Relativistic Corrections to the Light-Scattering Spectrum of a Plasma*

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(Received 5 September 1967; revised manuscript received 22 July 1968)

The light scattering spectrum of a plasma with drift-displaced Maxwellian velocity distribution is calculated to first order in $\vec{v}/c$ (c = velocity of light). The scattered intensity is obtained from the nonrelativistic Vlasov equation, and from the Lienard-Wiechert potentials produced by electrons which move with velocity $\vec{v}$, and are accelerated by the incident electric and magnetic radiation fields. For the incoherent spectrum, the relativistic effects are found equivalent to an apparent drift-velocity component parallel to the scattering vector $(\vec{k}^S - \vec{k}^i)$ of approximate magnitude $(5/3)v_e(v_e/c)\sin\frac{1}{2}\theta$, where v_e is the rms thermal electron velocity, θ the scattering angle, and $\vec{k}^S(\vec{k}^i)$ the wave vector of the scattered (incident) radiation. In the coherent spectrum the relativistic corrections are found proportional to the quantity $\omega/c\,|\vec{k}^s - \vec{k}^i|$. Because of the proportionality to the frequency shift ω, the corrections are negligible for the ion spectrum.

I. INTRODUCTION

Light scattering is used as a tool for plasma diagnostics under conditions in which the ratio $\vec{\beta} = \vec{v}/c$ of electron over light velocity may easily reach values of the order 0.1. Relativistic corrections are then practically important since they make the scattering spectrum $I^S(\omega)$ asymmetric even if the electron velocity distribution is isotropic. Thus $I^S(\omega) \neq I^S(-\omega)$, where

$$\omega = \omega^S - \omega^i \tag{1}$$

is the frequency shift between incident and scattered radiation. If this relativistic effect is ignored, an experimentally observed asymmetry may be erroneously interpreted in terms of drift velocities of the order of the thermal electron velocity. Wrong conclusions concerning drift instabilities may then be deduced from a scattering experiment.

Relativistic corrections to the scattering spectrum of a hot plasma have been calculated by Pappert[1] to the first order in β. However, we believe that Pappert's work contains a nontrivial conceptual error which will be discussed and corrected in this paper.

While Pappert's work shows that relativistic corrections to the collective scattering spectrum are small and practically less important than corrections to single-particle scattering, he nevertheless treats the scattering problem in a very general fashion which includes both types of scattering in the same formalism. For this reason he uses, instead of moving electrons, stationary volume elements in a continuous time dependent current distribution as sources of the scattering. Individual electrons never show up explicitly and we believe that this is responsible for Pappert's error of omission. In order to clarify this point, we shall calculate the scattering spectrum of a plasma, and correct to first order in β for the practically important case of a Maxwellian velocity distribution with drift. We shall do it, in the spirit of Pappert, so that the collective and single particle scattering are obtained from the same theory.

The scattering spectrum is determined by the space-time Fourier transform $f_{ek}(\omega)$ of the fine grained electron density of the plasma which is not perturbed by an incident radiation. Pappert obtains this function as a solution of the nonrelativistic Vlasov equation.[2] We shall follow this seemingly inconsistent procedure, since the relativistic corrections to the scattering cross section are most significant for single-particle scattering, while relativistic corrections to $f_{ek}(\omega)$ are nontrivial only in the collective regime. This is not to say that relativistic corrections to $f_{ek}(\omega)$ are not interesting since they may affect the shape of the side maximum in the collective spectrum which can be used for determining the ion temperature in a hot plasma. We shall discuss this problem in a subsequent paper.

II. THE GENERAL THEORY OF LIGHT SCATTERING BY A PLASMA

We consider a collisionless plasma containing N electrons in a scattering volume $V = L^3$ defined by the incident laser beam and the geometry of the detecting system. We assume L to be such that the time of flight L/v of a relativistic electron with velocity $\vec{v}$ through the scattering volume is much larger than the period ν^{-1} of the incident radiation. Line broadening associated with the finite emission time L/v is then negligible.

In a collisionless plasma, the electrons move along nearly straight orbits, which are only slightly perturbed by electric interactions. Deviations from the straight line motion are responsible for bremsstrahlung and for the long-range correlations between ions and electrons, but they are irrelevant for the elementary light scattering process. Thus for evaluating the radiation field scattered by the jth electron, we shall assume that this electron has a constant velocity $\vec{v}_j$, and that the scattering process is not influenced by the presence of the other charged particles in the plasma. This assumption is implicit in Pappert's Eq. (3) in which a term $(q_j\langle\vec{E}\rangle/m_j c)\cdot\nabla_p f_j^{(1)}$ has been neglected which would represent a perturbation of the radiation-induced first-order distribution function $f_j^{(1)}$ by the self-consistent plasma field $\langle\vec{E}\rangle$.

On the basis of this consideration we study first one-single electron with charge $-e$, rest mass m, velocity $\vec{v}_j$ and position

$$\vec{r}_j(t_j')=\vec{r}_j(o)+\vec{v}_j t_j', \tag{2}$$

exposed to the fields $\vec{E}^i(\vec{r}_j, t_j')$ and $\vec{H}^i(\vec{r}_j, t_j')$ of an incident linearly-polarized laser beam with wave vector $\hat{\imath}k^i=\hat{\imath}\omega^i/c$. We note that t'_j is the retarded time associated with the time t of observation at a distant point $\vec{R}\gg L$. It is given by the relation

$$t_j'=t-|\vec{R}-\vec{r}_j(t_j')|/c=[t-(R/c)+\hat{s}\cdot\vec{r}_j(0)/c](1-\hat{s}\cdot\vec{\beta}_j)^{-1}\ , \tag{3}$$

where use has been made of Eq. (2) and the Fraunhofer approximation

$$\lim_{r_j/R\to O}|\vec{R}-\vec{r}_j|=R-\hat{s}\cdot\vec{r}_j\ . \tag{4}$$

The $\hat{s}$ is a unit vector in direction of the scattered radiation, and $\theta=\cos^{-1}(\hat{\imath}\cdot\hat{s})$ is the scattering angle which is assumed fixed and independent of $\vec{r}_j(0)$ and $\vec{v}_j$ because of the condition $R>>L$.

