A $\lambda\varphi^4$ Quantum Field Theory without Cutoffs. I

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(Received 20 May 1968)

We construct the dynamics for a local quantum field theory in two-dimensional space-time. Our model describes a scalar field φ with a $\lambda\varphi^4$ self-interaction. We rely upon the fact that, in the Heisenberg picture, influence propagates at the speed of light. The resulting dynamics is independent of any cutoff, and hence the theory is formally Lorentz-covariant.

I. INTRODUCTION

In this paper we obtain the dynamics for a local quantum field theory in two-dimensional space-time. We study a scalar $\lambda\varphi^4$ self-interaction, and no cutoffs appear in the solution. Formally, our model is Lorentz-covariant, and a covariant local quantum field theory should result.

This $\lambda\varphi^4$ model is simple because the only momentum-space divergences are removed by Wick-ordering the Hamiltonian density. The vacuum energy has a divergence linear in the volume in each order of perturbation theory, as is expected in any theory with vacuum polarization. In perturbation theory the scattering matrix for this interaction is nontrivial. However, the perturbation series is known to diverge when summed to all orders,[1] so we cannot solve the theory in this manner.

We construct the dynamics in the Heisenberg picture which means that we have a map σ_t which takes the field operators from time zero to time t. The main advantage of the Heisenberg picture, is that the volume divergence does not enter in this formulation.

In order to prove that the theory exists, we introduce a spatial cutoff in the interaction Hamiltonian. Thus the formal Hamiltonian expression

$$H(g)=H_0+\lambda\int_{t=0} :\varphi^4(x):dx$$

is replaced by the operator on Fock space,

$$\begin{aligned}H(g)&=H_0+\int_{t=0} :\varphi^4(x):g(x)dx\\&=H_0+H_I(g)\,,\end{aligned}\tag{1.1}$$

where the space-dependent coupling constant $g(x)$ is a smooth, non-negative function which vanishes outside a large interval. We derive an estimate for fixed $g(x)$,

$$-[H_0^{1/2},[H_0^{1/2},H_I(g)]]\leqq\epsilon H_0^2+b\,,\tag{1.2}$$

where ϵ is any positive number and $b(\epsilon)$ a suitably large number. This estimate, along with the previously known estimate[2,3]

$$-H_I(g)\leqq\epsilon H_0+b\,,\tag{1.3}$$

and a knowledge of the self adjointness of H_0 and $H_I(g)$, allows us to prove that the total Hamiltonian $H(g)$ is self adjoint. The connection between estimates (1.2) and (1.3) and self adjointness leads to a general theory of singular perturbations developed elsewhere.[4]

We then discuss the Heisenberg picture dynamics

$$\sigma_t(A)=e^{iH(g)t}Ae^{-iH(g)t},$$

of an observable A associated with a bounded region of space at $t=0$. The locality of the Hamiltonian density is essential to our method. As physicists have long known, it ensures that influence propagates in the Heisenberg picture at the speed of light. A local Hamiltonian density, therefore, assures a local theory on the formal level. Guenin pointed out that in perturbation theory the finite propagation speed is not destroyed by a space cutoff.[5] Segal states this result as a rigorous theorem, under the hypothesis that the total and the interaction Hamiltonians both are self adjoint.[6] In other words $\sigma_t(A)$ does not depend on $g(x)$, provided that $g(x)$ equals a constant, i.e., the desired coupling constant λ, for x in some suitably large region. By choosing an appropriate $g(x)$ for each bounded region of space-time, we can patch together a time translation for any operator. A similar argument shows that the theory is local. Since we have proved that the Hamiltonian $H(g)$ is self adjoint, we have constructed a local quantum field theory without a cutoff.

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* Supported in part by the National Science Foundation under Grant No. GP-7477.

† Supported in part by the Air Force Office of Scientific Research.

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II. ESTIMATES ON THE HAMILTONIAN

Let $\mathfrak{F}$ be the Fock space for a massive, neutral scalar field in two-dimensional space-time. The elements of $\mathfrak{F}$ are sequences of functions on momentum space. Let the annihilation and creation operators be normalized by the relation

$$[a(k),a^*(k')]=\delta(k-k'), \tag{2.1}$$

so that the free-field Hamiltonian is

$$H_0=\int a^*(k)a(k)\omega(k)dk,$$

where

$$\omega(k)=(k^2+m^2)^{1/2}.$$

The $t=0$ field

$$\varphi(x)=\frac{1}{(2\pi)^{1/2}\sqrt{2}}\int e^{-ikx}[a^*(k)+a(-k)]\omega(k)^{-1/2}dk \tag{2.3}$$

is used to construct the spatially cut-off interaction Hamiltonian

$$H_I(g)=\int :\varphi^4(x):g(x)dx$$

$$=\sum_{j=0}^{4}\binom{4}{j}\int a^*(k_1)\cdots a^*(k_j)a(-k_{j+1})\cdots \times a(-k_4)\tilde{g}(\sum_{i=1}^{4}k_i)\prod_{i=1}^{4}\omega(k_i)^{-1/2}dk_i. \tag{2.4}$$

The total Hamiltonian for the spatially cut-off interaction is

$$H(g)=H_0+H_I(g). \tag{2.5}$$

We also need the number operator

$$N=\int a^*(k)a(k)dk, \tag{2.6}$$

and the domain

$$D_0=\bigcap_{n=0}^{\infty}D(H_0^n). \tag{2.7}$$

We let $\mathfrak{S}(R^1)$ denote the Schwartz space of infinitely differentiable functions, which, along with all their derivatives, vanish faster than any inverse polynomial at infinity.

