This gives corrections to the total and forward differential cross sections of about -0.4% and -0.8%, respectively. For purposes of comparison, the corresponding corrections due to the double-scattering term are about -6% and -12%.

V. DISCUSSION

We have considered a correction to the Glauber multiple-scattering expansion which, in potential theory, results from a three-body interaction. Because of this interaction the potential seen by a particle incident on a deuteron is not just the sum of the proton and neutron potentials but has an additional contribution which vanishes when the two nucleons are far apart.

We have attempted to estimate the importance of effects of this type by considering the possibility that the incident particle may interact with a pion being exchanged by the two nucleons. This correction reduces the total cross section, but only by about 0.4%, which is just outside the limits of present experimental accuracy.[13] Our expressions are probably not accurate at large angles, but suggest that three-body effects may become increasingly important as the momentum transfer increases.

The correction considered here is of course only one of many rather poorly understood corrections to the Glauber expansion, and our estimate of its size is quite crude. It should be kept in mind, however, if the uncorrected expansion (including spin dependence and phase variation of the input amplitudes) is unable to fit accurately the scattering from composite system.

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PHYSICAL REVIEW VOLUME 176, NUMBER 5 25 DECEMBER 1968

Regge Asymptotic Behavior and the Bethe-Salpeter Equation

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(Received 25 July 1968)

Making certain assumptions, we directly take the limit of the Bethe-Salpeter equation at high energy (s) for the elastic scattering amplitude $A(s,t)$ in $g\phi^3$ theory. For a large class of planar kernels behaving like $I(t)s^{-q}(\ln s)^p$, with integer $p\geq 0$ and $q\geq 1$, we obtain a Volterra equation for the asymptotic amplitude. The unique solution is of the Regge form $s^{\alpha(t)}$, with $\alpha(t)=-q+[p!K_q(t)I(t)]^{1/(p+1)}$. Here K_q is an explicit function, independent of g. This result agrees with that obtained by summation of leading asymptotic behaviors in each order of perturbation theory in all known cases. We explicitly justify our assumptions for the ladder approximation. In this case we obtain an integral equation for the exact $\alpha(t)$ which agrees with the usual result to order g^2. We conclude with a discussion of nonplanar kernels in our formalism.

I. INTRODUCTION

APART from the example of potential theory,[1] there is little theoretical justification for the assumption[2] that physical scattering amplitudes $A(s,t)$ possess Regge asymptotic behavior $s^{\alpha(t)}$. Most previous attempts[3–16] at supplying such a justification have relied on the technique of choosing an infinite set of Feynman diagrams $\{A_n, n=0,1,2,\cdots\}$, determining the asymptotic behavior $\tilde{A}_n$ of each, and finally summing $\sum_n \tilde{A}_n$ to obtain a hopefully good indication of the asymptotic behavior of the sum $\sum_n A_n$. This procedure has led to results which were sometimes in agreement[3–5,7,10,12–15] with Regge behavior and sometimes not.[3,6,8,9,11] An exception is the example of the Bethe-Salpeter (B-S) equation[17] in the ladder approximation, where meromorphy of the scattering amplitude in the complex angular-momentum half-plane $\{\mathrm{Re}\,l>-\frac{3}{2}\}$ has been established,[18] and where the *complete* set of terms of the form $s^{-1}(\ln s)^n$ has been summed.[14] Also, Bjorken[19] has established Regge behavior at $t=0$ for B-S amplitudes defined by "nice" kernels.

* Supported in part by the U. S. Air Force under Grant No. AFOSR 68-1453.
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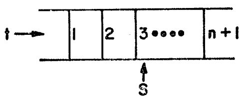


FIG. 1. The $(n+1)$-rung ladder diagram.

In this paper we employ a new method for investigating scattering amplitudes at high energy (HE).[20] We consider the B-S equation in ϕ^3 theory with a certain class of kernels and directly take its HE limit in a somewhat heuristic fashion. This gives us an integral equation for the asymptotic amplitude $\tilde{A}$ which we can solve. The solution will have Regge behavior with a leading trajectory function $\alpha(t)=-q+g^{r/(p+1)}V(t)$, where q, p, and r are integers and g^r is the lowest power of the coupling constant occurring in the kernel. The trajectory functions so obtained agree with the summation of leading HE behaviors in each order of g^2 in all known cases. In this sense our method should be considered as an alternative to the usual summation technique.

In the case of the ladder approximation, we are able to take the HE limit much more carefully and again obtain an integral equation whose solution has Regge behavior. $\alpha(t)$ is now determined by an integral equation which gives the usual result in order g^2 in agreement with both our heuristic approach and the summation technique.

In summary, we obtain the following new results: (i) an abstract characterization of a class of B-S kernels which lead to scattering amplitudes with Regge asymptotic behavior (this class includes those kernels given by Feynman diagrams for which $\sum_n \tilde{A}_n$ is of Regge type), (ii) an integral equation for the trajectory functions $\alpha(t)$ associated with the ladder kernel, and (iii) the complete expressions of order g^2 for $\alpha(t)$ for the kernels in (i).

For later reference we now review some results of the diagram summation technique in more detail. The simplest example involves the $(n+1)$-rung ladder diagram of Fig. 1. The corresponding scattering amplitude $A_L^{(n)}(s,t)$ has the HE behavior

$$A_L^{(n)}(s,t)\sim\tilde{A}_L^{(n)}(s,t)\equiv g^2[g^2K(t)]^n(1/n!)(\ln s)^n/s\,, \quad (1.1)$$

where $K(t)$ is the amplitude corresponding to the "contracted" graph of Fig. 2 with two-dimensional loop momentum:

$$K(t)=\frac{1}{16\pi^2}\int_0^1\frac{d\alpha}{m^2-\alpha(1-\alpha)t}. \quad (1.2)$$

The occurrence of K^n in (1.1) corresponds to the relevance of Fig. 3 to $\tilde{A}$. Summing the $\tilde{A}_L^{(n)}$ over all n gives the Regge form

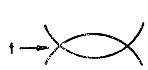


FIG. 2. Diagram associated with $K(t)$.

FIG. 3. Diagram associated with $K^n(t)$.

$$\sum_n \tilde{A}_L^{(n)}(s,t)=g^2s^{\alpha_L^{(1)}(t)}\,, \quad (1.3)$$

where

$$\alpha_L^{(1)}(t)=-1+g^2K(t). \quad (1.4)$$

This does not, of course, show that the HE behavior $\tilde{A}_L(s,t)$ of the ladder amplitude $A_L(s,t)\equiv\sum_n A_L^{(n)}(s,t)$ is given by (1.3). However, it has been shown[18] that the leading HE behavior of A_L is given by a Regge pole whose trajectory function α_L is a power series in g^2 given by (1.4) to order g^2. Furthermore, the lower-order terms $s^{-1}(\ln s)^{n-1}$, etc., for each power of g^2 in A_L^n have all been summed[14] and lead to Regge asymptotic behavior consistent with (1.4) in second order.

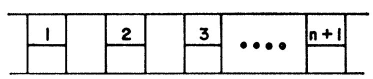


FIG. 4. The $(n+1)$-H exchange diagram.

A large class of other planar diagrams has been similarly summed, but only in the approximation of keeping the leading HE term in each order of g^2. An example is the $(n+1)$-H exchange graph of Fig. 4, which has been shown[3] to satisfy

$$A_H^{(n)}(s,t)\sim\tilde{A}_H^{(n)}(s,t)\equiv[H(t)]^{n+1}[K'(t)]^n\times\frac{1}{(2n+1)!}\frac{(\ln s)^{2n+1}}{s^2}, \quad (1.5)$$

where $H(t)$ is a function associated with the contracted graph of Fig. 5 and $K'(t)=-d/dm^2K(t)$. The factorization of the coefficient of $(\ln s)^{2n+1}/s^2$ is depicted in Fig. 6. Note, however, that, contrary to what is suggested by

FIG. 5. Diagram associated with $H(t)$.

Fig. 6, K' occurs in (1.5) rather than K. Summation gives the asymptotic behavior of $\sum_n \tilde{A}_H^{(n)}(s,t)$ to be that associated with a pair of Regge poles with trajectories

$$\alpha_H^{\pm(1)}(t)=-2\pm[H(t)K'(t)]^{1/2}. \quad (1.6)$$

The general class of planar graphs for which such summation has been shown[13,15] to lead to Regge asymptotic behavior is defined by the requirement that the lines which are contracted to find the HE behavior (d

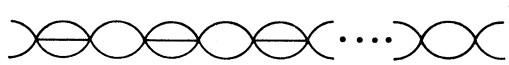


FIG. 6. Factorization of the asymptotic $(n+1)$-H exchange diagram.

[20] A related method has been employed by Fubini *et al.* [Nuovo Cimento **22**, 569 (1961); **25**, 626 (1962); **26**, 896 (1962); **26**, 247 (1962)] for the absorptive part of the amplitude in what is essentially the ladder approximation. They obtain an integral equation similar to our Eq. (2.16).