The incident radiation acts with a Minkowski force on the moving electron and produces an acceleration[3]

$$\dot{\vec{v}}_j(\vec{r}_j,t_j')=-(e/m)(1-\beta_j^2)^{1/2}[\vec{E}^i+\vec{\beta}_j\times\vec{H}^i-\vec{\beta}_j(\vec{\beta}_j\cdot\vec{E}^i)]\exp\{i[\vec{k}^i\cdot\vec{r}_j(0)-(\omega^i-\vec{k}^i\cdot\vec{v}_j)t_j']\}\ . \tag{5}$$

The scattered radiation field, $\vec{E}_j^S(\vec{R},t)$, emitted by the accelerated electron may be obtained from the Liénard-Wiechert potentials in the form[4]

$$\vec{E}_j^S(R,t)=-(e/S_j^3c^2)\vec{R}\times(\vec{R}_{v_j}\times\dot{\vec{v}}_j)\exp\{i[\vec{k}\cdot\vec{r}_j(0)-(\omega^i-\vec{k}^i\cdot\vec{v}_j)t_j']\}, \tag{6a}$$

$$=-(e/S_j^3c^2)\vec{R}\times(\vec{R}_{v_j}\times\dot{\vec{v}}_j)\exp\{i[k_j^SR-\omega_j^St-\vec{k}_j\cdot\vec{r}_j(0)]\}, \tag{6b}$$

where $S_j=R(1-\hat{s}\cdot\vec{\beta}_j)$, (7)

$$\vec{R}_{v_j}=R(\hat{s}-\vec{\beta}_j), \tag{8}$$

$$\vec{k}_j^S=\hat{s}k_j^S=\hat{s}\omega_j^S/c=\text{wave vector of scattered radiation}, \tag{9}$$

$$\vec{k}_j=\vec{k}_j^S-\vec{k}^i=\text{scattering vector}, \tag{10}$$

and $\omega_j^S=\omega^i+\omega_j=\omega^i+\vec{k}_j\cdot\vec{v}_j$ (11)

is the frequency of the scattered radiation. It is obtained together with the phase in Eq. (6b), by substituting t_j' of Eq. (3) into the phase angle of Eq. (6a), and noting that $\dot{\vec{v}}_j$ is an amplitude not depending on time.

Evaluating Eq. (6b) to a first order in $\vec{\beta}_j$ gives

$$\vec{E}_j^S(\vec{R},t)=-e[\vec{\epsilon}^{\,0}+\vec{\epsilon}_j'+0(\beta_j^2)]\exp\{i[k_j^SR-(\omega^i+\vec{k}_j\cdot\vec{v}_j)t-\vec{k}_j\cdot\vec{r}_j(0)]\}\ , \tag{12}$$

where $\vec{\epsilon}^{\,0}=-(e/mc^2R)[(\hat{s}\cdot\vec{E}^i)\hat{s}-\vec{E}^i]$ (13)

$$\vec{\epsilon}_j'=-(e/mc^2R)\{[-2(\hat{s}\cdot\vec{\beta}_j)+(\hat{\imath}\cdot\vec{\beta}_j)]\vec{E}^i-(\vec{\beta}_j\cdot\vec{E}^i)\hat{\imath}-(\hat{s}\cdot\vec{E}^i)\vec{\beta}_j+[(\hat{s}\cdot\hat{\imath})(\vec{\beta}_j\cdot\vec{E}^i)-(\hat{s}\cdot\vec{E}^i)(\vec{\beta}_j\cdot\hat{\imath})+3(\hat{s}\cdot\vec{E}^i)(\vec{\beta}_j\cdot\hat{s})]\hat{s}\} \tag{14a}$$

$$=(\vec{\beta}_j\cdot\vec{1})\vec{\epsilon}^{\,0},\quad \vec{1}=2\hat{s}-\hat{\imath},\quad \text{if}\quad (\vec{E}^i\cdot\hat{\imath})=(\vec{E}^i\cdot\hat{s})=0. \tag{14b}$$

$\vec{E}^i$ in Eqs. (5), (13), and (14a) represents the constant amplitude of the radiation field, and use has been made of the plane wave relation $\vec{H}^i = \hat{\imath} \times \vec{E}^i$. Eq. (14b) refers to the practically most important case, where the incident laser beam is linearly polarized normal to the scattering plane defined by the unit vectors $\hat{\imath}$ and $\hat{s}$. In (14b) a component of $\vec{\epsilon}_j'$ normal to $\vec{\epsilon}^0$ has been omitted since its contribution to the scattering intensity is of second order in $\vec{\beta}_j$. This component generates a depolarization of the scattered radiation.[5]

In order to extend the analysis to the case of scattering by a plasma of N electrons and N ions, we introduce the fine grained Boltzmann distribution function associated with the jth electron in the form

$$f_j(\vec{r}, \vec{v}, t) = \delta(v - v_j)\delta[\vec{r} - \vec{r}_j(0) - \vec{v}t] = V^{-1}\sum_{\vec{q}} f_{j,\vec{q}}(\vec{v}, t)\exp(i\vec{q}\cdot\vec{r}), \tag{15}$$

$$\text{where } f_{j,\vec{q}}(\vec{v}, t) = \delta(\vec{v} - \vec{v}_j)\exp\{-i\vec{q}\cdot[\vec{r}_j(0) + \vec{v}t]\}, \tag{16}$$

and $\vec{q}$ satisfies the boundary condition

$$q_i = 2\pi h_i/L, \quad h_i = 0, \ \pm 1, \ \pm 2, \ \ldots, \quad i = x, \ y, \ z.$$

We can now rewrite Eq. (12) in the form

$$\vec{E}_{\vec{k}_j}{}^S(\vec{R}, t) = Ve\int d\vec{v} f_{j,\vec{k}_j}(\vec{v}, t)[\vec{\epsilon}^0 + \vec{\epsilon}'(\vec{v})]\exp\{i[k_j{}^S R - \omega^i t]\}. \tag{17}$$

