S)\mathbf{P}, \quad (18)$$
$$\partial_\mu\mathbf{P}=-\lambda S\mathbf{A}_\mu-\lambda\mathbf{V}_\mu\times\mathbf{P}.$$

IV. EXAMPLES

In this section we should like to analyze in somewhat more detail several specific models.

Model (a)

Leaving out internal symmetry, we consider a choice of parameters in Eqs. (6) such that the only nonvanishing ordinary commutators are

$$(A^0,P)=i,$$
$$(V^k,M^{0l})=-ig^{kl}.$$

These are canonical commutation relations for a psuedoscalar and a vector field, respectively, with their conjugates. If we then choose

$$\lambda_S=\lambda_B=0,\quad \lambda_A=\lambda_M=1,\quad \lambda_V=m_V^2,\quad \lambda_P=m_P^2,$$

we obtain, as equations of motion,

$$\begin{aligned}
\partial_\mu V^\mu&=0,\\
\partial_\mu V_\nu-\partial_\nu V_\mu&=M_{\mu\nu},\\
\partial_\nu\tilde{M}^{\mu\nu}&=0,\\
\partial_\nu M^{\mu\nu}&=-m_V^2V^\mu, \quad (19)\\
\partial_\mu P&=A_\mu,\\
\partial_\mu A^\mu&=m_P^2P,\\
\partial_\mu A_\nu-\partial_\nu A_\mu&=0,\\
\partial_\mu S&=0,
\end{aligned}$$

which characterize a system of noninteracting pseudoscalar particles of mass m_P and noninteracting spin-one particles of mass m_V. Clearly S is an irrelevant, constant c number.

Model (b)

The next case is to take the nonvanishing ordinary commutators to be

$$(A^0,P)=iS,$$
$$(A^0,S)=-iP,$$
$$(V^k,M^{0l})=-ig^{kl}.$$

Again, the choice of $\lambda_M=1$, $\lambda_V=m_V^2$ leads to the first four of Eqs. (19). But with $\lambda_S=\lambda_B=0$, the other equations of motion are now

$$\begin{aligned}
\partial_\mu S&=-\lambda_A PA_\mu,\\
\partial_\mu P&=\lambda_A SA_\mu,\\
\partial_\mu A_\nu-\partial_\nu A_\mu&=0,\\
\partial_\mu A^\mu&=\lambda_P SP.
\end{aligned}$$

It is clear that the combination P^2+S^2 is a constant operator, which we may take to be equal to 1 by appropriate choice of λ_P. Then if we write

$$P=\sin(\lambda_A{}^{1/2}\phi),$$
$$S=\cos(\lambda_A{}^{1/2}\phi),$$
$$A^\mu=\phi^\mu/\lambda_A{}^{1/2},$$

we discover

$$(\phi^0,\phi)=i,$$
$$\partial_\mu\phi=\phi_\mu,$$
$$\partial_\mu\phi^\mu=\partial^2\phi=\tfrac{1}{2}\lambda_P\lambda_A{}^{1/2}\sin(2\lambda_A{}^{1/2}\phi),$$

which can also be obtained from the Lagrangian density

$$\mathcal{L}=-\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi^\mu\phi_\mu+\tfrac{1}{4}\lambda_P\cos(2\lambda_A{}^{1/2}\phi)$$

for a canonical, self-coupled pseudoscalar field.

It is clear that this kind of theory can be generalized to allow for the λ_P term in $T^{\mu\nu}$ to be any arbitrary function $f(P)$, so that

$$\partial_\mu A^\mu=\lambda_P f'(P)S,$$

or

$$\partial^2\phi=\lambda_P\lambda_A{}^{1/2}f'[\sin(\lambda_A{}^{1/2}\phi)]\cos(\lambda_A{}^{1/2}\phi)\equiv g'(\phi).$$

Hence an arbitrary self-coupled pseudoscalar interaction in $\mathcal{L}$ can be reproduced with the present formalism.

Model (c)

The last model we consider is for the case of broken chiral symmetry. We choose the function $f(S)$ appearing in Eq. (17) to be

$$f(S)=-CS,$$

so that

$$\partial_\mu\mathbf{A}^\mu=-C\mathbf{P}.$$

The procedure analogous to that followed with model (b) would be to introduce a canonical isovector field $\boldsymbol{\phi}$ and its conjugate $\boldsymbol{\phi}_\mu$, to write $\mathbf{P}=\boldsymbol{\phi}g(\boldsymbol{\phi}^2)$, and to ask that $\mathbf{A}_\mu$ be linearly related to $\boldsymbol{\phi}_\mu$. However, this procedure does not work because of the complication of isotopic spin.[17]

There are several features that allow for some simplification. First, Eqs. (13) and (14) tell us that $S^2+\mathbf{P}^2$ is a constant c number. Secondly, all of the equations

[16] This equivalence has been pointed out in the context of Lagrangian models by L. S. Brown, Phys. Rev. **163**, 1802 (1967).