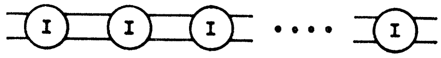


FIG. 7. Schematic B-S amplitude.

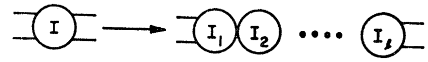


FIG. 8. Diagram associated with the HE behavior of the kernels that we consider.

lines[13] or t paths[7]) do not intersect. In the context of B-S amplitudes (represented graphically in Fig. 7), this means that the kernels I have this property. Then the contracted diagram corresponding to the HE behavior of I will have the form shown in Fig. 8. For simplicity we shall explicitly consider only the case where the extreme blobs I_1 and I_l are point vertices. Then the high-s behavior of I is independent of the external masses and some of our equations simplify. Explicit expressions for the trajectory functions [analogous to (1.4) and (1.6)] have not heretofore been given for the above general cases.

The remaining sections of this paper will serve to partially justify and extend these results. In Sec. II we give a heuristic derivation of the integral equation for the asymptotic ladder amplitude $\tilde{A}_L$ whose unique solution is of the Regge form. In Sec. III we give a careful derivation of an integral equation for $\tilde{A}_L$ whose solution is of the Regge form and which reduces to that of Sec. II in the approximation of taking $\alpha(t)$ to order g^2. An integral equation for the asymptotic B-S amplitude defined by the H kernel is discussed in Sec. IV. The general planar kernel is treated in Sec. V and the general trajectory function $\alpha(t)$ is exhibited. Nonplanar kernels are discussed in Sec. VI together with our conclusions. The Appendix contains some relevant mathematical results.

II. LADDER APPROXIMATION: HEURISTIC DISCUSSION

In order to display the essential simplicity of our method, we present in this section a heuristic discussion of what is involved. The justification of our procedure will be contained in the more rigorous discussion of Sec. III.

The B-S equation for the scattering amplitude with the initial particles off shell and the final particles on shell as shown in Fig. 9 is

$$A(s,t;p_1^2,p_2^2)=I(s,t;p_1^2,p_2^2,1,1)-\frac{i}{(2\pi)^4}\times\int d^4k\,\frac{I(k_1^2,t;\,p_1^2,p_2^2,k_2^2,k_4^2)A(k_3^2,t;\,k_2^2,k_4^2)}{(k_2^2-1+i\epsilon)(k_4^2-1+i\epsilon)}. \quad (2.1)$$

Our conventions are those of Ref. 21. Here $I(s,t;\,p_1^2,p_2^2,k_2^2,k_4^2)$ is the off-shell two-particle irreducible kernel, k_2 and k_4 are the momenta of the two internal lines, while k_1 and k_3 are the momentum transfers for I and A, respectively:

$$k_1=k-p_1,\quad k_2=k,\quad k_3=k-p_3,\quad k_4=k-p_1-p_2,$$
$$s=(p_1-p_3)^2,\quad t=(p_1+p_2)^2. \quad (2.2)$$

The HE limit is $s\rightarrow+\infty$ for fixed $t<0$. For convenience we have set the mass equal to unity. We also simplify the notation by setting

$$\kappa_i=k_i^2,\quad \rho_i=p_i^2,\quad i=1,2,3,4. \quad (2.3)$$

Specializing to the ladder approximation, we have

$$I(s,t;\rho_i)=-g^2/(s-1+i\epsilon). \quad (2.4)$$

We now change variables from the loop momentum k to the κ_i. The Jacobian is

$$J=\theta(D)/4D^{1/2},\quad D=-\det|2k_i\cdot k_j|. \quad (2.5)$$

The function D can be found explicitly and has the form

$$D=-s^2\Delta(\kappa_2,\kappa_4,t)+E(s,t,\kappa_i,\rho_i), \quad (2.6)$$

where Δ is the triangle function

$$\Delta(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz \quad (2.7)$$

and E is a polynomial in s of degree 1.

Our integral equation now becomes

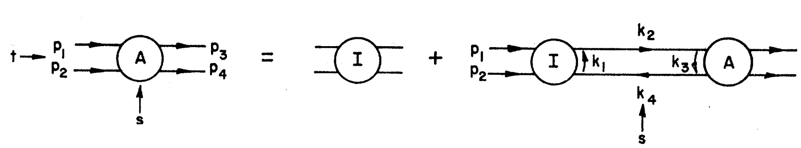
$$A(s,t;\rho_1,\rho_2)=\frac{-g^2}{s-m^2}+\frac{ig^2}{4(2\pi)^4}\times\int\frac{d\kappa_1d\kappa_2d\kappa_3d\kappa_4A(\kappa_3,t;\kappa_2,\kappa_4)}{(\kappa_1-1+i\epsilon)(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\frac{\theta(D)}{D^{1/2}}. \quad (2.8)$$

Assuming that the κ_i integrations are suitably convergent, we see from (2.6) that, for large s, Δ can only vanish at the edge of the integration region specified by $\theta(D)$. It then follows from (2.6) and (2.8) that, for large s, $A(s,t;\rho_1,\rho_2)$ is independent of the masses ρ_1 and ρ_2. Thus we have

$$A(s,t;\rho_1,\rho_2)\sim A'(s,t), \quad (2.9)$$

for large s. We mean by (2.9) that $A-A'$ vanishes for $s\rightarrow\infty$ faster than A or A'. Thus Eq. (2.8) becomes

FIG. 9. The B-S equation.



[21] R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, England, 1966).

for large s

$$A'(s,t)=-\frac{g^2}{s}+\frac{ig^2}{4(2\pi)^4}\int\prod_{i=1}^{4}d\kappa_i \times\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{(\kappa_1-1+i\epsilon)(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\frac{\theta(D)}{D^{1/2}}. \quad (2.10)$$

Now we assume that the major contribution to the integral in (2.10) comes from $\kappa_3\gg\kappa_2,\kappa_4$. This will be justified by the final solution and, more generally, by the analysis of Sec. III. Then, in view of (2.9), Eq. (2.10) becomes

$$A''(s,t)=-\frac{g^2}{s}+\frac{ig^2}{4(2\pi)^4}\int\prod_{i=1}^{4}d\kappa_i \times\frac{A''(\kappa_3,t)}{(\kappa_1-1+i\epsilon)(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\frac{\theta(D)}{D^{1/2}}, \quad (2.11)$$

where $A\sim A''$ for large s.

Our next step is to perform the κ_1 integration in (2.11). A standard Landau-type analysis shows that for large s the only important singularity of

$$\int\prod_{i=2}^{4}d\kappa_i\frac{A'(\kappa_3,t)}{(\kappa_1-1+i\epsilon)(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\frac{\theta(D)}{D^{1/2}} \quad (2.12)$$

in the lower half κ_1 plane comes from the pole at $\kappa_1=1$ and there is sufficient convergence to close the κ_1-integration contour there. After the κ_1 integration we arrive at

$$A'''(s,t)=-\frac{g^2}{s}+\frac{1}{4(2\pi)^3}g^2\int\prod_{i=2}^{4}d\kappa_i \times\frac{A'''(\kappa_3,t)}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\frac{\theta(D)}{D^{1/2}}, \quad (2.13)$$

where $A\sim A'''$.

Next we write

$$D^{1/2}\cong s(-\Delta)^{1/2}+E/2s(-\Delta)^{1/2} \quad (2.14)$$

and use theorem 1 of the Appendix to write[22]

$$\int_{-\infty}^{\infty}d\kappa_3\frac{A'''(\kappa_3,t)}{D^{1/2}}\cong\frac{1}{s(-\Delta)^{1/2}}\int_1^s d\kappa_3\,A'''(\kappa_3,t), \quad (2.15)$$

where we have chosen the lower limit to be 1 for convenience.[23] Thus we finally arrive at the integral equation

$$\tilde{A}(s,t)=-\frac{g^2}{s}-g^2K(t)\frac{1}{s}\int_1^s d\kappa\,\tilde{A}(\kappa,t), \quad (2.16)$$

where

$$K(t)=\frac{1}{32\pi^3}\int\frac{d\kappa_2d\kappa_4\theta(-\Delta)}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(-\Delta)^{1/2}} \quad (2.17)$$

and approximately

$$A\sim\tilde{A}$$

for large s. Note that Eqs. (1.2) and (2.17) define the same function $K(t)$.[24]

Equation (2.16) is of the Volterra type[25] and has the unique solution

$$\tilde{A}(s,t)=-g^2s^{-1+g^2K(t)}, \quad (2.18)$$

which is precisely the same function as (1.3) obtained by summing the asymptotic behaviors of the ladders in each order of g^2.