Finally we form the time Fourier transform, or rather Laplace transform[6] $\vec{E}_{\vec{k}j}{}^S(\vec{R}, \omega^S)$ of the scattered field (17). We wish that the Fourier transform satisfy the relation

$$\vec{E}_{\vec{k}_j}{}^S(\vec{R}, t) = (2\pi)^{-1}\int_{-\infty}^{\infty}\vec{E}_{\vec{k}_j}{}^S(\vec{R}, \omega^S)\exp(-i\omega^S t)d\omega^S, \tag{18}$$

and note that $\vec{k}_j$ is not a constant parameter, but varies with ω^S according to the Eqs. (9) and (10). Furthermore, we see from (16) that the scattered field (17), contains the phase

$$\exp(-i\vec{k}^S\cdot\vec{v}t) = \exp(-i\omega^S\hat{s}\cdot\vec{\beta}t), \tag{19}$$

and this together with the condition (18), introduces a Jacobian $(1 - \hat{s}\cdot\vec{\beta})$ into the conventional integral representation of the time Fourier transform in which $\vec{k}_j$ would not depend on ω^S. Thus

$$\vec{E}_{\vec{k}_j}{}^S(\vec{R}, \omega^S) = \int_{-\tau/2}^{\tau/2}(1 - \hat{s}\cdot\vec{\beta})\vec{E}_{\vec{k}_j}{}^S(\vec{R}, t)\exp(i\omega^S t)dt \tag{20a}$$

$$= \lim_{\gamma\to 0}\int_0^{\infty}(1 - \hat{s}\cdot\vec{\beta})\vec{E}_{\vec{k}_j}{}^S(\vec{R}, t)\exp[(i\omega^S - \gamma)t]dt \tag{20b}$$

$$= -eV\int d\vec{v}(1 - \hat{s}\cdot\vec{\beta})[\vec{\epsilon}^0 + \vec{\epsilon}'(\vec{v})]f_{j,\vec{k}_j}(\vec{v}, \omega_j)\exp(ik_j{}^S R), \tag{20c}$$

$$\text{where } f_{j,\vec{k}_j}(\vec{v}, \omega_j) = 2\pi\delta(\vec{v} - \vec{v}_j)\delta(\omega_j - \vec{k}_j\cdot\vec{v})\exp[-i\vec{k}_j\cdot\vec{r}_j(0)] \tag{21}$$

is the time Fourier transform or Laplace transform of the Fourier component $f_{j,\vec{k}_j}(\vec{v}, t)$ associated with the frequency $\omega_j = \omega^S - \omega^i$.

It should be noted that Eq. (20a) gives the δ function (21) in the familiar form

$$\delta(\omega_j - \vec{k}_j\cdot\vec{v}) = \lim_{\tau\to\infty}\frac{1}{\pi}\frac{\sin[\frac{1}{2}\tau(\omega_j - \vec{k}_j\cdot\vec{v})]}{(\omega j - \vec{k}j\cdot\vec{v})}, \tag{22}$$

while Eq. (20b) leads to the expression

$$\delta(\omega_j - \vec{k}_j\cdot\vec{v}) = \lim_{\gamma\to 0}\frac{i}{\pi[(\omega - \vec{k}_j\cdot\vec{v}) + i\gamma]} = \frac{i(\omega - \vec{k}_j\cdot\vec{v}) + \gamma}{\pi[(\omega - \vec{k}_j\cdot\vec{v})^2 + \gamma^2]}. \tag{23}$$

The real part of (23) is indeed a δ function. The imaginary part is an odd function of the variable $u \equiv \omega - \vec{k}_j\cdot\vec{v}$ and hence, the neighborhood of the point $u = 0$ does not contribute to integrals of the form $\int F(u)u/(u^2 + \gamma^2)du$. On the basis of this property, the imaginary part of (23) may be neglected as small compared to the real part.

Comparing Eqs. (20a) and (20b) it is seen that γ may be interpreted as the reciprocal of the time τ which has been selected by the observer at $\vec{R}$ for spectral analysis. For a scattering experiment with a pulsed laser, that is essentially the duration of the pulse of scattered light seen by the observer at $\vec{R}$. In view of Eq. (3), τ differs from the pulse length τ' seen by the plasma, and is obtained from τ' by multiplication with the factor $(1-\hat{s}\cdot\vec{\beta}_j)$. This difference has to be taken into account when the Laplace transform is evaluated on the basis of Eq. (6a) in terms of the variable t'. As shown in the Appendix the resulting Laplace transform is the same as that given by Eq. (20).

Another problem associated with the physical interpretation of γ is the containment effect discussed by Pechacek and Trivelpiece.[7] We ignore this effect, and this implies essentially that the time of flight of an electron through the scattering volume is larger than the duration of the laser pulse.

We consider now the radiation field scattered by N electrons in a plasma. Because of the Coulomb interactions the electron velocities $\vec{v}_j(t)$ are slowly varying functions of time, but it may be safely assumed that the $\vec{v}_j$ are practically constant over several radiation periods ν^{-1}. Thus the elementary light scattering process is the same as that presented for one particle with constant $\vec{v}_j$, and a relativistic correction to the scattered field is required which, at any time, is determined by the instantaneous electron velocities. However, since these velocities are time dependent, and since Eq. (2) is not valid if $t' \gg \nu^{-1}$, the radiation scattered by one electron contains many frequencies in addition to that given by Eq. (11), and the spectrum of the radiation scattered by the ensemble of N electrons in a plasma has a complicated form which is not necessarily a direct image of the electron velocity distribution. Whether or not that is the case depends on the magnitude of the parameter α, Eq. (58), which determines the coherent (collective) or incoherent (single-particle) nature of the scattering.

We can now introduce the fine grained Boltzmann distribution function of the N-electron system by the relation

$$f_e(\vec{r},\vec{v},t)d\vec{v} = \sum_{\{j\}\vec{v},\,d\vec{v}} f_j(\vec{r},\vec{v},t), \tag{24}$$

where the summation label $\{j\}\vec{v}, d\vec{v}$ indicates summation over all electrons with velocity between $\vec{v}$ and $\vec{v}+d\vec{v}$. Summation over all j may then be replaced by a velocity integral,

$$\int_{-\infty}^{\infty} f_e(\vec{r},\vec{v},t)d\vec{v} = \sum_{j=1}^{N} f_j(\vec{r},\vec{v},t). \tag{25}$$