Theorem 2.1: For any $\epsilon>0$ and for fixed $g\in\mathfrak{S}(R^1)$ there is a constant b such that as bilinear forms on $D_0\times D_0$

$$-[H_0^{1/2},[H_0^{1/2},H_I(g)]]\leqq\epsilon H_0^2+b \tag{2.8}$$

and

$$-[N,[N,H_I(g)]]\leqq\epsilon N^3+b. \tag{2.9}$$

Let us state two lemmas, which then yield the theorem.

Lemma 2.2: Let W be an operator of the form

$$W=\int w(k)a^*(k_1)\cdots a(-k_m)dk. \tag{2.10}$$

Then

$$\|(N+I)^{-j/2}W(N+I)^{-(m-j)/2}\| \leqq \text{const}\,\|w\|_{L^2},\quad |j|\leqq m \tag{2.11}$$

$$\|(H_0+I)^{-1}[H_0^{1/2},[H_0^{1/2},W]](H_0+I)^{-1}(N+I)^{-(m-4)/2}\| \leqq\text{const}\,\|\omega^{1/2}(\sum_{i=1}^{m}k_i)w\|_{L^2}, \tag{2.12}$$

and

$$\|[H_0^{1/2},[H_0^{1/2},W]](N+I)^{-m/2}\| \leqq\text{const}\,\|\sum_{i=1}^{m}\omega(k_i)w\|_{L^2}. \tag{2.13}$$

Lemma 2.3: Let W be as above, and suppose that $w(k)=0$ whenever $|k_i|>\kappa$ for any i. Then

$$\|[H_0^{1/2},[H_0^{1/2},W]](N+I)^{-(m-2)/2}\| \leqq\text{const}\,\kappa^2\|w\|_{L^2}. \tag{2.14}$$

Proof of Theorem 2.1: Introduce the $t=0$ field φ_κ with an ultraviolet cutoff:

$$\varphi_\kappa(x)=\frac{1}{(2\pi)^{1/2}\sqrt{2}}\int_{-\kappa}^{\kappa}e^{-ikx}[a^*(k)+a(-k)][\omega(k)]^{-1/2}dk.$$

The interaction $H_{I,\kappa}(g)$ is defined by

$$H_{I,\kappa}=\int :\varphi_\kappa^4(x):g(x)dx, \tag{2.15}$$

and formally

$$H_I=\lim_{\kappa\to\infty}H_{I,\kappa}. \tag{2.16}$$

If we write H_I as a sum of five operators of the form W in (2.10), then by Lemma 2.2 taken for the case $m=4$

$$\|(H_0+1)^{-1}[H_0^{1/2},[H_0^{1/2},W]](H_0+1)^{-1}\| \leqq\text{const}\,\|\omega^{1/2}(\sum_{i=1}^{4}k_i)w\|_{L^2}. \tag{2.17}$$

Since the kernel $w(k)$ has an over-all factor $\tilde{g}(\sum_{i=1}^4 k_i)$, where $\tilde{g}$ is the Fourier transform of the spatial cutoff $g(x)$, the fast decrease of $\tilde{g}(k)$ ensures that $\omega^{1/2}(\sum_{i=1}^4 k_i)w$ is in L^2. Thus the kernel for the corresponding cutoff interaction term w_κ approximates w in the sense that

$$\|\omega^{1/2}(\sum_{i=1}^{4}k_i)(w-w_\kappa)\|_{L^2}\to 0, \tag{2.18}$$

as $\kappa\to\infty$. This is true for each W making up $H_I(g)$, so we infer that there exists a κ_0 such that for $\kappa>\kappa_0$,

$$\|(H_0+I)^{-1}[H_0^{1/2},[H_0^{1/2},H_I-H_{I,\kappa}]]\times(H_0+I)^{-1}\|\leqq\tfrac{1}{2}\epsilon. \quad (2.19)$$