That the solution to Eq. (2.16) is the sum (1.3) of the terms $\tilde{A}_L^{(n)}$ given in (1.1) can also be seen, as follows: Since

$$\frac{1}{s}\int_1^s d\kappa\frac{(\ln\kappa)^n}{\kappa}=\frac{1}{n+1}\frac{(\ln s)^{n+1}}{s}, \quad (2.19)$$

the nth term in the Neumann series for (2.16) is precisely $\tilde{A}_L^{(n)}$, and since (2.16) is a Volterra integral equation, the sum of this series is its exact solution.[25] In view of this, Eq. (2.16) can be looked upon as a simple tool for the derivation of the rigorous relation (1.1). Whereas for the ladder graphs (1.1) is itself rather easy to derive, for more complicated planar graphs the analogs of Eq. (2.16) offer much simpler derivations of results, such as (1.1), than do the usual explicit treatments.

Our treatment above is incomplete for two reasons. Our derivation of Eq. (2.16) was only heuristic and the solution to (2.16) only corresponded to the exact trajectory function to order g^2. Both of these defects will be remedied in Sec. III, where a more exact integral equation will be derived whose solution is a Regge amplitude with a trajectory expressed as a power series in g^2 which agrees with that given above in order g^2.

[22] The theorem actually gives an upper limit of the form $sf(\kappa_2,\kappa_4,t)$ in this case. Using this more correct limit would only change our final trajectory function $\alpha(t)$ in orders g^4 and higher. See Sec. III.

[23] Clearly the choice of this lower limit cannot affect the HE behavior of A'''. Furthermore, we have neglected terms of order $1/s$ so that our inclusion of the inhomogeneous term in (2.13) is also done for convenience only. The fixing of the lower limit and inclusion of the inhomogeneous term will simply give us an equation with a unique solution. Alternatively, the solution can be made unique by the requirement that it reduce to the Born term in second order.

[24] That (1.2), (2.17), and (3.35) define the same function $K(t)$ can be seen by starting for the expression for the Feynman graph of Fig. 2 with two-dimensional loop momentum:

$$K(t)=-\frac{i}{16\pi^3}\int\frac{d^2k}{(k^2-m^2+i\epsilon)[(k-p)^2-m^2+i\epsilon]},\quad p^2=t.$$

By introducing Feynman parameters one is led to (1.2). If, instead, one changes variables to $\kappa_2=k^2$, $\kappa_4=(k-p)^2$, one is led to (3.35). Finally, if the k_0 contour is rotated by 90° first and then the variables $\kappa_2=\tilde{k}^2$, $\kappa_4=(\tilde{k}-\tilde{p})^2$ are introduced, where $\tilde{k}$ and $\tilde{p}$ are Euclidean 2-vectors, one obtains (2.17). Therefore they are all equivalent.

[25] See, for example, W. V. Lovitt, *Linear Integral Equations* (Dover Publications, Inc., New York, 1950).

III. LADDER APPROXIMATION: ANALYTIC DISCUSSION

Our starting point is Eq. (2.8) of Sec. II. Keeping in mind that as a result of (2.6) and (2.8) the asymptotic behavior of $A(s,t;\rho_1,\rho_2)$ is independent of the masses, we set the masses ρ_1 and ρ_2 equal to unity. Equation (2.8) then becomes

$$A(s,t)=-\frac{g^2}{s-1}+\frac{ig^2}{4(2\pi)^4}\int\prod_{i=1}^{4}d\kappa_i \times\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{(\kappa_1-1+i\epsilon)(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\frac{\theta(D)}{D^{1/2}}. \quad (3.1)$$

In this case the Jacobian simplifies somewhat and we present it in a form which is convenient for doing the κ_1 integral first:

$$D(s,t;\kappa_i;\rho_i=1)=A\kappa_1^2+B\kappa_1+C, \quad (3.2)$$

$$\begin{aligned}A&=-t(t-4),\\ B&=4st(\kappa_3-1)+2t(t-4)\kappa_3+2st^2-2st(\kappa_2+\kappa_4),\\ C&=-s^2\Delta(\kappa_2,\kappa_4,t)+2st^2\kappa_3-2st\kappa_3(\kappa_2+\kappa_4)\\ &\quad-t(t-4)\kappa_3^2-4st(\kappa_2+\kappa_3+\kappa_4-1)\\ &\quad+4s(\kappa_2-\kappa_4)^2+4st\kappa_2\kappa_4.\end{aligned} \quad (3.3)$$

There are no approximations in these expressions.

We now wish to evaluate the κ_1 integral

$$F\equiv\int\frac{d\kappa_1}{\kappa_1-1+i\epsilon}\frac{\theta(A\kappa_1^2+B\kappa_1+C)}{(A\kappa_1^2+B\kappa_1+C)^{1/2}}. \quad (3.4)$$

Since t is negative, $A=-t(t-4)<0$ and the integral will be zero unless

$$B^2-4AC>0. \quad (3.5)$$

Changing variables to $u=\kappa_1-1$, we have

$$F=\int\frac{du}{u+i\epsilon}\frac{\theta(A'u^2+B'u+C')}{(A'u^2+B'u+C')^{1/2}}, \quad (3.6)$$

where

$$A'=A,\quad B'=2A+B,\quad C'=A+B+C. \quad (3.7)$$

There are now three cases of interest as shown in Fig. 10. The integral can be done in each case to give

$$F=[\pi/(-C')^{1/2}]\theta(B^2-4AC). \quad (3.8)$$

Referring to Fig. 10, we note that in case (a), $C'<0$ and $B'>0$, F will be real and negative, since the square root in the integrand is positive and $u<0$ throughout the region of integration. Similarly, in case (c), $C'<0$ and $B'<0$, F will be real and positive. In case (b), $C'>0$, we see from (3.8) that F will be pure imaginary. But the sign of the $i\epsilon$ in (3.6) shows that F must have a negative imaginary part in this case. Summarizing, F

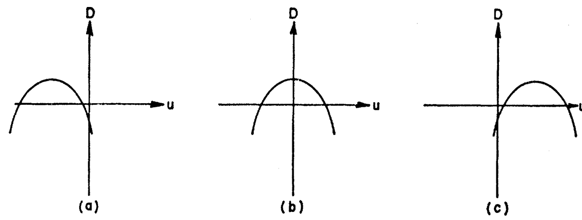


FIG. 10. The function $D=A'u^2+B'u+C'$. (a) $C'<0$, $B'>0$. (b) $C'>0$. (c) $C'<0$, $B'<0$.

must satisfy the following conditions:

$$\begin{aligned}&\text{(a)}\quad C'<0,\quad B'>0\Rightarrow F=-|F|,\\ &\text{(b)}\quad C'>0\Rightarrow F=-i|F|,\\ &\text{(c)}\quad C'<0,\quad B'<0\Rightarrow F=+|F|.\end{aligned} \quad (3.9)$$

We wish to do the κ_3 integral next, so we exhibit C' as a quadratic function of κ_3:

$$\begin{aligned}C'&=a\kappa_3^2+b\kappa_3+c,\\ a&=-t(t-4),\\ b&=2st^2-2st(\kappa_2+\kappa_4)+O(s^0),\\ c&=-s^2\Delta(\kappa_2,\kappa_4,t)+O(s).\end{aligned} \quad (3.10)$$

Only the leading high-s term for each coefficient need be kept, since we are taking the limit as $s\to\infty$. We have the following integral:

$$L=\int\frac{d\kappa_3\,A(\kappa_3,t;\kappa_2,\kappa_4)}{[-(a\kappa_3^2+b\kappa_3+c)]^{1/2}}\theta(B^2-4AC). \quad (3.11)$$

Now B^2-4AC can be written in the following way:

$$B^2-4AC=s^2(a'\kappa_3^2+b'\kappa_3+c')+O(s), \quad (3.12)$$

$$\begin{aligned}a'&=16t^2,\\ b'&=16t^2(t-\kappa_2-\kappa_4-2),\\ c'&=16t[\Delta(\kappa_2,\kappa_4,t)+t(\kappa_2\kappa_4+\kappa_2+\kappa_4-t+1)].\end{aligned} \quad (3.13)$$

Again we need only the leading s term. Since $a'>0$, $B^2-4AC>0$ for $\kappa_3\to\pm\infty$. The only question is whether the parabola in Eq. (3.12) has zeros or not. To this end we calculate $b'^2-4a'c'$ and find

$$b'^2-4a'c'=(16t)^2t(t-4)\Delta(\kappa_2,\kappa_4,t). \quad (3.14)$$

Thus, for $\Delta<0$, $b'^2-4a'c'<0$ and there will be no zeros. In this case B^2-4AC is positive for all real κ_3 and the integral in Eq. (3.11) goes over the whole real κ_3 axis. However, for $\Delta>0$, $b'^2-4a'c'>0$ and there will be a gap in the κ_3 integration between the points

$$\kappa_\pm=\tfrac{1}{2}\{t(\kappa_2+\kappa_4+2-t)\pm[t(t-4)\Delta(\kappa_2,\kappa_4,t)]^{1/2}\}. \quad (3.15)$$

It is therefore necessary to treat the two cases $\Delta<0$ and $\Delta>0$ separately.