The Fourier component with the general wave vector $\vec{q}$ of $f_e(\vec{r},\vec{v},t)$ is

$$f_{e\vec{q}}(\vec{v},t)d\vec{v} = \int_V f_e(\vec{r},\vec{v},t)\exp(-i\vec{q}\cdot\vec{r})d\vec{r}\,d\vec{v} = \sum_{\{j\}\vec{v},\,d\vec{v}} f_{j,\vec{q}}(\vec{v},t), \tag{26}$$

and the time Fourier transform of this Fourier component is for fixed $\vec{q}$,

$$f_{e\vec{q}}(\vec{v},\omega) = \int_0^{\infty} f_{e\vec{q}}(\vec{v},t)\exp[(i\omega-\gamma)t]dt = \sum_{\{j\}\vec{v},\,d\vec{v}} f_{j,\vec{q}}(\vec{v},\omega). \tag{27}$$

The space-time Fourier transform of the scattered electrical field, $\vec{E}_{\vec{k}_\omega}{}^{s}(\vec{R},\omega^s)$ associated with the frequency ω^s, and the wave vector

$$\vec{k}_\omega = \vec{k}_\omega{}^{s} - \vec{k}^{i} = \vec{k} + \hat{s}\omega/c, \tag{28}$$

$$\vec{k} = (\omega^i/c)(\hat{s}-\hat{\imath}), \tag{29}$$

is now easily obtained from Eq. (20c) by summation over all j.

Utilizing the Eqs. (24) to (27) we get

$$\lim_{\gamma\to 0} \vec{E}_{\vec{k}_\omega}{}^{s}(\vec{R},\omega^s) = -e\int_{-\infty}^{\infty} d\vec{v}(1-\hat{s}\cdot\vec{\beta})[\vec{\epsilon}^{\,0} + \vec{\epsilon}'(\vec{v})] f_{e\vec{k}_\omega}(\vec{v},\omega), \tag{30}$$

and this is essentially identical with Pappert's Eqs. (12), (16), and (17) which were derived in a fairly different manner. But the relation between the spectral intensity distribution $I(\vec{R},\omega^s)$ of the scattered radiation, and the space-time Fourier transform $\vec{E}(\vec{R},\omega^s)$ of the scattered field, is more complicated than that employed by Pappert, if one takes into account that the scattering sources are moving point changes.

This is most clearly seen for the case of single-particle scattering, where the total scattering intensity produced by an N-electron system is the sum of contributions of N independent electrons. The spectral intensity distribution $I(\vec{R},\omega^s)$ is then a contribution of those electrons the velocity of which satisfies the relation (11). Each of these electrons scatters with intensity

$$I_{\vec{k}_\omega}{}^{s}(\vec{R},\vec{v}_j) = (c/4\pi)\langle|\vec{E}_{\vec{k}_j}{}^{s}(\vec{R},t)|^2\rangle_{\text{av}}, \tag{31}$$

where $\vec{E}(\vec{R},t)$ is given by Eq. (17) and the brackets denote a time average. The spectral intensity distribution for the N-electron system can now be obtained by integrating (31) over the normalized electron velocity distribution $F_e(v)$ and selecting the required frequency by means of a suitable δ function.[5] This gives

$$I_{\vec{k}_\omega}{}^{s}(\vec{R},\omega^s) = N\int_{-\infty}^{\infty} I_{\vec{k}_\omega}{}^{s}(\vec{R},\vec{v})\delta(\omega^s-\omega^i-\vec{k}_\omega\cdot\vec{v})F_e(\vec{v})d\vec{v} \tag{32}$$

Since the spectral intensity distribution (32) has been immediately derived from the scattered radiation field (17) without using the Fourier transform (30), it does not contain the Jacobian $(1-\hat{s}\cdot\vec{\beta})$.

This is the point where our work differs conceptually from that of Pappert who defines a continuous spectral intensity distribution in terms of the Fourier transform (30). Using our terminology, Pappert's result is essentially

$$I_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s) = \lim_{\gamma\to 0} \frac{c}{4\pi}\frac{\gamma}{\pi} |E_{\vec{k}_\omega}(\vec{R}, \omega^s)|^2, \tag{33}$$

and this differs from the correct result (32) by the factor $(1-\hat{s}\cdot\beta)^2$.

In spite of the failure of Eq. (33), it is possible and convenient to preserve its familiar analytical form. To this purpose we follow Rosenbluth and Rostoker[8] and introduce the rough grained Fourier transform

$$\lim_{\Delta\omega\to 0} \overline{\vec{E}}_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s) = (2\pi)^{-1} \int_{\omega^s - \frac{1}{2}\Delta\omega}^{\omega^s + \frac{1}{2}\Delta\omega} \vec{E}_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s)|^2 d\omega^s . \tag{34}$$

Since $\vec{E}(\vec{R}, \omega^s)$ will be shown to be a sum of delta functions of the type (21), the net result of the integration (34) will be that the Jacobian $(1-\hat{s}\cdot\vec{\beta}|$ which has been introduced on taking the time Fourier transform, will be removed on taking the inverse transform (34). Thus

$$\overline{\vec{E}}_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s) = -e\int_{-\infty}^{\infty} d\vec{v}[\epsilon^0 + \epsilon'(v)] f_{e\vec{k}_\omega}(\vec{v}, \omega), \tag{35}$$

and this corresponds to the spectral intensity distribution[6, 8]

$$I_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s) = \lim_{\gamma\to 0} \frac{c}{4\pi}\frac{\gamma}{\pi} |\overline{\vec{E}}_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s)|^2. \tag{36}$$

Eq. (36) gives the correct result for the case of single-particle scattering, and it appears to us that the use of the rough grained Fourier transform is the most convenient device for presenting scattering by moving point charges in terms of continuous distribution functions.