By Lemma 2.3, we have the relations

$$-[H_0^{1/2},[H_0^{1/2},H_{I,\kappa}]]\leqq\text{const}\,\kappa_0^2(N+I)\leqq\tfrac{1}{2}\epsilon H_0^2+\text{const}, \quad (2.20)$$

on the domain $D_0\times D_0$. Thus combining (2.20) with the estimate on the tail (2.19),

$$-[H_0^{1/2},[H_0^{1/2},H_I]]=-[H_0^{1/2},[H_0^{1/2},H_I-H_{I,\kappa}]]-[H_0^{1/2},[H_0^{1/2},H_{I,\kappa}]]\leqq\epsilon H_0^2+b,$$

which is (2.8). In order to prove (2.9), we expand $H_I(g)$ once more as a sum of terms W of the form (2.10). Since $[N,W]=\text{const}\,W$, and W is a fourth-order operator with an L^2 kernel, we infer from (2.11) that

$$|(\psi[N,[N,W]]\psi)|\leqq\text{const}\,(\psi,(N+I)^2\psi)\leqq\tfrac{1}{5}\epsilon(\psi,N^3\psi)+\tfrac{1}{5}b(\psi,\psi). \quad (2.21)$$

As $H_I(g)$ has 5 such terms,

$$[N,[N,H_I]]\leqq\epsilon N^3+b,$$

which completes the proof of the theorem.

Proof of Lemmas 2.2 and 2.3: Let θ be an n-particle wave function. Then $(a(k)\theta)(k_1,\cdots,k_{n-1})$ is an $(n-1)$-particle state defined by

$$(a(k)\theta)(k_1,\cdots,k_{n-1})=(\sqrt{n})\theta(k,k_1,\cdots,k_{n-1}). \quad (2.22)$$

Using (2.22) and the Schwarz inequality, it is easy to establish the bound (2.11).

Note that

$$([H_0^{1/2},a(k)]\theta)(k_1,\cdots,k_{n-1})=\lambda_1(k;k_1,\cdots,k_{n-1})(a(k)\theta)(k_1,\cdots,k_{n-1})\equiv(a_1(k)\theta)(k_1,\cdots,k_{n-1}), \quad (2.23)$$

where

$$\lambda_1(k;k_1,\cdots,k_{n-1})=[\omega(k)+\sum_{i=1}^{n-1}\omega(k_i)]^{1/2}-[\sum_{i=1}^{n-1}\omega(k_i)]^{1/2}. \quad (2.24)$$

For convenience, we have defined a modified annihilation operator $a_1(k)$, which decreases the particle number by one.

For $x\geq0$, we see that

$$(1+x)^{1/2}-1\leq x^{1/2},$$

so

$$|\lambda_1(k;k_1,\cdots,k_{n-1})|\leq[\omega(k)]^{1/2}. \quad (2.25)$$

Furthermore, for $x\geq0$,

$$(1+x)^{1/2}-1\leq\tfrac{1}{2}x,$$

so

$$|\lambda_1(k;k_1,\cdots,k_{n-1})|\leq\tfrac{1}{2}\omega(k)[\sum_{i=1}^{n-1}\omega(k)]^{-1/2}\leq\tfrac{1}{2}\omega(k)\frac{1}{m^{1/2}(n-1)^{1/2}}. \quad (2.26)$$

In the same way

$$([H_0^{1/2},[H_0^{1/2},a(k)]]\theta)(k_1,\cdots,k_{n-1})=\lambda_2(k;k_1,\cdots,k_{n-1})(a(k)\theta)(k_1,\cdots,k_{n-1})\equiv(a_2(k)\theta)(k_1,\cdots,k_{n-1}). \quad (2.27)$$

Here

$$|\lambda_2(k;k_1,\cdots,k_{n-1})|\leq\begin{cases}\omega(k)\\ \text{const}\,\omega^2(k)/n\end{cases}. \quad (2.28)$$

Taking adjoints,

$$[H_0^{1/2},a^*(k)]=-a_1^*(k), \quad (2.29)$$

and

$$[H_0^{1/2},[H_0^{1/2},a^*(k)]]=a_2^*(k). \quad (2.30)$$

Note that the double commutation $[H_0^{1/2},[H_0^{1/2},W]]$ can be expressed as a sum of two kinds of terms. There will be m contributions arising from double commutators of $H_0^{1/2}$ with one of the m creation or annihilation operators, and these will be of the form of the original W in (2.10), but they will have one $a(k)$ or one $a^*(k)$, replaced by an $a_2(k)$ or $a_2^*(k)$. There will also be $\frac{1}{2}m(m-1)$ terms arising from the commutator $H_0^{1/2}$ with two distinct creators or annihilators of W. These terms will be of the form of the right-hand side of (2.10) but will involve two of the modified operators $a_1(k)$ or $-a_1^*(k)$.

In order to prove (2.12), we use the definitions (2.22)–(2.24) and (2.27). In order to estimate the matrix element $|(\psi,[H_0^{1/2},[H_0^{1/2},W]]\theta)|$, we use the bound (2.25) and the first of the inequalities (2.28). As in the proof of (2.11), the Schwarz inequality then yields

$$\|[H_0^{1/2},[H_0^{1/2},W]](N+I)^{-m/2}\|\leq\text{const}\,\|\sum_{i=1}^{m}\omega(k_i)w\|_{L^2}, \quad (2.31)$$

which is (2.13). The proof of (2.14) proceeds in a similar fashion, using the second set of bounds for the $|\lambda_j|$.