A. $\Delta(\kappa_2,\kappa_4,t)<0$

$$L_-=\int_{-\infty}^{\infty}d\kappa_3\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{[-(a\kappa_3^2+b\kappa_3+c)]^{1/2}}. \quad (3.16)$$

Since $c\cong -s^2\Delta$ and $a<0$, the zeros of the square root will be real. However, going back to Eq. (3.6) and keeping a finite $i\epsilon$ in the integrand, we find after applying the standard analysis that the branch points are really both in the upper half-plane. To choose the correct branch of the square root we must apply the boundary conditions (3.9). Between the branch points, $C'=a\kappa_3^2+b\kappa_3+c>0$ and the square root must have the phase $e^{i\pi/2}$. Furthermore, B' is given by

$$B'=\kappa_3[4st+2t(t-4)]+O(1)\,, \tag{3.17}$$

so that $B'<0$ for $\kappa_3\to+\infty$ and $B'>0$ for $\kappa_3\to-\infty$. This means that the square root must be positive for $\kappa_3\to+\infty$ and negative for $\kappa_3\to-\infty$. A cut drawn between the branch points will reproduce these phases. The positions of the branch points are given by

$$[-b\pm(b^2-4ac)^{1/2}]/2a=sf_\pm+O(\sqrt{s})\,, \tag{3.18}$$

where

$$f_\pm(\kappa_2,\kappa_4,t)=\{t(\kappa_2+\kappa_4-t)\pm[4t(\Delta+t\kappa_2\kappa_4)]^{1/2}\}/[-t(t-4)]. \tag{3.19}$$

To go further we must make two assumptions about the analytic structure of $A(\kappa_3,t;\kappa_2,\kappa_4)$. Each individual ladder graph has only a right-hand cut in κ_3 for κ_2 and κ_4 near 1. We assume that $A(\kappa_3,t;\kappa_2\kappa_4)$, which is the infinite sum of such graphs, has no new singularities in the upper half-plane coming from the sum. Our other assumption is that we can neglect complex singularities which occur in perturbation theory when κ_2 and κ_4 are varied. Some justification for this is provided by the fact that the κ_2 and κ_4 integrals have good convergence properties because of the presence of the propagator poles and the eventual $[\Delta(\kappa_2,\kappa_4,t)]^{1/2}$ in the denominator. Therefore most of the contribution comes from κ_2,κ_4 near 1, where there are no complex singularities.

Under these assumptions the analytic structure of the integrand of L in the κ_3 plane is shown in Fig. 11. The contour goes over the right-hand cut because it is the physical amplitude that appears in the integral. Assuming that $A(\kappa_3,t;\kappa_2,\kappa_4)\to 0$ as $\kappa_3\to\infty$, we can close the contour in the upper half-plane and get

$$L_-=\frac{-2i}{(-a)^{1/2}}\int_{sf_+}^{sf_-}\frac{d\kappa_3\,A(\kappa_3,t;\kappa_2,\kappa_4)}{[(\kappa_3-sf_+)(sf_--\kappa_3)]^{1/2}}, \tag{3.20}$$

where the square root is now defined to be positive.

B. $\Delta(\kappa_2,\kappa_4,t)>0$

In this case the κ_3 integral goes from $-\infty$ to κ_- and from κ_+ to $+\infty$, with $\kappa_\pm$ given in (3.15). We have

$$L_+=\left(\int_{-\infty}^{\kappa_-}d\kappa_3+\int_{\kappa_+}^{\infty}d\kappa_3\right)\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{[-(a\kappa_3^2+b\kappa_3+c)]^{1/2}}. \tag{3.21}$$

For $\Delta(\kappa_2,\kappa_4,t)$ sufficiently positive, $b^2-4ac<0$ and the branch points of the square root will be off the real axis at complex conjugate points in the κ_3 plane. The boundary conditions (3.9) can be satisfied by drawing the cut between the branch points and through the gap in the κ_3-integration contour. This is illustrated in Fig. 12.

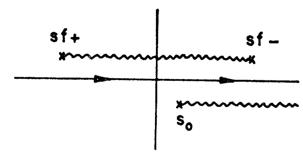


FIG. 11. Singularity structure for $\Delta<0$.

To check that this is consistent with the singularity structure of Fig. 11 for $\Delta<0$, we fix $b\propto(\kappa_2+\kappa_4-t)$ and let Δ decrease through zero. For $b<0$ the situation will be as shown in Fig. 12, with the branch points to the left of the gap. As Δ is decreased, the branch points will come to the real axis and move apart along the real axis. The branch point in the lower half-plane will move to the right as shown in Fig. 13. As Δ decreases further, the gap closes as the lower branch point approaches it. At $\Delta=0$ the gap has closed and the lower branch point has just gotten through into the upper half-plane.

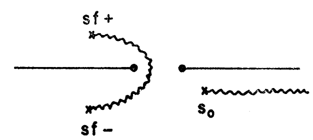


FIG. 12. Singularity structure for $\Delta>0$.

To evaluate the integrals in (3.21) we consider the configuration of Fig. 13 and add and subtract an integration contour that goes around the lower branch cut to get

$$L_+=\left(\int_{C_1}d\kappa_3-\int_{C_2}d\kappa_3\right)\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{[-(a\kappa_3^2+b\kappa_3+c)]^{1/2}}, \tag{3.22}$$

where C_1 and C_2 are shown in Fig. 14. The contour C_1 can be closed in the upper half-plane and gives the integral around the cut. The contour C_2 gives the integral around the lower branch cut plus a small integral along the real axis. This last piece can be neglected, since $\kappa_\pm$ are s-independent and this integral will give a term behaving as s^{-1}, while the dominant terms will turn out to be s^α and $\alpha>-1$. After evaluating the

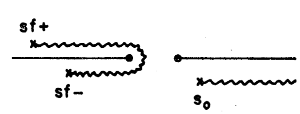


FIG. 13. Movement of the branch points as Δ decreases.

(a) (b)

FIG. 14. Contours C_1 and C_2. (a) C_1. (b) C_2.

discontinuities, we get

$$L_+=\frac{-2i}{(-a)^{1/2}}\int_{sf_+}^{sf_-}\frac{d\kappa_3\,A(\kappa_3,t;\kappa_2,\kappa_4)}{[(\kappa_3-sf_+)(sf_--\kappa_3)]^{1/2}}-\frac{2}{(-a)^{1/2}}\times\int_{sf_-}^{h}d\kappa_3\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{[(\kappa_3-sf_+)(\kappa_3-sf_-)]^{1/2}},$$
$$h=h(\kappa_2,\kappa_4,t)=\tfrac{1}{2}t(\kappa_2+\kappa_4+2-t)\,, \tag{3.23}$$

where both square roots are real and positive. The result for configurations other than Fig. 13 is obtained by analytic continuation.

Putting together our results for $\Delta<0$ and $\Delta>0$, the integral equation (3.1) becomes for large s

$$\tilde{A}(s,t)=\frac{-g^2}{s}+\frac{g^2\pi}{2(2\pi)^4}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(-a)^{1/2}}\times\int_{sf_+}^{sf_-}d\kappa_3\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{[(\kappa_3-sf_+)(sf_--\kappa_3)]^{1/2}}-\frac{ig^2\pi}{2(2\pi)^4}\times\int\frac{d\kappa_2 d\kappa_4\theta(\Delta(\kappa_2,\kappa_4,t))}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(-a)^{1/2}}\times\int_{sf_-}^{h}d\kappa_3\frac{A(\kappa_3,t;\kappa_2,\kappa_4)}{[(\kappa_3-sf_+)(\kappa_3-sf_-)]^{1/2}}. \tag{3.24}$$

A simplification occurs in the integral from sf_+ to sf_- because we can replace $A(\kappa_3,t;\kappa_2,\kappa_4)$ with its asymptotic form $\tilde{A}(\kappa_3,t)$ under the integral. For $\Delta>0$ this is obvious, because sf_+ and sf_- have the same sign and the integral is only over the asymptotic region, since s is very large. For $\Delta<0$ the contour goes through $\kappa_3=0$ but it can be distorted into a semicircle in the upper half-plane as shown in Fig. 15. On this contour, A can be replaced with its asymptotic form and the contour can then be shrunk back to the real axis. The integral from sf_- to h in (3.24) can also be simplified by breaking it up into two parts. We integrate from sf_- to $l(\kappa_2,\kappa_4,t)$,

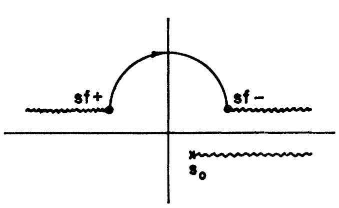


FIG. 15. Distortion of the integration contour for $\Delta<0$.

where $A(\kappa_3,t;\kappa_2,\kappa_4)$ can be replaced with its asymptotic form for $\kappa_3<l(\kappa_2,\kappa_4,t)<0$. The piece from $l(\kappa_2,\kappa_4,t)$ to $h(\kappa_2,\kappa_4,t)$ can be dropped, because it only gives a term proportional to s^{-1}.