We are now ready to calculate the scattering intensity essentially by the method of Salpeter[6] in which $f_{e\vec{k}}(\vec{v}, t)$ is obtained as a solution of the nonrelativistic Vlasov equation. Following Salpeter, whose paper should be consulted for details, we introduce the notation

$$\sigma_{\vec{k}_\omega}(\vec{v}, t) = \sum_r \sigma_{r\vec{k}_\omega}(\vec{v}, t) = \sum_r e_r f_{r\vec{k}_\omega}(\vec{v}, t), \quad r = i, e; \quad e_r = \pm e, \tag{37}$$

$$q_{\vec{k}_\omega}(\vec{v}, \omega) = \sum_r q_{r\vec{k}_\omega}(\vec{v}, \omega) = \sum_r \int_0^\infty dt\, \sigma_{r\vec{k}_\omega}(\vec{v}, t)\exp(i\omega - \gamma)t$$

$$= \sum_r [i\sigma_{r\vec{k}_\omega}(\vec{v}, 0) - \frac{4\pi n e^2}{k_\omega} Q_{\vec{k}_\omega}(\omega)\frac{1}{m_r}\partial F_r(\vec{v})/\partial v_\omega](\omega - \vec{k}_\omega v_\omega + i\gamma)^{-1} \tag{38}$$

$$v_\omega = \vec{k}_\omega\cdot\vec{v}/k_\omega \tag{39}$$

$$Q_{\vec{k}_\omega}(\omega) = \sum_r Q_{r\vec{k}_\omega}(\omega) = \sum_r \int_{-\infty}^{\infty} q_{r\vec{k}_\omega}(\vec{v}, \omega) d\vec{v}, \tag{40}$$

where $F_r(\vec{v})$ is the velocity distribution function of the particle species r normalized to unity, and $r = i$ (e) labels the ions (electrons) with number density n.

Using the Eqs. (14b) and (28)–(36), and omitting the obvious labels $\vec{k}_\omega$ and ω, we get, to first order in β, after performing the integration (35),

$$|\overline{\vec{E}}_{\vec{k}_\omega}{}^{s}(\vec{R}, \omega^s)|^2 = (\epsilon^0)^2|Q_e + Q_e'|^2 \simeq (\epsilon^0)^2[|Q_e|^2 + 2R_e(Q_e^* Q_e')], \tag{41}$$

where, for the practically most important situation (14b).

$$Q_e' = \int_{-\infty}^{\infty} d\vec{v} q_{e\vec{k}_\omega}(\vec{v}, \omega)(\vec{\beta}\cdot\vec{1}) \tag{42}$$

In order to proceed further without specifying the velocity distribution, we introduce the quantities

$$G_r = -(4\pi n e^2/k_\omega)\int_{-\infty}^{\infty} d\vec{v}[\partial F_r(\vec{v})/\partial v_\omega]/m_r(\omega - k_\omega v_\omega + i\gamma), \tag{43}$$

$$H_r = (4\pi ne^2/k_\omega)\int_{-\infty}^{\infty} d\vec{v}[\partial F_r(\vec{v})/\partial v_\omega](\vec{\beta}\cdot\vec{l})/m_r(\omega - k_\omega v_\omega + i\gamma), \tag{44}$$

$$S_r = \sum_j \exp(-ik_\omega z_{rj\omega})/(\omega - k_\omega v_{rj\omega} + i\gamma), \tag{45}$$

$$\Sigma_r = \sum_j (\vec{\beta}_{rj}\cdot\vec{l})\exp(ik_\omega z_{rj\omega})/(\omega - k_\omega v_{rj\omega} + i\gamma), \tag{46}$$

$$z_{rj\omega} = \vec{k}_\omega\cdot\vec{r}_{rj}/k_\omega, \tag{47}$$

and get, following essentially Salpeter's methods,

$$Q_e = -ie[(1-G_i)S_e - G_e S_i](1 - G_e - G_i)^{-1} \tag{48}$$

and $$Q_e' = -ie\Sigma_e - QH_e = -ie[(1-G_e-G_i)\Sigma_e + (S_i - S_e)H_e](1-G_e-G_i)^{-1} \tag{49}$$

According to the Eqs. (45)–(49), the space-time Fourier transforms Q_e and Q_e' are proportional to the delta functions (21) in the form (23). Hence it is here where the transition from the fine grained Fourier transform $E(\vec{R}, \omega^S)$ to the rough grained Fourier transform $\overline{E}(\vec{R}, \omega^S)$ removes Pappert's erroneous factor $(1 - \hat{s}\cdot\vec{\beta})$, and where the discussion of basic principles may be concluded. The rest is straightforward though lengthy algebra.

For evaluating squares we note that mixed products $S_i S_e$ and $\Sigma_e S_i$ are negligibly small compared with products $S_i{}^2$, $\Sigma_e S_e$ etc., if γ tends to zero.[6] Hence in this limit

$$|Q_e|^2 = e^2|1 - G_e - G_1|^{-2}[|1-G_i|^2|S_e|^2 + |G_e|^2|S_i|^2], \tag{50}$$

and $$2R_e(Q_e{}^*Q_e') = 2e^2|1-G_e-G_i|^{-2}R_e[(1-G_i)(1-G_e{}^*-G_i{}^*)S_e\Sigma_e{}^* - (1-G_i)H_e{}^*|S_e|^2 - G_eH_e{}^*|S_i|^2|], \tag{51}$$

where[6] $$|S_r|^2 = \int_{-\infty}^{\infty} dv_\omega F_r(v_\omega)/[(\omega - k_\omega v_\omega)^2 + \gamma^2] = (N\pi/\gamma k_\omega)F_r(\omega/k_\omega) \tag{52}$$

$$S_e\Sigma_e{}^* = \int_{-\infty}^{\infty} dv_\omega(\vec{\beta}\cdot\vec{l})F_e(v_\omega)/[(\omega - k_\omega v_\omega)^2 + \gamma^2] = [(\omega l_\omega/ck_\omega) + \vec{w}_{en}\cdot\vec{l}/c]\,|S_e|^2, \tag{53}$$

$$l_\omega = \vec{k}_\omega\cdot\vec{l}/k_\omega, \tag{54}$$

and $\vec{w}_{en}$ is the electron drift velocity normal[9] to $\vec{k}$ in systems with anisotropic velocity distributions $F_e(\vec{v})$. The one-dimensional velocity distribution $F_r(v_\omega)$ is the distribution function for the components v_ω of $\vec{v}$.