In order to prove (2.12), we investigate the operator

$$X=(H_0+I)^{-1}W(H_0+I)^{-1}(N+I)^{-(m-4)/2}. \quad (2.32)$$

By following the proof of Lemma 2.41 and Theorem 2.43 of Ref. 7, we find that

$$\|X\|\leq\text{const}\,\|E^{-1/2}w\|_{L^2},$$

where

$$E=\begin{cases}\sup_{i\neq j}\omega(k_i)\omega(k_j) & \text{if } m>1\\ \omega(k_i) & \text{if } m=1.\end{cases}$$

[7] J. Glimm, Commun. Math. Phys. 5, 343 (1967).

Replacing the operator W by $\hat{W}=[H_0^{1/2},[H_0^{1/2},W]]$, we find that for the corresponding $\hat{X}$,

$$\|\hat{X}\|\leq \text{const } \|E^{-1/2}\sum_{i=1}^{m}\omega(k_i)w\|_{L^2}.$$

Each $\omega(k_i)$ is dominated by a constant multiple of $E^{1/2}$, unless k_i is very much larger than the magnitude of all the other momenta, or unless $m=1$. The former can occur only when $\sum_{i=1}^{m} k_i$ is large in magnitude, and in either case

$$\omega(k_i)\leqq \text{const } E^{1/2}\omega^{1/2}(\sum_{i=1}^{m} k_i).$$

Thus

$$\|\hat{X}\|\leqq \text{const } \|\omega^{1/2}(\sum_{i=1}^{m} k_i)w\|_{L^2},$$

which was to be proved.

III. SELF ADJOINTNESS OF THE INTERACTION HAMILTONIAN

For a real spatial cutoff $g(x)$ in the Schwartz space $\mathfrak{S}(R^1)$, the interaction part of the Hamiltonian $H_I(g)$ is self adjoint. This result was proved by Lanford, and also by Doplicher and Jaffe. Statement of these theorems are given in Wightman.[8] With a slightly different formulation, the result has been announced by Segal.[6]

We initially define $H_I(g)$ on the domain

$$D_0=\bigcap_{n=0}^{\infty} D(H_0^n),$$

and we will prove that its closure is self adjoint. It follows from (2.11) that H_I is well defined on D_0.

Theorem 3.1: If $g\in\mathfrak{S}(R^1)$ is real, then

$$H_I(g)=\int_{t=0} :\varphi^4(x):g(x)dx \tag{3.1}$$

is essentially self adjoint on D_0.

Let us introduce a domain D_1 obtained by applying any polynomial of the $t=0$ fields $\varphi(f_i)$, $f_i\in\mathfrak{S}(R^1)$, to the no-particle state Ω_0. Clearly $D_1\subset D_0$, and any vector Ω in D_1 is an entire vector for $\varphi(f)$, which means that the power series

$$\sum_{n=0}^{\infty}\frac{\|\varphi(f)^n\Omega\|}{n!}z^n \tag{3.2}$$

defines an entire function of z. Since D_1 is dense in Fock space, a result of Nelson,[9] shows that for real f, $\varphi(f)$ is essentially self adjoint on D_1. A similar argument can be made for the canonically conjugate $t=0$ fields $\pi(f)$.

Let $\mathfrak{M}$ denote the von Neumann algebra of operators generated by the spectral projections of all the $t=0$ fields $\varphi(f)$, $f\in\mathfrak{S}(R^1)$. The algebra $\mathfrak{M}$ is maximal Abelian. In other words, a bounded operator which commutes with all operators in $\mathfrak{M}$ is itself in $\mathfrak{M}$.

Let us consider $\varphi(f)$ for supp $f\subset O\subset R^1$, where O is an open region of space. (The support of a function is the smallest closed set outside of which the function vanishes identically.) Define $\mathfrak{A}(O)$ as the von Neumann algebra of operators generated by the spectral projections of all the fields $\varphi(f)$ and $\pi(f)$, supp $f\subset O$.

Since

$$\varphi(x,t)=\int\left(\Delta(x-y,t)\pi(y)-\frac{\partial\Delta}{\partial t}(x-y,t)\varphi(y)\right)dy, \tag{3.3}$$

and $\Delta(x,t)$ vanishes outside the light cone, we infer that

$$e^{iH_0t}\mathfrak{A}(O)e^{-iH_0t}\subset\mathfrak{A}(O_t), \tag{3.4}$$

where O_t is the region O expanded by t.

Theorem 3.2: If $g(x)\in\mathfrak{S}(R^1)$ is real and has its support in an open interval O, then for the H_I of (3.1)

$$\exp(iH_I(g)t)\in\mathfrak{A}(O)\cap\mathfrak{M}.$$

We now prove two lemmas.