The integral equation then becomes

$$\tilde{A}(s,t)=\frac{-g^2}{s}-\frac{ig^2\pi}{2(2\pi)^4}\int\frac{d\kappa_2 d\kappa_4\theta\,(\Delta)}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(-a)^{1/2}}\times\int_{sf_-}^{l}\frac{d\kappa_3\tilde{A}(\kappa_3,t)}{[(\kappa_3-sf_+)(\kappa_3-sf_-)]^{1/2}}+\frac{g^2\pi}{2(2\pi)^4}\times\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(-a)^{1/2}}\times\int_{sf_+}^{sf_-}d\kappa_3\frac{\tilde{A}(\kappa_3,t)}{[(\kappa_3-sf_+)(sf_--\kappa_3)]^{1/2}}. \tag{3.25}$$

We now claim that the solution to this is of the form

$$\tilde{A}(s,t)=-g^2 s^{\alpha(t)} \tag{3.26}$$

and that it is unique.[26] To see that this is the solution we substitute it in (3.25). First look at the second term on the right side of (3.25). We expand the square root in a power series as follows:

$$\frac{1}{(\kappa_3-sf_+)^{1/2}}=\frac{1}{(-sf_+)^{1/2}}\sum_{i=0}^{\infty}a_i\left(\frac{\kappa_3}{-sf_+}\right)^i,$$
$$\frac{1}{(\kappa_3-sf_-)^{1/2}}=\frac{1}{(-sf_-)^{1/2}}\sum_{j=0}^{\infty}a_j\left(\frac{\kappa_3}{-sf_-}\right)^j, \tag{3.27}$$
$$\frac{1}{[(\kappa_3-sf_+)(\kappa_3-sf_-)]^{1/2}}=\frac{1}{s(f_+f_-)^{1/2}}\sum_{n=0}^{\infty}\left(\frac{\kappa_3}{s}\right)^n A_n,$$

where

$$a_i=\frac{(2i-1)!!}{i!2^i},$$
$$A_n=\sum_{i=0}^{n}\frac{a_i a_{n-i}}{(-f_+)^i(-f_-)^{n-i}}. \tag{3.28}$$

Note that

$$-af_+f_-=\Delta. \tag{3.29}$$

The power series (3.27) converges for κ_3 in the range of integration and there is no difficulty with the end point $\kappa_3=f_-$, because the integral of the square root converges. Putting this in the sf_--l integral and integrating

[26] Since (3.25) is only an equation for the asymptotic amplitude $\tilde{A}$, we mean that we have a solution if, when we substitute it on the right-hand side, we get back the left-hand side plus terms which are of lower order as $s\to\infty$. The uniqueness is ensured by requiring that $\tilde{A}(s,t)$ reduce to the Born term $-g^2s^{-1}$, in first order of g^2. Note that $\lambda s^\alpha \ln^p(s)$ would satisfy the first criterion of a solution but that the second allows only $-g^2s^\alpha$.

term by term, we get

$$-g^2s^{\alpha(t)}\left(\sum_{n=0}^{\infty}\frac{ig^2}{32\pi^2}\times\int\frac{d\kappa_2 d\kappa_4\theta(\Delta)A_n f_-^{\alpha(t)+n+1}}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(\Delta)^{1/2}[\alpha(t)+n+1]}\right)+O(s^{-1}),\quad(3.30)$$

where the s^{-1} term comes from the upper limit $l(\kappa_2,\kappa_4,t)$ and can be neglected as long as $\alpha(t)>-1$.

The last term on the right side of (3.25) also can be handled by expanding the square root in a power series, although care must be taken because different power series are needed for different parts of the region on integration. Again a factor $-g^2s^{\alpha(t)}$ will be common to all the terms and will therefore cancel on both sides of (3.25). There is no other s dependence left, and therefore $s^{\alpha(t)}$ is a solution, provided that all the series that are left converge.[27]

We are left with an integral equation for $\alpha(t)$:

$$1=\sum_{n=0}^{\infty}\left(\frac{ig^2}{32\pi^3}\times\int\frac{d\kappa_2 d\kappa_4\theta(\Delta)A_n f_-^{\alpha(t)+n+1}}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(\Delta)^{1/2}[\alpha(t)+n+1]}\right)+\text{contribution from last term}.\quad(3.31)$$

The contribution from the last term could be found explicitly as indicated above, but it will not contribute to the order-g^2 expression for $\alpha(t)$. To see this, separate out the $n=0$ term from (3.31) and multiply through by $\alpha(t)+1$:

$$\alpha(t)+1=\frac{ig^2}{32\pi^3}\int\frac{d\kappa_2 d\kappa_4\theta(\Delta)f_-^{\alpha(t)+1}}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(\Delta)^{1/2}}+[\alpha(t)+1]\sum_{n=1}^{\infty}\frac{ig^2}{32\pi^3}\times\int\frac{d\kappa_2 d\kappa_4\theta(\Delta)A_n f_-^{\alpha(t)+n+1}}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)(\Delta)^{1/2}[\alpha(t)+n+1]}+[\alpha(t)+1]\times\text{contribution from last term in (3.25)}.\quad(3.32)$$

[27] Since $\int_0 dx/x^{1/2}$ converges at $x=0$, we could change the lower limit in (3.25) from sf_- to $sf_--\delta$, $\delta\to+0$. The power series (3.30) would then converge uniformly in the range of integration, and therefore the integrated series converges, provided that the integrals all exist. Noting that both f_- and f_+ are linear in κ_2 and κ_4, the integrals in (3.33) will converge at $\kappa_2,\kappa_4\to\infty$ for $\alpha(t)<0$. This defines the region of validity of (3.33) as $-1<\alpha(t)<0$. One might also worry about the integral failing to converge at the boundary $\Delta=0$, since the A_n defined by (3.28) contain some factors of $\Delta^{-m/2}$, $0<m\leq n$. This divergence is spurious, since it is not present in (3.25), and it can be removed by a technique described in Secs. IV and V which is essentially integration by parts. It is understood that we do this wherever necessary in (3.31).

As $g^2\to0$, $\alpha(t)\to-1$. To order g^2 the second term in (3.32) drops out and the last term contributes only if there is a pole at $\alpha(t)=-1$ from the last term in (3.25). The only relevant integral is

$$\frac{1}{s(f_+f_-)^{1/2}}\int_{-sf_-}^{sf_-}d\kappa_3\kappa_3^{\alpha(t)}=\frac{s^{\alpha(t)}[f_-^{\alpha(t)+1}-(-f_-)^{\alpha(t)+1}]}{(f_+f_-)^{1/2}[\alpha(t)+1]}.\quad(3.33)$$

As $\alpha(t)\to-1$, there is no pole because the numerator vanishes, and therefore this piece gives no contribution to order g^2. It will contribute to order g^4 and higher.

Finally, to get the trajectory function to order g^2 from the first term in (3.32) we must set $f_-^{\alpha(t)+1}=f_-^{\alpha^{(0)}(t)+1}=1$ under the integral. This gives

$$\alpha_L^{(1)}=-1+g^2K(t),\quad(3.34)$$

$$K(t)=\frac{i}{32\pi^3}\times\int\frac{d\kappa_2 d\kappa_4\theta(\Delta(\kappa_2,\kappa_4,t))}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)\Delta^{1/2}(\kappa_2,\kappa_4,t)},\quad(3.35)$$

which agrees with the heuristic result of Sec. II and with the standard result from summing diagrams.[24]

IV. *H* KERNEL

Having shown how Eq. (2.18) arises in a well-defined approximation from the correct integral equation, we return in this section to the heuristic type of arguments employed in Sec. II and consider the planar H kernel. More rigorous arguments analogous to those of Sec. III will not be given here, but, upon iteration, our equation will be seen to reproduce the correct HE behavior in each order. Thus, on the one hand, we can justify summation of leading behaviors for these graphs, and, on the other hand, we can discuss the general case (Sec. V). The ultimate justification of our approach must, of course, be based on more careful analysis and, as we shall show in Sec. VI, such analysis is essential for an understanding of nonplanar kernels.

We first consider the B-S equation corresponding to the H-exchange graphs of Fig. 4. The appropriate kernel $I_H(s,t;\rho_i)$ is given by the graph of Fig. 4 for $n=0$ and, according to (1.5), satisfies

$$I_H(s,t;\rho_i)\sim H(t)s^{-2}\ln s,\quad(4.1)$$

for large s. Now, I_H satisfies a dispersion relation with only a right-hand cut:

$$I_H(s,t;\rho_i)=\int_0^{\infty}da\frac{\sigma(a,t;\rho_i)}{s-a}.\quad(4.2)$$

The relation (4.2) is strictly valid only for the masses ρ_i sufficiently small. We shall, however, assume that it is correct for all ρ_i, relying on the damping of relevant integrals for large ρ_i.