III. EVALUATION OF FORMULAS FOR MAXWELLIAN VELOCITY DISTRIBUTIONS WITH DRIFT

We consider velocity distributions of the form

$$F_r(\vec{v}) = (m_r/2\pi K_B T_r)^{\frac{3}{2}} \exp[-(m_r/2K_B T_r)(\vec{v} - \vec{w}_r)^2], \tag{55}$$

where K_B is the Boltzmann constant, and T_r and $\vec{w}_r$ are respectively the temperature and drift velocities of the particle species r. It follows that

$$\partial F_r(\vec{v})/\partial v_\omega = [m_r F_r(\vec{v})/K_B T_r](v_\omega - w_{r\omega}) \tag{56}$$

and $$G_r = k_\omega \alpha_{r\omega}{}^2 \int_{-\infty}^{\infty} dv_\omega (v_\omega - w_{r\omega})F_r(v_\omega)/(\omega - k_\omega v_\omega + i\gamma), \tag{57}$$

where $w_{r\omega}$ is the component of $\vec{w}_r$ parallel to $\vec{k}_\omega$, and

$$\alpha_{r\omega} = (4\pi ne^2/k_\omega{}^2 K_B T_r)^{1/2}. \tag{58}$$

We now introduce the Doppler-shifted frequency

$$\omega_{rw} = \omega - k_\omega w_{r\omega} \tag{59}$$

and denote quantities belonging to an undisplaced Maxwell distribution with a superscript zero. We find then immediately

$$G_r(\omega) = k_\omega \alpha_{r\omega}{}^2 \int_{-\infty}^{\infty} \frac{(v_\omega - w_{r\omega})F_r{}^0(v_w - w_{r\omega})d(v_\omega - w_{r\omega})}{\omega_{rw} - k_\omega(v_w - w_{r\omega}) + i\gamma} = G_r{}^0(\omega_{rw}), \tag{60}$$

and $$|S_r{}^0(\omega)|^2=\int_{-\infty}^{\infty}\frac{F_r{}^0(v_\omega-w_{r\omega})d(v_\omega-w_{r\omega})}{[\omega_{r\omega}-k_\omega(v_\omega-w_{r\omega})]^2+\gamma^2}=\frac{N\pi}{\gamma k_\omega}F_r{}^0\left(\frac{\omega_{rw}}{k_\omega}\right)=|S_r{}^0(\omega_{rw})|^2. \tag{61}$$

For evaluating Eq. (51) we still need the quantity H_e defined in Eq. (44), and since H_e has not been used by Salpeter we shall discuss the required velocity integral in some detail.

It is convenient to introduce cartesian vector components by the relation

$$\vec{A}=\sum_{j=1}^{3}A_j\hat{n}_j, \tag{62}$$

where $\vec{A}$ is an arbitrary vector, $A_1=A_\omega$ its component parallel to $\vec{k}_w$, and A_2, A_3 the components normal to $\vec{k}_\omega$. In terms of these components we get

$$H_e=k_\omega\alpha_{e\omega}{}^2\int_{-\infty}^{\infty}\frac{(v_\omega-w_{e\omega})\prod_{j=1}^{3}d(v_j-w_{ej})F_e{}^0(v_j-w_{ej})\left(\sum_{j=1}^{3}\beta_j l_j\right)}{\omega_{ew}-k_\omega(v_\omega-w_{e\omega})+i\gamma}. \tag{63}$$

H_e is now a sum of three terms, and we consider first the term with $j=1$. Adding and subtracting $l_\omega w_{e\omega}/c$ it has the form

$$H_{e1}=J_e+M_e, \tag{64}$$

with $$J_e=k_\omega\alpha_{e\omega}{}^2\frac{l_\omega}{c}\int_{-\infty}^{\infty}\frac{(v_\omega-w_{e\omega})^2F_e{}^0(v_\omega-w_{e\omega})d(v_\omega-w_{e\omega})}{\omega_{ew}-k_\omega(v_\omega-w_{e\omega})+i\gamma}, \tag{65}$$

and $$M_e=k_\omega\alpha_{e\omega}{}^2\frac{l_\omega w_{e\omega}}{c}\int_{-\infty}^{\infty}\frac{(v_\omega-w_{e\omega})F_e{}^0(v_\omega-w_{e\omega})d(v_\omega-w_{e\omega})}{\omega_{ew}-k_\omega(v_\omega-w_{e\omega})+i\gamma}=\frac{l_\omega w_{e\omega}}{c}G_e. \tag{66}$$

The general form of J_e is

$$J_e=\frac{Al_\omega}{c}\int_{-\infty}^{\infty}\frac{x^2F^0(x)dx}{B-x}=\frac{Al_\omega}{c}\int_0^{\infty}x^2F^0(x)\left(\frac{1}{B+x}+\frac{1}{B-x}\right)dx=\frac{2ABl_\omega}{c}\int_0^{\infty}\frac{x^2F^0(x)}{B^2-x^2}dx. \tag{67}$$

The intermediate formula in Eq. (67) is a consequence of the symmetry $F^0(x)=F^0(-x)$. Equation (67) should be compared with the function G_e which has the general form

$$G_e=-A\int_{-\infty}^{\infty}\frac{xF^0(x)dx}{B-x}=-A\int_0^{\infty}xF^0(x)\left(\frac{1}{B-x}-\frac{1}{B+x}\right)dx=-2A\int_0^{\infty}\frac{x^2F^0(x)}{B^2-x^2}dx. \tag{68}$$

Using $B=\omega_{ew}/k_\omega$ we find

$$J_e=-(l_\omega\omega_{ew}/ck_\omega)G_e. \tag{69}$$

We come now to the terms in Eq. (63) with $j=2, 3$. They are products of two one-dimensional velocity integrals, one of which is G_e and the other is essentially the average of v_j formed with a one-dimensional displaced Maxwell distribution. Hence

$$H_{ej}=(l_jw_{ej}/c)G_e,\quad j=2,\ 3. \tag{70}$$

Combining the Eqs. (63), (64), (66), (69) and (70) we get

$$H_e=(-l_\omega\omega_{ew}/ck_\omega+\vec{l}\cdot\vec{w}/c)G_e. \tag{71}$$

Substituting Eq. (71) into Eq. (51) gives

$$2\mathrm{Re}(Q_e{}^*Q_e{}')=2[(\omega l_\omega/ck_\omega)+\vec{w}_{en}\cdot\vec{l}/c]-(\vec{w}_{en}\cdot\vec{l}/c)P_e \tag{72}$$

with $$P_e=4e^2|1-G_e-G_i|^{-2}\,\mathrm{Re}\,[(1-G_i)G_e{}^*|S_e|^2+|G_e|^2|S_i|^2], \tag{73}$$

and from Eqs. (36) and (41)

$$I_{\vec{k}_\omega}{}^{s}(\omega^s)=I_{\vec{k}_\omega}{}^{0s}(\omega^s)[1+(2\omega l_\omega/k_\omega c)+2\vec{w}_{en}\cdot\vec{l}/c]-(C\gamma/2\pi^2c)\epsilon_0{}^2(\vec{w}_{en}\cdot\vec{l})P_e. \tag{74}$$

$I^0(\omega^s)$ is the zeroth order intensity distribution in which terms of first and higher orders in ω/ck are neglected. This is not yet identical with the conventional intensity distribution by Salpeter[6] and others[8] in which $\vec{k}_\omega$ is approximated by $\vec{k}$. The effect of taking into account the difference between $\vec{k}_\omega$ and $\vec{k}$ will be referred to as a Doppler shift correction.