Lemma 3.3: Let T be any operator with domain D_1 such that

$$TD_1\subset D(\varphi(f)^n), \tag{3.5}$$

$$TD_1\subset D(T|_{D_1}{}^*), \tag{3.6}$$

and

$$[T,\varphi(f)^n]D_1=0. \tag{3.7}$$

Then

$$\mathfrak{M}D_1\subset D(T|_{D_1}{}^-) \tag{3.8}$$

and

$$[T^-,\mathfrak{M}]D_1=0. \tag{3.9}$$

Proof: For $\Omega\in D_1$, from (3.5) and (3.7) we have

$$T\varphi(f)^n\Omega=\varphi(f)^nT\Omega.$$

But by (3.6), for real f

$$\|T\varphi(f)^n\Omega\|^2=(T\Omega,\varphi(f)^{2n}T\Omega)=(T^*T\Omega,\varphi(f)^{2n}\Omega)$$
$$\leqq\|T^*T\Omega\|\,\|\varphi(f)^{2n}\Omega\|.$$

Thus the convergent power series (3.2) shows that for Ω in D_1,

$$T^-e^{i\varphi(f)}\Omega=e^{i\varphi(f)}T\Omega. \tag{3.10}$$

It is clear that (3.10) is still valid with $\exp(i\varphi(f))$ replaced by strong limits of sums of such exponentials, and hence (3.8) and (3.9).

Lemma 3.4: Let $\mathfrak{M}$ be a maximal Abelian algebra of bounded operators on a Hilbert space $\mathfrak{H}$ with a cyclic vector Ω_0. Let T be a symmetric operator with domain

[8] A. S. Wightman, in *1964 Cargèse Summer School Lectures*, edited by M. Lévy (Gordon and Breach, Science Publishers, Inc., New York, 1967).

[9] E. Nelson, Ann. Math. **70**, 572 (1959); H. J. Borchers and W. Zimmermann, Nuovo Cimento **31**, 1047 (1963).

$\mathfrak{M}\Omega_0$, and let T commute with $\mathfrak{M}$. Then T is essentially self adjoint.

Proof: Without loss of generality, $\mathfrak{M}=L_\infty(X)$ and $\mathfrak{H}=L_2(X)$ for some measure space X, and Ω_0 is the function 1. Let $t=T\Omega_0$. Then $t\in L_2$ and T is multiplication by t, with domain L_∞. Let $f\in L_2$ and suppose $tf\in L_2$ also and let

$$f_n(x)=\begin{cases} f(x) & \text{if } |f(x)|\leqq n \\ 0 & \text{otherwise.}\end{cases}$$

Then $f_n\in L_\infty=D(T)$ and $f_n\to f$, $tf_n\to tf$ in L_2 norm by the bounded convergence theorem. Thus $\{f,tf\}$ is in the graph of the closure of T. Thus the closure of T is self adjoint, and T is essentially self adjoint.

Remark: Let T_n be a sequence of operators with the property of T in the lemma. Then $T_n\to T$ strongly on the domain $\mathfrak{M}\Omega_0$ if and only if $T_n\Omega_0\to T\Omega_0$.

Proof of Theorems 3.1 and 3.2: We apply the Lemmas 3.3 and 3.4 with the case $T=H_I(g)$, $\mathfrak{M}$ in Lemma 3.4 as in Lemma 3.3, the Hilbert space Fock space, and Ω_0 the Fock no-particle state. The hypotheses (3.5) and (3.6) can be verified by a direct computation.[10] Thus $H_I(g)$ is essentially self adjoint on $D_1\subset D_0$, and hence $H_I(g)$ is essentially self adjoint on D_0.

If we assume that supp $g\subset O$, then as O is an open interval, supp $g\subset O_1$ where O_1 is O contracted by some small amount $\epsilon>0$. Since $H_I(g)$ commutes with $\mathfrak{M}$, and $\mathfrak{M}$ is maximal Abelian, $\exp(iH_I(g)t)\in\mathfrak{M}$. Furthermore the argument in the proof of Lemma 3.3, can be repeated to show that $H_I(g)$ commutes with $\mathfrak{A}(O_1')$, where O_1' is the complement of the closure of O_1.[11] Since $\mathfrak{A}(R^1)$ is irreducible and $H_I(g)$ commutes with $\mathfrak{A}(O_1')$, $\exp(iH_I(g)t)\in\mathfrak{A}(O_2)$ where O_2 is O_1 expanded by any amount $\epsilon'>0$.[12] Taking $\epsilon'<\epsilon$, we have $\exp(iH_I(g)t)\in\mathfrak{A}(O)$, which completes the proof.

IV. SELF ADJOINTNESS OF THE TOTAL HAMILTONIAN

Theorem 4.1: (a) For real $g(x)\in\mathfrak{S}(R^1)$, the total Hamiltonian $H(g)=H_0+H_I(g)$ is self adjoint with the domain $D(H(g))=D(H_0)\cap D(H_I(g))$.