[17] Canonical realizations of this theory have been given by K. Bardakci and M. B. Halpern, Phys. Rev. **172**, 1542 (1968), and by H. Sugawara and M. Yoshimura, *ibid.* **173**, 1419 (1968). However, unlike our models (a) and (b), in these cases one cannot invert the procedure and express the canonical fields in terms of the currents.

of motion are invariant under a common scale transformation of S and $\mathbf{P}$, with the exception of Eq. (18). But a redefinition of C can compensate for this. Thus, we are free to choose the constant $B=S^2+\mathbf{P}^2$ to fit our own convenience, and may furthermore consider S to be given directly in terms of $\mathbf{P}$.

One can look at sum rules derived from application of the commutation relations to the spectral representations of certain two-point functions. Consider, for example,

$$\langle 0|[\mathbf{a}\cdot\mathbf{P}(x),\mathbf{b}\cdot\mathbf{P}(0)]|0\rangle=\mathbf{a}\cdot\mathbf{b}\int_0^\infty ds\,\rho_P(s)\Delta(s;x)\,, \tag{20}$$

where $\Delta(s,x)$ is the invariant commutator function for a free scalar field of mass $\sqrt{s}$, and where $\rho_P(s)\geqq 0$. Computing $\langle 0|[\mathbf{a}\cdot\partial_0\mathbf{P}(x),\mathbf{b}\cdot\mathbf{P}(0)]_{\mathrm{ET}}|0\rangle$ on the left-hand side of Eq. (20), using the equations of motion and commutation relations, and on the right-hand side, using the fact that $\partial_0\Delta(s,x)|_{x^0=0}=-i\delta(\mathbf{x})$, we obtain the sum rule

$$\mathbf{a}\cdot\mathbf{b}\int_0^\infty ds\,\rho_P(s)=\lambda\langle 0|(\mathbf{a}\cdot\mathbf{b})(S^2+\mathbf{P}^2)-(\mathbf{a}\cdot\mathbf{P})(\mathbf{b}\cdot\mathbf{P})|0\rangle$$
$$=\lambda\mathbf{a}\cdot\mathbf{b}\{B-\tfrac{1}{3}\langle 0|\mathbf{P}^2|0\rangle\}\,.$$

Since $B=S^2+\mathbf{P}^2\geqq\mathbf{P}^2$ (assuming that the formal positive-definite property of S^2 is valid), we may write $\langle 0|\mathbf{P}^2|0\rangle=B\alpha$, where $0\leqq\alpha\leqq 1$, and so obtain

$$\int_0^\infty ds\,\rho_P(s)=\lambda B(1-\tfrac{1}{3}\alpha)\,.$$

Now we would expect that B is infinite. But in fact, according to the scale invariance, it is ours to choose, and so we may take $B=1$.

Similarly, writing

$$\langle 0|[\mathbf{a}\cdot\mathbf{A}_\mu(x),\mathbf{b}\cdot\mathbf{A}_\nu(0)]|0\rangle=\mathbf{a}\cdot\mathbf{b}\int ds[\rho_A(s)(g_{\mu\nu}-\partial_\mu\partial_\nu/s)-C^2\rho_P(s)\partial_\mu\partial_\nu/s^2]\Delta(s,x)\,,$$

$$\langle 0|[\mathbf{a}\cdot\mathbf{V}_\mu(x),\mathbf{b}\cdot\mathbf{V}_\nu(0)]|0\rangle=\mathbf{a}\cdot\mathbf{b}\int ds\times\rho_V(s)(g_{\mu\nu}-\partial_\mu\partial_\nu/s)\Delta(s,x)\,,$$

from the Schwinger-term commutators involving $\mathbf{A}^\mu$ and $\mathbf{V}^\mu$, we obtain

$$\lambda^{-1}=\int_0^\infty ds[\rho_A(s)/s+C^2\rho_P(s)/s^2]\,,$$

$$\lambda^{-1}=\int_0^\infty ds\,\rho_V(s)/s\,,$$

which, taken together, constitute Weinberg's first sum rule.[18] Weinberg's second sum rule,

$$\int_0^\infty ds[\rho_V(s)-\rho_A(s)]=0\,,$$

comes from the identity

$$\langle 0|[\mathbf{a}\cdot(\partial_\mu\mathbf{V}_\nu-\partial_\nu\mathbf{V}_\mu),\mathbf{b}\cdot\mathbf{V}_\lambda]_{\mathrm{ET}}|0\rangle=\langle 0|[\mathbf{a}\cdot(\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu),\mathbf{b}\cdot\mathbf{A}_\lambda]_{\mathrm{ET}}|0\rangle\,.$$

There is an additional sum rule based on

$$\langle 0|(\mathbf{a}\cdot\mathbf{A}_0,\mathbf{b}\cdot\mathbf{P}|0\rangle=i\mathbf{a}\cdot\mathbf{b}\langle 0|S|0\rangle\,,$$

which is

$$C\int ds\,\rho_P(s)/s=\langle 0|S|0\rangle\,,$$

and which also holds in the σ model.

Clearly, all of the usual model-independent results of broken chiral invariance will hold here as well. What is needed to proceed further in this model is to find a small parameter such that there exists a perturbation theory in this parameter. Efforts in this direction are continuing.

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[18] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).