Since $I_H \sim s^{-2}$, we have

$$\int_0^\infty da\,\sigma(a,t;\rho_i)=0\,.$$

Thus application of theorem 2 of the Appendix to (4.2) gives

$$I_H(s,t;\rho_i)\sim s^{-2}\int_0^s da\,\sigma(a,t;,\rho_i)a\,. \qquad (4.3)$$

Comparison with (4.1) now gives

$$\int_0^s da\,\sigma(a,t;\rho_i)a=H(t)\ln s\,. \qquad (4.4)$$

With the kernel (4.2) the B-S equation (2.1) becomes

$$A_H(s,t;\rho_1,\rho_2)=I_H(s,t,\rho_1,\rho_2)-\frac{i}{4(2\pi)^4}\int da$$
$$\times\int\frac{d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4\sigma(a,t;\rho_1,\rho_2,\kappa_2,\kappa_4)A_H(\kappa_3,t;\kappa_2,\kappa_4)}{(\kappa_1-a+i\epsilon)(\kappa_2-1)(\kappa_4-1)}\frac{\theta(D)}{D^{1/2}}. \qquad (4.5)$$

We assume that, as before, the κ_1-integration contour can be closed in the lower half-plane and that only the propagator pole is important for large s. Thus our first HE approximation to (4.5) becomes

$$A_H'(s,t;\rho_1,\rho_2)=H(t)\frac{\ln s}{s^2}+\frac{1}{4(2\pi)^3}$$
$$\times\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1)(\kappa_4-1)}\int d\kappa_3\int da\,\sigma(a,t;\rho_1,\rho_2,\kappa_2,\kappa_4)$$
$$\times A_H'(\kappa_3,t;\kappa_2,\kappa_4)\left(\frac{\theta(D)}{D^{1/2}}\right)_{\kappa_1=a}. \qquad (4.6)$$

We next want to use (2.14) and apply theorem 4 of the Appendix, so that we need the term in E proportional to $sa\kappa_3$. One finds

$$(E)_{\kappa_1=a}=4sta\kappa_3+\cdots,$$

so that (2.12) becomes

$$D^{1/2}\cong s(-\Delta)^{1/2}+2t(-\Delta)^{-1/2}a\kappa_3\,. \qquad (4.7)$$

Thus (4.6) becomes essentially ($\kappa=\kappa_3$)

$$A_H''(s,t;\rho_1,\rho_2)=H(t)\frac{\ln s}{s^2}+\frac{1}{4(2\pi)^3}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1)(\kappa_4-1)}$$
$$\times\int_1^s d\kappa\int_0^{s/\kappa}da\,\sigma(a,t;\rho_1,\rho_2,\kappa_2,\kappa_4)$$
$$\times A_H''(\kappa,t;\kappa_2,\kappa_4)\frac{2ta\kappa\theta(-s^2\Delta-\eta)}{s^2(-\Delta)^{3/2}}, \qquad (4.8)$$

where the remainder η in the argument of θ has been kept in order to avoid the spurious divergence of the κ_2,κ_4 integration at $\Delta=0$.

In view of (4.4), this becomes

$$A_H''(s,t;\rho_1,\rho_2)=H(t)\frac{\ln s}{s^2}+\frac{2tH(t)}{4(2\pi)^3s^2}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1)(\kappa_4-1)}$$
$$\times\frac{\theta(-s^2\Delta-\eta)}{(-\Delta)^{3/2}}\int_1^s d\kappa\,A_H''(\kappa,t;\kappa_2,\kappa_4)\kappa\ln(s/\kappa)\,. \qquad (4.9)$$

Since the right-hand side of (4.9) is independent of the masses ρ_1 and ρ_2, we have $A_H''(s,t;\rho_1,\rho_2)\sim\tilde{A}_H(s,t)$ for large s. We now introduce the final assumption, to be verified by the solution, that the integration region with $\kappa\gg\kappa_2,\kappa_4$ gives the major contribution to (4.9) for large s. Then we obtain as our final integral equation

$$\tilde{A}_H(s,t)=H(t)\frac{\ln s}{s^2}+K_2(t)$$
$$\times H(t)\frac{1}{s^2}\int_1^s d\kappa\,\tilde{A}_H(\kappa,t)\kappa\ln(s/\kappa)\,, \qquad (4.10)$$

where[28]

$$K_2(t)=\lim_{s\to\infty}\frac{2t}{4(2\pi)^3}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-m^2)(\kappa_4-m^2)}\frac{\theta(-s^2\Delta-\eta)}{(-\Delta)^{3/2}} \qquad (4.11)$$
$$=\lim_{s\to\infty}\frac{1}{4(2\pi)^3}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-m^2)(\kappa_4-m^2)}$$
$$\times\theta(-s^2\Delta-\eta)\left(-\frac{\partial}{\partial\kappa_2}-\frac{\partial}{\partial\kappa_4}\right)(-\Delta)^{-1/2}$$
$$=\lim_{s\to\infty}\frac{1}{4(2\pi)^3}\int d\kappa_2 d\kappa_4\frac{\theta(-s^2\Delta-\eta)}{(-\Delta)^{1/2}}$$
$$\times\left(\frac{\partial}{\partial\kappa_2}+\frac{\partial}{\partial\kappa_4}\right)(\kappa_2-m^2)^{-1}(\kappa_4-m^2)^{-1}$$
$$=\frac{-1}{4(2\pi)^3}\int d\kappa_2 d\kappa_4\frac{\theta(-\Delta)}{(-\Delta)^{1/2}}\frac{\partial}{\partial m^2}$$
$$\times(\kappa_2-m^2)^{-1}(\kappa_4-m^2)^{-1} \qquad (4.12)$$
$$=-\frac{\partial}{\partial m^2}K(t)$$
$$=K'(t)\,. \qquad (4.13)$$

In deriving (4.13) from (4.11) we have neglected the contribution from the apparent $\delta(-s^2\Delta-\eta)$ term, since, as is clear from (4.6), the neighborhood of $\Delta=0$ does not contribute to the asymptotic behavior of A_H. In (4.12) the singularity at $\Delta=0$ is integrable, so that the $\eta\to 0$ limit can be taken directly.

[28] Previously we set the mass m equal to unity. We now reinstate the symbol m for purposes of differentiation.

Our integral equation (4.10) is again Volterra and has the unique solution

$$\tilde{A}_H(s,t)=\tfrac{1}{2}H(t)[R(t)]^{-1}(s^{-2+R(t)}-s^{-2-R(t)}), \quad (4.14)$$

where

$$R(t)=[K_2(t)H(t)]^{1/2}. \quad (4.15)$$

Thus, in view of (4.13), we obtain complete agreement with the result (1.6) obtained by summation of leading HE behaviors in each order. Correspondingly, iteration of (4.10) n times gives the result (1.5).

V. GENERAL PLANAR KERNEL

The analysis for an arbitrary planar kernel is a simple generalization of that given in Sec. IV. Thus consider the amplitude $I(s,t;\rho_1,\rho_2,\kappa_2,\kappa_4)$ corresponding to an arbitrary sum of two-particle irreducible planar Feynman diagrams. Since planar diagrams in φ^3 theory have no left-hand cuts in s, we assume that I satisfies a dispersion relation

$$I(s,t;\rho_1,\rho_2,\kappa_2,\kappa_4)=\int_0^\infty da\frac{\sigma(a,t;\rho_1,\rho_2,\kappa_2,\kappa_4)}{s-a}, \quad (5.1)$$

and has a large-s behavior of the form

$$I(s,t;\rho_1,\rho_2,\kappa_2,\kappa_4)\sim I(t)s^{-q}(\ln s)^p, \quad (5.2)$$

with integer $p\geq 0$ and $q\geq 1$. In (5.1) we have neglected the complex singularities which occur when the masses are sufficiently large and positive. We believe that this is justified by virtue of the rapid convergence of the mass integrations. In (5.2) we have assumed that the asymptotic behavior is independent of the masses. This is done for simplicity only, since the alternative case would also result in Regge behavior with an integral equation for the trajectory function.

We see that the class of kernels for which we shall establish Regge asymptotic behavior is larger than the class (described in Sec. I) given by summation of Feynman diagrams for two reasons. First, we make no explicit restrictions on the d lines and, second, our kernels need not even be given by Feynman diagrams, as long as (5.1) and (5.2) are satisfied.