(b) The total Hamiltonian $H(g)$ is essentially self adjoint on the domain

$$D_0=\bigcap_{n=0}^{\infty} D(H_0{}^n).$$

In order to prove the self adjointness of H, we combine the estimates of Sec. II, the self adjointness of $H_I(g)$ proved in Sec. III, and a singular perturbation theory developed elsewhere.[4] We need the following result which is a special case of Theorem 8 of Ref. 4.

[10] A. M. Jaffe, J. Math. Phys. **7**, 1250 (1966).
[11] A. M. Jaffe, Ann. Phys. (N. Y.) **32**, 127 (1965).
[12] H. Araki, J. Math. Phys. **5**, 1 (1964).

Theorem 4.2: Under the hypotheses (i)–(iii) below, the operator $H=H_0+H_I$ is self adjoint.

(i) Both H_0 and H_I are self adjoint. The domain

$$D_0=\bigcap_{n=0}^{\infty} D(H_0{}^n)$$

is contained in the domain of H_I, and H_I is essentially self adjoint on D_0.

(ii) Let $N\geqq 0$ be a positive self adjoint operator, commuting with H_0, and such that $N\leqq\text{const } H_0$. Suppose that the operators $(I+N)^{-1}H_I(I+N)^{-1}$ and $(I+N)^{+1}H_I(I+N)^{-3}$ are bounded.

(iii) Suppose that for any $\epsilon>0$, there exists a number b such that as bilinear forms on $D_0\times D_0$,

$$-H_I\leqq\epsilon N+bI\,, \tag{4.1}$$

$$-[H_0{}^{1/2},[H_0{}^{1/2},H_I]]\leqq\epsilon H_0{}^2+bI\,, \tag{4.2}$$

and

$$-[N,[N,H_I]]\leqq\epsilon N^3+bI\,. \tag{4.3}$$

Proof of Theorem 4.1: In order to prove that H is self adjoint, we apply Theorem 4.2 in the case that H_0 is the free Hamiltonian, N is the number operator, and H_I is the interaction Hamiltonian $H_I(g)$. Thus we need to verify (i–iii). Condition (i) was dealt with in Theorem 3.1, while condition (ii) is a consequence of (2.11). In Refs. 2, and 3, it is shown that for any $\epsilon>0$, there is a number b such that

$$-H_I(g)\leqq\epsilon H_0+bI\,.$$

By following that proof, but using the smoothing operator $\exp(-tN)$, in place of $\exp(-tH_0)$, one arrives at the estimate (4.1) required in (iii). The remaining estimates (4.2) and (4.3) were established in Theorem 2.1. Thus we conclude from Theorem 4.2 that $H(g)$ is self adjoint on the domain $D(H_0)\cap D(H_I(g))$.

We now show that $H(g)$ is essentially self adjoint on D_0. We first show that $H(g)$ is essentially self adjoint on $D_2=D(H_0)\cap D(N^2)$. By (2.11) it is clear that the domain of $H(g)$ contains D_2. For $\psi\in D(H(g))=D(H_0)\cap D(H_I(g))$, consider $\psi_n\in D_2$ defined by

$$\psi_n=n(nI+N)^{-1}\psi\,. \tag{4.4}$$

Thus $\|\psi_n-\psi\|+\|H_0\psi_n-H_0\psi\|\to 0$ as $n\to\infty$. We study

$$H_I\psi_n-H_I\psi=-N(nI+N)^{-1}H_I\psi+n[H_I,\,(nI+N)^{-1}]\psi\,.$$

Since $N(nI+N)^{-1}$ is a uniformly bounded sequence converging to zero on the dense set $D(N)$, it converges to zero and $\|N(nI+N)^{-1}H_I\psi\|\to 0$ as $n\to\infty$. But for the other term

$$\begin{aligned} n[H_I,\,(nI+N)^{-1}]\psi &=[H_I,\,(nI+N)^{-1}](nI+N)n(nI+N)^{-1}\psi \\ &=(nI+N)^{-1}[N,H_I]n(nI+N)^{-1}\psi \\ &=(nI+N)^{-1}(I+N)(I+N)^{-1}[N,H_I] \\ &\qquad\times(I+N)^{-1}n(nI+N)^{-1}(I+N)\psi\,. \end{aligned}$$

Note that as $n\to\infty$, $n(nI+N)^{-1}(I+N)$ converges strongly to $(I+N)\psi$, that by (2.11), $(I+N)^{-1}[N,H_I]\times(I+N)^{-1}$ is bounded, and that $(nI+N)^{-1}(I+N)$ converges strongly to zero. Thus $\|[H_I,(nI+N)^{-1}]\psi\|\to 0$ as $n\to\infty$, and so $\|H_I\psi_n-H_I\psi\|\psi\to 0$. Thus we see that $H(g)$ is the closure of $H(g)$ restricted to D_2, so $H(g)$ is essentially self adjoint on D_2.