IV. SUMMARY OF RESULTS IN TERMS OF SALPETER'S APPROXIMATION

Following Salpeter[6] we represent the total zeroth-order scattering spectrum as a sum of two terms,

$$I_{\vec{k}_\omega}^{0s}(\vec{R}, \omega^s) = \sum_r I_{\vec{k}_\omega}^{0r}(\vec{R}, \omega^s), \tag{75}$$

where the strongly shifted "electron spectrum" has the form

$$I_{\vec{k}_\omega}^{0e}(\vec{R}, \omega^s) = \frac{C}{\omega_e} \frac{|1 - G_i(x_i)|^2 e^{-x_e^2}}{|1 - G_e(x_e) - G_i(x_i)|^2}, \tag{76}$$

and the "weakly shifted" ion spectrum is

$$I_{\vec{k}_\omega}^{0i}(\vec{R}, \omega^s) = \frac{C}{\omega_i} \frac{|G_e(x_e)|^2 e^{-x_i^2}}{|1 - G_e(x_e) - G_i(x_i)|^2} \tag{77}$$

In these equations we have used the notation

$$C = I_0 N e^4 / \pi^{1/2} m_e^2 c^4 R^2, \tag{78}$$

$$x_r = \omega_{rw}/\omega_r = (\omega/\omega_r) - (3^{1/2} w_{rw}/2^{1/2} v_r), \tag{79}$$

$$\omega_r = (2/3)^{1/2} k_\omega v_r = \text{Doppler shift frequency}, \tag{80}$$

$$v_r = (3K_B T_r/m_r)^{1/2} = \text{rms thermal velocity}, \tag{81}$$

and I_0 is the intensity of the incident radiation. We are interested in the asymptotic behavior of the spectrum for large x_i (electron spectrum) and small x_e (ion spectrum). To this purpose we write $G_r(x_r)$ in the form[6]

$$G_r(x_r) = -\alpha_r^2[1 - f(x_r) - i\pi^{1/2} x_r e^{-x_r^2}] \tag{82}$$

$$\text{with } f(x_r) = 2x_r e^{-x_r^2} \int_0^{x_r} e^{t^2} dt, \tag{83}$$

and use the asymptotic forms

$$e^{-x_r^2} \simeq 1, \quad x_r < 0.1; \qquad 0, \quad x_r \gg 1 \tag{84}$$

$$f(x_r) \simeq 0, \quad x_r < 0.1; \qquad 1, \quad x_r \gg 1 \tag{85}$$

This gives

$$I_{\vec{k}_\omega}^{0e}(\vec{R}, \omega^s) = (C/\omega_e)\Gamma_\alpha(x_e) \tag{86}$$

$$\text{and } I_{\vec{k}_\omega}^{0i}(\vec{R}, \omega^s) = \frac{C}{\omega_i}\left(\frac{\alpha_e^2}{1+\alpha_e^2}\right)^2 \Gamma_\beta(x_i) \tag{87}$$

$$\text{with } \Gamma_\alpha(x_e) = \frac{e^{-x_e^2}}{[1+\alpha_e^2 - \alpha_e^2 f(x_e)]^2 + \pi\alpha_e^4 x_e^2 \exp(-2x_e^2)}, \tag{88}$$

$$\Gamma_\beta(x_i) = \frac{e^{-x_i^2}}{[1+\beta^2 - \beta^2 f(x_i)]^2 + \pi\beta^4 x_i^2 \exp(-2x_i^2)}, \tag{89}$$

$$\text{and} \qquad \beta = (T_e/T_i)\alpha_e^2/(1+\alpha_e^2). \tag{90}$$

The Eqs. (86) and (87) are approximately based on the asymptotic forms (84) and (85), and this implies that the ratio $w_{e\omega}/v_e$ should be smaller than 0.1 in order that the expression (87) for the ion spectrum be valid. But there is no such restriction on the electron spectrum (86) which can be used for arbitrarily large electron drift $w_{e\omega}$ as long as no drift instabilities are generated.

We can now study the effect of the relativistic radiation corrections and the difference between $\vec{k}_\omega$ and $\vec{k}$. To this purpose we use the Eqs. (14b), (28), and (29) and find, to first order in ω/ck,

$$k_\omega = k(1 + \hat{k}\cdot\hat{s}\,\omega/ck), \tag{91}$$

$$\text{and } 2l_\omega \omega/ck_\omega = 2\vec{l}\cdot\hat{k}\omega/ck, \tag{92}$$

$$\text{with } \hat{k} = \vec{k}/k = (\hat{s} - \hat{\imath})/2\sin\tfrac{1}{2}\theta. \tag{93}$$

Now, in the ion spectrum, ω is of the order ω_i and ω/ck is of the order v_i/c. Since we consider in this paper only cases where $v_e/c \leq 0.1$, v_i/c is of the order 10^{-3} and is completely negligible. Thus the zeroth-order ion spectrum is a good approximation, and the ion spectrum need not be further discussed.