It follows from (5.1) and (5.2) that

$$\int_0^\infty da\,\sigma(a,\cdots)a^n=0 \quad \text{for } 0\leq n\leq q-1, \quad (5.3)$$

and so, by theorem 2 of the Appendix, we have

$$\int_0^s da\,\sigma(a,\cdots)a^{q-1}\cong I(t)(\ln s)^p. \quad (5.4)$$

Now we proceed exactly as for the H kernel and write, as our first HE approximation to the B-S equation,

$$A'(s,t;\rho_1,\rho_2)=I(t)s^{-q}(\ln s)^p+\frac{1}{4(2\pi)^3}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1)(\kappa_4-1)}$$
$$\times\int d\kappa\int da\,\sigma(a,\cdots)A'(\kappa,t;\kappa_2,\kappa_4)\left(\frac{\theta(D)}{D^{1/2}}\right)_{\kappa_1=a}. \quad (5.5)$$

We next use

$$D\cong -s^2\Delta(\kappa_2,\kappa_4,t)+4sta\kappa \quad (5.6)$$

and write

$$D^{-1/2}\cong s^{-1}(-\Delta)^{-1/2}\left[1+\sum_{r=1}^\infty\frac{(2r-1)!!}{r!}\left(-\frac{2ta\kappa}{s\Delta}\right)^r\right]. \quad (5.7)$$

This expansion is, of course, not valid for $\Delta\sim 0$ and we must keep in mind that this region is not important in determining the asymptotic behavior of the integral (5.5). Equation (5.3) and theorem 4 tell us that the $r=q-1$ term in (5.7) gives the leading contribution, so that (5.5) becomes

$$A''(s,t;\rho_1,\rho_2)=I(t)s^{-q}(\ln s)^p+\frac{1}{4(2\pi)^3}\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1)(\kappa_4-1)}$$
$$\times\int_1^s d\kappa\int_0^{s/\kappa}da\sigma(a,\cdots)A''(\kappa_3,t;\kappa_2,\kappa_4)s^{-1}(-\Delta)^{-1/2}$$
$$\times\frac{(2q-3)!!}{(q-1)!}\left(\frac{-2ta\kappa}{s\Delta}\right)^{q-1}\theta(-s^2\Delta-\eta), \quad (5.8)$$

where the η term in θ is again a symbolic prescription to ignore contributions from the region $\Delta\sim 0$. Our final step is to use (5.4) in (5.8), note that $A''(s,t;\rho_1,\rho_2)\sim\tilde{A}(s,t)$ for large s, and assume that the integration region with $\kappa\gg\kappa_2,\kappa_4$ gives the leading contribution. Thus we obtain

$$\tilde{A}(s,t)=I(t)s^{-q}(\ln s)^p+K_q(t)I(t)s^{-q}$$
$$\times\int_1^s d\kappa\left(\ln\frac{s}{\kappa}\right)^p\kappa^{q-1}\tilde{A}(\kappa,t), \quad (5.9)$$

where

$$K_q(t)=\lim_{s\to\infty}\frac{(2q-3)!!}{(q-1)!}\frac{(2t)^{q-1}}{4(2\pi)^3}$$
$$\times\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-m^2)(\kappa_4-m^2)}\frac{\theta(-s^2\Delta-\eta)}{(-\Delta)^{q-1/2}}. \quad (5.10)$$

In order to put (5.10) in a form which is manifestly independent of nonintegrable singularities at $\Delta=0$, we use the relation

$$\left(\frac{\partial}{\partial\kappa_2}+\frac{\partial}{\partial\kappa_4}\right)\Delta=-4t$$

to write

$$\left(-\frac{\partial}{\partial\kappa_2}-\frac{\partial}{\partial\kappa_4}\right)^{q-1}(-\Delta)^{-1/2}$$

$$=(2q-3)!!(2t)^{q-1}(-\Delta)^{-q+1/2}, \quad (5.11)$$

and we integrate by parts:

$$K_q(t)=\frac{1}{4(2\pi)^3(q-1)!}\int d\kappa_2 d\kappa_4\frac{\theta(-\Delta)}{(-\Delta)^{1/2}}$$

$$\times\left(\frac{\partial}{\partial\kappa_2}+\frac{\partial}{\partial\kappa_4}\right)^{q-1}(\kappa_2-m^2)^{-1}(\kappa_4-m^2)^{-1}$$

$$=\frac{1}{(q-1)!}\left(\frac{-\partial}{\partial m^2}\right)^{q-1}K(t). \quad (5.12)$$

In accordance with our remarks above, we have omitted the $\delta(-s^2\Delta-\eta)$ terms, since they only give contributions corresponding to $\Delta\sim 0$.

Our general integral equation (5.9) is of the Volterra type and its unique solution is of the Regge form. We shall only exhibit the leading trajectory function $\alpha(t)$. Using

$$\int^s d\kappa\,\kappa^{-1+Q}\left(\ln\frac{s}{\kappa}\right)^m\sim m!\frac{s^Q}{Q^{m+1}}, \quad (5.13)$$

we find that the solution to (5.9) is asymptotically proportional to $s^{\alpha(t)}$, with

$$\alpha(t)=-q+R(t), \quad (5.14)$$

where

$$R(t)=[q!K_q(t)I(t)]^{1/(p+1)}. \quad (5.15)$$

Thus, in our approximation, every kernel I satisfying (5.1) and (5.2) defines a B-S amplitude $A(s,t)$ with a Regge asymptotic behavior $\tilde{A}(s,t)\sim s^{\alpha(t)}$, with $\alpha(t)$ given by (5.14), (5.15), and (5.12). This might be expected on the grounds that the absence of a left-hand cut in (5.1) and the fast decrease ($q\geq 1$) in (5.2) represent simple generalizations of potential scattering from nonsingular potentials—a situation which is known to Reggeize. Moreover, it is known that inclusion of a left-hand cut[29] or more singular kernels[8] need *not* lead to Regge asymptotic behavior.

The essential reason why the solution to (5.9) has the Regge form is that the power of κ inside the integral is 1 less than the power of s^{-1} outside the integral. Indeed, the integral

$$s^{-n}\int^s d\kappa\left(\ln\frac{s}{\kappa}\right)^p\kappa^{q-1}\tilde{A}(\kappa)$$

will lead to Regge asymptotic behavior only if $n=q$. The occurrence of $n=q$ in our integral equation can be traced back to Eqs. (5.6) and (5.7) for D, where the ratios κ^r/s^{r+1} occur. The structure of D is, of course, intimately connected to the two-particle structure of the B-S equation.

[29] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters 2, 239 (1962).

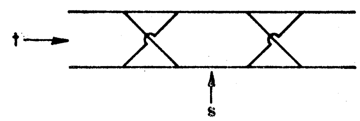


FIG. 16. Iterated X diagram.

VI. NONPLANAR KERNELS AND CONCLUSIONS

We have given a heuristic argument to show that the B-S amplitude corresponding to a planar kernel with mass-independent HE behavior will Reggeize. The trajectory functions so obtained agree with the known results of summing the asymptotic forms of iterated planar graphs. In fact, when the asymptotic integral equation (5.9) is iterated, it produces exactly the HE behavior for the iterated planar graphs that is obtained in the usual analysis.[3,5,7] This is the main justification of the heuristic method.

Because of the simple analytic structure of planar graphs (only right-hand cuts) we believe that a more rigorous justification for the Regge behavior in the planar case could be given along the lines of the argument in Sec. III for the ladder. This is not true for nonplanar kernels. *A priori* one would expect the heuristic argument to show that nonplanar kernels also produce Regge asymptotic behavior, since all that was really needed in Sec. V was the HE behavior of the kernel. However, a closer inspection of the analytic behavior of the nonplanar kernel shows that the heuristic argument will not work.

To see what goes wrong, consider the simpler example of obtaining the HE behavior of the iterated X diagram shown in Fig. 16. In the Feynman parameter analysis, one finds that the naive "edge" contribution gives an HE behavior $s^{-2}\ln^5 s$, while the correct behavior, s^{-1}, is obtained from a "pinch" of the Feynman parameter hypercontour.[3,9,11] A similar effect occurs in our formalism. The diagram is proportional to

$$G=\int\frac{\prod_{i=1}^4 d\kappa_i\, X(\kappa_1,t;\rho_1,\rho_2,\kappa_2,\kappa_4)}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}$$

$$\times X(\kappa_3,t;\kappa_2,\kappa_4,\rho_3,\rho_4)\frac{\theta(D)}{D^{1/2}}, \quad (6.1)$$

where

$$X(s,t;\rho_1,\rho_2,\rho_3,\rho_4)=\int_{s_R}^{\infty}\frac{\sigma(s',t;\rho_1,\rho_2,\rho_3,\rho_4)ds'}{s-s'+i\epsilon}$$

$$+\int_{-\infty}^{s_L}\frac{\sigma(s',t;\rho_1,\rho_2,\rho_3,\rho_4)ds'}{s-s'-i\epsilon}. \quad (6.2)$$

The crucial point is the sign of the $i\epsilon$'s. The κ_1 and κ_3 integrations in (6.2) must go over the right-hand cuts of X and under the left-hand cuts. (Of course, we still neglect complex singularities.) The fact that one must go under the left-hand cut can be seen directly from the Mandelstam representation for the Feynman amplitude X.