Let D_2 be a Hilbert space with the norm

$$(\|\psi\|_{D_2})^2=\|\psi\|^2+\|H_0\psi\|^2+\|N^2\psi\|^2.$$

From (2.11) we infer that

$$\|H\psi\|\leqq \text{const}\|\psi\|_{D_2},$$

so that $H(g)$ is essentially self adjoint on any subset of D_2 which is dense in the Hilbert space D_2. For $\psi\in D_2$, $\psi_\lambda=\exp(-\lambda H_0)\psi\in D_0=\bigcap_{n=0}^{\infty}D(H_0{}^n)$, and $\|\psi_\lambda-\psi\|_{D_1}\to 0$ as $\lambda\to 0$. Thus $H(g)$ is essentially self adjoint on D_0.

V. REMOVING THE SPATIAL CUTOFF AND LOCALITY

For the reader's convenience, we sketch a proof of Segal's theorem[6] that the self adjointness of $H(g)$ allows the removal of the spatial cutoff. In fact, if A is a bounded function of the free fields localized in a bounded region of space at $t=0$, then

$$\sigma_t(A)=e^{itH(g)}Ae^{-itH(g)}$$

is independent of $g(x)$ provided that $g(x)=\lambda$, the desired coupling constant, on a sufficiently large interval, depending on t. Furthermore, if A is localized in the region of space O, then $\sigma_t(A)$ is localized in the region O_t, where O_t is the region O expanded by t. (We have taken the velocity of light to be one.) In other words, the time translation σ_t gives rise to a local theory. If one chooses for the operator A a spectral projection of the $t=0$ field $\varphi(f)$, one can piece together the time translation operator for the fields themselves.

In section IV, we showed that $H=H_0+H_I$, which is sum of two self adjoint operators, is itself self adjoint. As a consequence of this fact, the Trotter product formula[13] says that for all ψ

$$e^{itH}\psi=\lim_{n\to\infty}(e^{itH_0/n}e^{itH_I/n})^n\psi.$$

Thus

$$\sigma_t(A)\psi=\lim_{n\to\infty}(e^{iH_0t/n}e^{iH_It/n})^nA\,(e^{-iH_It/n}e^{-iH_0t/n})^n\psi.$$

Let O be the region defined by $|x|<M$, $t=0$, and let $A\in\mathfrak{A}(O)$, where $\mathfrak{A}(O)$ is defined in Sec. III.

Given an arbitrary, positive ϵ, split $g(x)$ into two infinitely differentiable parts

$$g(x)=g_1(x)+g_2(x),$$

where supp $g_1(x)\subset O_\epsilon$, and supp $g_2\bigcap O_{\epsilon/2}$ is empty. Write

$$H_I(g)=H_I(g_1)+H_I(g_2),$$

so that as a consequence of theorems 3.1 and 3.2, $H_I(g_1)$ and $H_I(g_2)$ commute, and

$$\exp[iH_I(g)t/n]=\exp[iH_I(g_1)t/n]\exp[iH_I(g_2)t/n].$$

Furthermore,

$$\exp[iH_I(g_1)t/n]\in\mathfrak{A}(O_\epsilon),$$

and $\exp[iH_I(g_2)t/n]$ commutes with $\mathfrak{A}(O_{\epsilon/4})$. Therefore,

$$A_1(t)=\exp(iH_0t/n)\exp[iH_I(g)t/n]A\times\exp[-iH_I(g)t/n]\exp(-iH_0t/n)$$

depends on $g(x)$ only in the region O_ϵ, and by the free propagation property (3.4),

$$A_1\in\mathfrak{A}(O_{(t/n)+\epsilon}).$$

We continue step by step, and after n steps we conclude that

$$A_n(t)=[\exp(iH_0t/n)\exp(iH_I(g)t/n)]^nA\times[\exp(-iH_I(g)t/n)\exp(-H_0t/n)]^n$$

depends on $g(x)$ only in the region $O_{t+n\epsilon}$, and

$$A_n(t)\in\mathfrak{A}(O_{t+n\epsilon}).$$

Since ϵ can be chosen arbitrarily, $A_n(t)$ depends on $g(x)$ only in the region $\bar{O}_t$, the closure of O_t, and

$$A_n(t)\in\bigcap_{\epsilon>0}\mathfrak{A}(O_{t+\epsilon}).$$

Thus $A_n(t)$ commutes with any local observable B localized in open region of space O' such that O' and O_t are disjoint. As this is true for each n, it is true for

$$\sigma_t(A)=\text{strong}\lim_{n\to\infty}A_n(t).$$

Hence $\sigma_t(A)$ is *local*, and it depends on $g(x)$ only in the region $\bar{O}_t$, where we choose $g(x)=\lambda$. We therefore conclude that the spatial cutoff has been removed and the resulting theory is local.

VI. THEORY IN A BOX

We can consider a somewhat different cutoff theory, namely the $g\varphi^4$ theory in a periodic box. This gives a cut-off interaction which is translation invariant, and therefore it is useful for the study of the vacuum state.[14] In a finite interval, but with no ultraviolet cutoff, we prove that the total Hamiltonian is self adjoint and has a complete set of normalizable eigenstates.