In the electron spectrum ω/ck is of the order v_e/c and that is not negligible compared to unity in high-temperature fusion plasmas. Simple analytical expressions for the electron spectrum can be obtained in the regime of single particle scattering which is characterized by the condition $G_r \simeq \alpha_r \ll 1$. In this case we get from Eqs. (73), (74), (76), and (80)

$$\lim_{\alpha_r \to 0} I_{\vec{k}_\omega}^{e}(\vec{R}, \omega^s) = (3/2)^{\frac{1}{2}} C k_\omega^{-1} v_e^{-1} \exp(-x_e^2) \times [1 + (2l_\omega \omega/ck_\omega + 2\vec{w}_{en}\cdot\vec{l}/c], \tag{94}$$

and in order to get the ω-dependence of the scattering intensity explicitly we have to express $\vec{k}_\omega$ in terms of $\vec{k}$ and ω everywhere in Eq. (94). Using the symbol

$$\omega/kc = \beta_\omega, \tag{95}$$

and the Eqs. (28), (29), (79) to (81), and (91) to (93) we get

$$x_e = -(3/2)^{1/2} v_e^{-1}[(w_e - w/k)(1 - \beta_\omega \sin\tfrac{1}{2}\theta) + (\vec{w}_e\cdot\hat{s})\beta_\omega], \tag{96}$$

where w_e is the component of $\vec{w}_e$ parallel to $\vec{k}$. Similarly we get from Eqs. (91) to (93)

$$k_\omega^{-1}[1 + 2l_\omega \omega/ck_\omega + 2\vec{w}_{en}\cdot\vec{l}/c] = k^{-1}(1 + 5\beta_\omega \sin\tfrac{1}{2}\theta + 2\vec{w}_{en}\cdot\vec{l}/c) \tag{97}$$

The Doppler-shift corrections and relativistic corrections contained in Eqs. (96) and (97) have almost the same effect as replacing the true drift

w_e by an apparent drift w_e' in the uncorrected Salpeter formulas. In order to find w_e' we note that the scattered intensity is a frequency function of the form

$$F(\omega) = \exp[-x_e^2(\omega)]\,(1 + a + b\omega)\,, \tag{98}$$

$$\text{where } a = 2\vec{w}_{en}\cdot\vec{l}/c, \quad b = 5\sin\tfrac{1}{2}\theta/kc\,. \tag{99}$$

We find the maximum of $F(\omega)$ from the equation

$$-2x_e(\partial x_e/\partial\omega)(1 + a + b\omega) + b = 0 \tag{100}$$

which, to a first order in β, has the solution

$$\omega_{\max} = k[w_e(1 - \beta_w \sin\tfrac{1}{2}\theta) + \tfrac{5}{3}v_e\beta_v \sin\tfrac{1}{2}\theta]$$

$$\equiv kw_e' \tag{101}$$

$$\text{where } \beta_w = w_e/c, \quad \beta_v = v_e/c\,. \tag{102}$$

In the absence of a true drift w_e the apparent drift is

$$w_e' = 4\times 10^{-5} v_e T^{1/2} \sin\tfrac{1}{2}\theta. \tag{103}$$

Some concluding remarks about the practically important case $\alpha_\gamma \simeq 1$ are still required. In this case electron-ion correlations have a marked influence on the scattering spectrum, but the characteristic width of the spectrum is still determined by the Doppler shift associated with the electron motion. Thus relativistic effects are important, but simple analytical formulas and numerical results cannot be obtained. One has to go back to Eqs. (74) and (86), and in evaluating $\Gamma_\alpha(x_e)$, one has to take into account the difference between $\vec{k}$ and $\vec{k}_\omega$ at the appropriate places.

It should also be noted that the detailed formulas presented in Chapts. III and IV apply only to the case in which the incident laser beam is linearly polarized normal to the scattering plasma. To our knowledge no other experimental setup has been used in the past, and for this reason, the lengthy calculations required for analyzing the more general polarization situation are practically not very important. They are most conveniently performed for the simple case of incoherent single-particle scattering for which relativistic corrections are most pronounced anyway.

ACKNOWLEDGMENT

One of the authors (O. Theimer) wishes to acknowledge the hospitality of the Physics Division of the Aspen Institute for Humanistic Studies, where part of this work was done.

APPENDIX A

We shall evaluate the space time Fourier transform of the scattered field as seen by an observer at $\vec{R}$, using the retarded time t_j' as integration variable. Consequently, we employ Eq. (6a) in which the phase is presented explicitly as a function of t_j', and evaluate it to the first order in β_j. Then we take the Laplace transform as defined by Eq. (18) and obtain

$$E_j^S(\vec{R}, \omega^S) = -e\int_0^\infty dt(\vec{\epsilon}^0 + \vec{\epsilon}_j{}')\exp\{i[\vec{k}^i\cdot\vec{r}_j(0) - (\omega^i - \vec{k}^i\cdot\vec{v}_j)t_j' + (\omega^S + i\gamma)t]\}. \tag{A1}$$

We perform the integration in terms of the variable t_j', and to this purpose we have to express t as a function of t', by means of the relation (3). We get

$$\lim_{\gamma\to 0} \vec{E}_j^S(\vec{R}, \omega^S) = -e\int_0^\infty dt'(1 - \hat{s}\cdot\vec{\beta}_j)[\vec{\epsilon}^0 + \vec{\epsilon}_j{}']\exp\{i[(k_j^S - \gamma/c)R - \vec{k}_j\cdot\vec{r}_j(0)] - \gamma(1 - \hat{s}\cdot\vec{\beta}_j)t_j'\}$$

$$= -(e/\gamma)[\vec{\epsilon}^0 + \vec{\epsilon}_j{}']\exp\{i[k_j^S R - \vec{k}_j\cdot\vec{r}_j(0)]\}, \tag{A2}$$

where $\gamma/c \ll k_j^S$ has been neglected.

The same result is obtained if the Laplace transform (18) is taken of Eq. (12) and the t integration is carried out without introducing the Boltzmann distribution function. It should be noted that the Jacobian $(1 - \hat{s}\cdot\vec{\beta}_j)$ of the transformation from t to t' is canceled after integration by the same factor $(1 - \hat{s}\cdot\vec{\beta}_j)$ in the expression $(1 - \hat{s}\cdot\vec{\beta}_j)t'_j$, and this is the formal reason why the t integration and the t' integration give, as they must, the same result.

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*This work has been supported in part by the U. S. Office of Naval Research, Contract Nonr-3531(02).

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[9]Actually $\vec{w}_{en}$ is the drift velocity normal to $\vec{k}_\omega$, however, since $\vec{w}_{en}$ occurs in a term which is small of first order in β, the difference between $\vec{k}$ and $\vec{k}_\omega$ may be neglected.

[10]Equation (53) has been obtained using the components w_{ej} of Eq. (62) and the reasoning leading to Eq. (70).