The high-s behavior of X is known to be[3,9,11]

$$X(s,t;\rho_i)\sim J(t)[\ln^2(s)]/s^2, \tag{6.3}$$

where we need not specify $J(t)$, and by theorem 2 of the Appendix we have

$$\int_{-\infty}^{s_L}\sigma(s',t;\rho_i)ds'+\int_{s_R}^{\infty}\sigma(s',t;\rho_i)ds'=0, \tag{6.4}$$

$$\int_{-s}^{s_L}s'\sigma(s',t;\rho_i)ds'+\int_{s_R}^{s}s'\sigma(s',t;\rho_i)ds' =J(t)\ln^2(s). \tag{6.5}$$

Putting the spectral forms into (6.1), we obtain

$$I=\int\frac{d\kappa_2 d\kappa_4}{(\kappa_2-1+i\epsilon)(\kappa_4-1+i\epsilon)}\int_{-\infty}^{\infty}ds'ds''\sigma(s')\sigma(s'')\times\int\frac{d\kappa_1 d\kappa_3\theta(D)}{(\kappa_1-s'\pm i\epsilon)(\kappa_3-s''\pm i\epsilon)D^{1/2}}, \tag{6.6}$$

where we have suppressed some of the arguments. If we proceeded naively and neglected the $i\epsilon$'s, the last two integrals can be done and yield a factor of $\ln s$ and an appropriate square-root denominator. When this square root is expanded, the first term gives

$$\int_{-\infty}^{\infty}\sigma(s')ds'\int_{-\infty}^{\infty}\sigma(s'')ds'=0$$

[from (6.4)] and the next term gives a factor of $\ln^2 s$ [from (6.5)] for each of the s' and s'' integrals. The end result is $s^{-2}\ln^5(s)$, which is wrong.

Being more careful, we have the following integrals to evaluate:

$$\int_{s_R}^{\infty}ds'\sigma(s')\int\frac{d\kappa_1}{\kappa_1-s'+i\epsilon}\frac{\theta(a\kappa_1^2+b\kappa_1+c)}{(a\kappa_1^2+b\kappa_1+c)^{1/2}}+\int_{-\infty}^{s_L}ds'\sigma(s')\int\frac{d\kappa_1}{\kappa_1-s'-i\epsilon}\frac{\theta(a\kappa_1^2+b\kappa_1+c)}{(a\kappa_1^2+b\kappa_1+c)^{1/2}}. \tag{6.7}$$

The κ_1 intergals can be done exactly as in Sec. III, with the result

$$\pi\theta(b^2-4ac)/[-(as'^2+bs'+c)]^{1/2}.$$

However, the branch of the square root is different in the two terms of (6.7) because of the different signs of $i\epsilon$. In particular, when the square root is pure imaginary, it will have opposite signs in the two parts. Remembering that c has a term $s^2\Delta$, we get

$$-i\pi\int_{s_R}^{s_{f-}}ds'\frac{\sigma(s')}{(as'^2+bs'+c)^{1/2}}+i\pi\int_{s_{f+}}^{s_L}\frac{ds'\sigma(s')}{(as'^2+bs'+c)^{1/2}}\sim\frac{i\pi}{(c)^{1/2}}\left(\int_{s_R}^{\infty}ds'\,\sigma(s',t;\rho_i)-\int_{-\infty}^{s_L}ds'\,\sigma(s',t;\rho_i)\right). \tag{6.8}$$

Instead of cancelling as in the naive approach, these integrals will combine. Furthermore, the quantity in the large parentheses in (6.8) depends crucially on the behavior of the spectral function for nonasymptotic values of s', and so the information in (6.5) is not sufficient to evaluate it. The explicit form of the spectral function is needed for this, including its dependence on the masses ρ_i.

Since c is a quadratic function of s'' and of κ_3, we can apply the same reasoning to the κ_3 and s'' integrations in (6.6). As a result, there will be an s^{-1} that survives from the $s^2\Delta$ part of D and thus gives the entire high-s dependence of the double X graph as expected. The coefficient of s^{-1} could be calculated from (6.8) if we use the exact spectral function. The spectral function is known for the X but this method would not be useful for more complicated nonplanar kernels where only asymptotic properties of the spectral function [like (6.5)] are known.

This example shows how the left-hand cut invalidates the procedure of Sec. V for nonplanar kernels. The actual asymptotic behavior of the B-S amplitude with the X as a kernel is a more difficult problem because it requires the explicit spectral function, but it should be amenable to our techniques.

Finally, let us remark that the formalism that we have presented can be easily adapted to include spin and also self-energy and vertex corrections. In a future paper we shall discuss this together with the related problem of Reggeization in vector-spinor theory.

ACKNOWLEDGMENTS

We are grateful to Professor Alex Dragt for helpful discussions and useful suggestions. We thank Professor Maurice Lévy for several helpful conversations.

APPENDIX

In this Appendix we shall prove some elementary theorems concerning the asymptotic behavior of functions defined by integrals. The spectral functions $\sigma(a)$ will be assumed to behave asymptotically like $a^p(\ln a)^q$ for some real numbers p and q. For functions $f(s)$ and

$g(s)$ we shall write $f(s)\sim g(s)$ if $\lim_{s\to+\infty}[f(s)/g(s)]=\text{const.}$

We first prove

Theorem 1. If $|\int_0^\infty da\,\sigma(a)a^{-1}|<\infty$ and $\int_0^\infty da\,\sigma(a)\neq 0$, then

$$\int_0^\infty da\frac{\sigma(a)}{s+a}\sim\frac{1}{s}\int_0^s da\,\sigma(a).$$

Proof. The result is obvious if $|\int_0^\infty da\,\sigma(a)|<\infty$, and so we assume that $\int_0^\infty da\,\sigma(a)=\infty$. Then $\int_0^s da\,\sigma(a)\to\infty$, but

$$\int_0^s da\,\sigma(a)\frac{s}{s+a}=\tau(s)\int_0^s da\,\sigma(a),$$

where $\frac{1}{2}\leq\tau(s)\leq 1$. Therefore, $\int_0^s da\,\sigma(a)\sim s\int_0^s da\,\sigma(a)\times(s+a)^{-1}$ or $s^{-1}\int_0^s da\,\sigma(a)\sim\int_0^s da\,\sigma(a)(s+a)^{-1}$. Since $\int_0^s da\,\sigma(a)(s+a)^{-1}\sim\int_0^\infty da\,\sigma(a)(s+a)^{-1}$, the result follows.

An immediate corollary is

Theorem 2. If $\int_0^\infty da\,\sigma(a)a^j=0$ for $0\leq j<k$ and $\int_0^\infty da\,\sigma(a)a^k\neq 0$, then

$$\int_0^\infty da\frac{\sigma(a)}{s+a}\sim\frac{1}{s^{k+1}}\int_0^s da\,\sigma(a)(-a)^k.$$

Proof. By theorem 1 we have

$$\frac{1}{s^k}\int_0^\infty da\frac{\sigma(a)(-a)^k}{s+a}\sim\frac{1}{s^{k+1}}\int_0^s da\,\sigma(a)(-a)^k,$$

and using

$$\frac{1}{s+a}-\left(-\frac{a}{s}\right)^k\frac{1}{s+a}=\frac{1}{s}\sum_{n=0}^{k-1}\left(-\frac{a}{s}\right)^n,$$

it follows from the hypotheses that

$$\int_0^\infty da\frac{\sigma(a)}{s+a}-\int_0^\infty da\frac{\sigma(a)}{s+a}\left(-\frac{a}{s}\right)^k$$
$$=\frac{1}{s}\sum_{n=0}^{k-1}\int da\,\sigma(a)\left(-\frac{a}{s}\right)^n=0.$$

A similar theorem for double integrals is

Theorem 3. If σ_1 and σ_2 satisfy the conditions of theorem 1, then

$$\int_0^\infty d\kappa\int_0^\infty da\frac{\sigma_1(\kappa)\sigma_2(a)}{s+\kappa a}\sim\frac{1}{s}\int_0^s d\kappa\int_0^{s/\kappa}da\,\sigma_1(\kappa)\sigma_2(a).$$

Proof. We have

$$\int_0^\infty d\kappa\int_0^\infty da\frac{\sigma_1\sigma_2}{s+\kappa a}\sim\int_0^s d\kappa\int_0^\infty da\frac{\sigma_1\sigma_2}{s+\kappa a}$$
$$\sim\int_0^s d\kappa\frac{\sigma_1}{\kappa}\int_0^\infty da\frac{\sigma_2}{s/\kappa+a}$$
$$\sim\int_0^s d\kappa\frac{\sigma_1}{\kappa}\frac{\kappa}{s}\int_0^{s/\kappa}da\,\sigma_2$$
$$=\frac{1}{s}\int_0^s d\kappa\int_0^{s/\kappa}da\,\sigma_1\sigma_2.$$

Finally, the generalization is

Theorem 4. If σ_1 satisfies the conditions of theorem 1 and σ_2 satisfies the conditions of theorem 2, then

$$\int_0^\infty d\kappa\int_0^\infty da\frac{\sigma_1(\kappa)\sigma_2(a)}{s+\kappa a}$$
$$\sim\frac{1}{s^{k+1}}\int_0^s d\kappa\int_0^{s/\kappa}da\,\sigma_1(\kappa)\sigma_2(a)(-a\kappa)^k.$$