The theory in volume V is constructed by taking a Fock space $\mathfrak{F}_V$ of functions defined on the momentum

[13] H. F. Trotter, Proc. Am. Math. Soc. **10**, 545 (1959); E. Nelson, J. Math. Phys. **5**, 332 (1964).

[14] A. M. Jaffe and R. T. Powers, Commun. Math. Phys. **7**, 218 (1968); J. Glimm and A. M. Jaffe (to be published).

space lattices Γ_V, $\Gamma_V\times\Gamma_V$, etc., where

$$\Gamma_V=\{k:\ k=2\pi n/V,\ n=0,+1,\pm 2,\cdots\}. \quad (6.1)$$

Thus the free Hamiltonian is

$$H_{0,V}=\sum_{k\in\Gamma V} a_V^*(k)a_V(k)\omega(k), \quad (6.2)$$

and the $t=0$ field is

$$\varphi_V(x)=\frac{1}{(2V)^{1/2}}\times\sum_{k\in\Gamma_V} e^{-ikx}[a_V^*(k)+a_V(-k)]\omega(k)^{-1/2}. \quad (6.3)$$

The interaction Hamiltonian $H_{I,V}$ is defined by

$$H_{I,V}=g\int_V :\varphi_V(x)^4:dx, \quad g\geqq 0 \quad (6.4)$$

and $H_{I,V}$ is self adjoint as was shown in Sec. III for $H_I(g)$. The total Hamiltonian in a box is

$$H_V=H_{0,V}+H_{I,V}.$$

It is possible to regard $\mathfrak{F}_V$ as a subspace of $\mathfrak{F}$ defined by functions on R, $R\times R$, etc., which are piecewise constant between lattice sites. The correspondence between a_V and a is

$$a_V(k)=\left(\frac{V}{2\pi}\right)^{1/2}\int_0^{2\pi/V} a(k+l)dl, \quad k\in\Gamma_V. \quad (6.5)$$

Therefore the estimates of Sec. II are valid for H_V, and so H_V is self adjoint on the domain $D(H_{0,V})\bigcap D(H_{I,V})$, and H_V is essentially self adjoint on the domain

$$D_0=\bigcap_{n=0}^{\infty} D(H_{0,V}^n).$$

Theorem 6.1: The spectrum of H_V is discrete with finite multiplicity, so H_V has a complete set of normalizable eigenstates.

Proof: Let $\Gamma_{\kappa,V}=\Gamma_V\bigcap\{k:|k|\leqq\kappa\}$, so that an ultraviolet cut-off field $\varphi_{\kappa,V}(x)$ is obtained from φ_V by summing (6.4) over $\Gamma_{\kappa,V}$. Then a sharply cut-off Hamiltonian $H_{\kappa,V}=H_{0,V}+H_{I,\kappa,V}$ comes from

$$H_{I,\kappa,V}=g\int_V :\varphi_{\kappa,V}(x)^4:dx.$$

The operator $H_{\kappa,V}$ has pure discrete spectrum with finite multiplicity.[15] Furthermore, $H_{I,\kappa,V}$ is the sum of five expressions of the form $W_{\kappa,V}$ of (2.10), such that $\|w_{\kappa,V}-w_{\kappa',V}\|_{L^2}\rightarrow 0$ as $\kappa,\kappa'\rightarrow\infty$. Therefore the bounded operators $(I+N_V)^{-1}H_{I,\kappa,V}(I+N_V)^{-1}$ converge in norm as $\kappa\rightarrow\infty$ to $(I+N_V)^{-1}H_{I,V}(I+N_V)^{-1}$. We now appeal to Corollary 10 of Ref. 4 to infer that the resolvents of $H_{\kappa,V}$ converge in norm as $\kappa\rightarrow\infty$ to the resolvent of H_V. Since each $R_{\kappa,V}=(H_{\kappa,V}+c)^{-1}$ is compact, so is their uniform limit R_V, and the theorem is proved.

We further see that the projections onto each eigenvalue $\lambda_n(\kappa,V)$ of $H_{\kappa,V}$ converge as $\kappa\rightarrow\infty$. This is a consequence of Theorem IV. 3.16 of Kato.[16] Thus, in particular the vacuum vectors for $H_{\kappa,V}$ converge in Fock space to a vacuum vector for H_V.

VII. CONCLUSIONS

We have shown the existence of a local time translation for a two-dimensional $\lambda\varphi^4$ theory without cutoffs. The method relies on basic inequalities satisfied by the first and second power of the Hamiltonian. Similar estimates of all the higher powers of the Hamiltonian proved useful to investigate the existence of vacuum-expectation values in a $\lambda\varphi^4$ theory with a sharp momentum-space cutoff.[15] It does not seem likely, in our case, that H_0^3 is dominated by higher powers of H, since the ground state of H must be in the domain of H^n, for all n, and the first-order perturbation correction to the Fock vacuum is in $D(H_0)$, but not in $D(H_0^{3/2})$.

[15] A. M. Jaffe, Ph.D. thesis, Princeton University (to be published).

[16] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1966).